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TANGENT MODULUS AND THE STRENGTH OF STEEL COLUMNS IN TESTS

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By O. H. Basquin

ABSTRACT

The tangent modulus describes the semielastic action of a material when subjected to excessive stress, being defined as the ratio of the rates of increase of stress and of strain at that stress. In 1889 Engesser suggested that if this modulus is used in Euler's formula in place of Young's modulus, the resulting formula should be appropriate for estimating the strength of columns of ordinary proportions. To test the accuracy of this proposition, the author has made a study of more than 200 column tests, which were completed by the Bureau of Standards in 1916 and whose general results have been reported by the American Society of Civil Engineers and the American Railway Engineering Association. Tests of columns of 50 L/R are used to determine the tangent modulus values for each type of column, and from these values estimates are made for the strength of longer columns of the same types with the following average errors: 620 lbs./in.^{2} for columns of 85 L/R, 1,190 lbs./in.^{2} for 120 L/R, and 2,100 lbs./in.^{2} for 155 L/R. Some long columns in their tests appear to show smaller tangent moduli than do similar short columns. While Engesser's formula is not completely satisfactory, it is regarded as superior to that proposed by Karman in 1910 as an improvement upon the former. As related problems, short discussions are given of the direction of deflection in buckling, eccentricity of loading, end restraint, effect of loading upon tangent moduli, and time effect.

CONTENTS

I. INTRODUCTION

The importance of safe and economical design of steel structures and, in particular, columns or compression members has caused the Bureau of Standards to undertake a number of extensive experimental investigations of various phases of the subject.

First.—In cooperation with the American Bridge Co. and the Bethlehem Steel Co. the bureau tested 18 full size compression members designed as appropriate for two large bridges and made from various grades of steel. These tests are reported in Technologic Papers of the Bureau of Standards, No. 101.

Second.—In cooperation with the American Bridge Co. the bureau tested 170 structural steel angles as compression members for use as legs and lattice members in steel towers. These tests are reported in Technologic Papers of the Bureau of Standards, No. 218.

Third.—In cooperation with the American Society of Civil Engineers and the American Railway Engineering Association, the bureau tested more than 200 steel columns of such designs as were thought appropriate for use in steel frame buildings and smaller truss bridges. The general results of these tests have been published by the societies concerned. The present paper is a study of these data which has been made from the standpoint of a theory that attempts to define the general conditions for column strength.

Fourth.—In cooperation with the American Bridge Co. and the Bethlehem Steel Co. the bureau has tested 69 large columns of H-shaped section for the purpose of comparing fabricated and solid rolled sections of the same cross-sectional area. A report on this investigation is nearly completed.

Fifth.—In cooperation with the Delaware River Bridge Joint Commission the bureau has tested 14 web members for the towers of the new Camden Bridge, the object being to determine the best relation between web thickness and web width. A report on this investigation is nearly completed.

Sixth.—There is now being carried on an investigation of the general relation between the strength of columns and the properties of the materials of which they are composed. This investigation is planned not only to clear up outstanding difficulties in predicting the strength of steel columns of ordinary properties, but to determine column formulas applicable to extreme ranges of size and shape, and to other materials than steel.

These investigations have been planned by different groups of engineers with different specific objects in view, and for this reason unity of presentation would be difficult to obtain. Each writer is given the fullest liberty to express his own opinions, so long as they are pertinent and not obviously wrong. The bureau will welcome the receipt of criticisms of these papers and of suggestions which may lead to a better understanding of column strength.

II. PROBLEM

In cooperation with committees of the American Society of Civil Engineers and the American Railway Engineering Association, the Bureau of Standards has tested a large number of steel columns.[1] In the summer of 1916, through the kindness of Director S. W. Stratton, the writer enjoyed the exceptional privilege of studying the complete data of the tests then completed. It

[1] Proc. Am. Ry. Eng. Assn., **16**, p. 636, 1915; ibid., **19**, p. 789, 1918; Trans. Am. Soc. C. E., **46**, p. 401, 1910; Proc. A. S. C. E., 1913; Trans. Am. Soc. C. E., **83**, p. 1583, 1919–1920.

has been understood that one of the principal objects of the bureau in making these tests was to make a beginning in an attempt to gain a complete understanding of column action in tests through study of stress-strain data alone. The tangent modulus appeared to offer advantages over the more common stress-strain curve, and the writer's work has been largely confined to an attempt to interpret the data from the standpoint of the tangent modulus. The time of one summer vacation proved entirely inadequate to complete a satisfactory survey of the mass of material available, and for similar reasons this report has been delayed for many years. It is felt that a still more thorough study of the data would furnish many valuable conclusions.

III. TANGENT MODULUS AND ENGESSER'S THEORY OF COLUMN STRENGTH

1. DEFINITION OF TANGENT MODULUS

In Figure 1 the curve *OABC* approximately represents the average stress-strain curve for steel columns of 50 L/R tested in the usual manner with flat ends. At small stresses this curve follows the straight line *OG* (fig. 1), which represents 30,000,000 lbs./in.^{2} as the value of Young's modulus of elasticity. At a stress which is somewhat smaller than 15,000 lbs./in.^{2}, the curve leaves the straight line *OG*, and as the stress continues to increase, the slope of the curve with respect to the horizontal axis continues to decrease. To estimate the value of this slope at any point of the curve, such as at *B* whose ordinate is 28,000 lbs./in.^{2}, one may draw a line *DE* tangent to the curve at *B*; parallel to *DE* the line *OF* is drawn through the origin, and the point *F* on this line shows a strain of 0.10 per cent and a stress of 13,500 lbs./in.^{2}, indicating that the slope of *DE* to the horizontal axis is represented by

$$\frac{13{,}500 \text{ lbs./in.}^2}{0.001} = 13{,}500{,}000 \text{ lbs./in.}^2$$

This slope is known as the *tangent modulus* at this stress. Below the proportional limit the tangent modulus has the same value as Young's modulus; above the proportional limit the tangent modulus has a smaller value than Young's modulus; its value varies with the stress used in its determination, generally decreasing as the stress increases.

If the column were provided with such lateral support as wou d prevent buckling either as a whole or in some of its parts, and if the loading were extended to give a strain of 2 or 3 per cent, the tangent modulus would show greater values toward the end of the test than in the yield stress range, but this subsequent increase in the tangent modulus has little application to ordinary column testing, except in the interpretation of the action of short, heavy columns when the manner of failure is emphasized.

The tangent modulus at any stress may be defined as the slope, with respect to the horizontal axis, of the stress-strain curve at

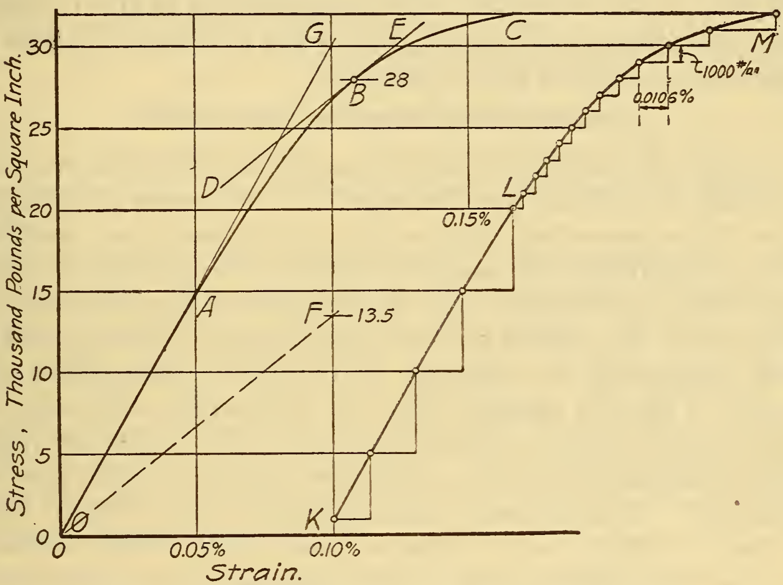

FIG. 1.—*Average compressive stress-strain curve for columns of 50 L/R, illustrating a theoretical method and a practical method for determining the tangent modulus*

the stress in question, it being understood that the test is conducted in a regular way under gradually increasing stress. We may use E' to denote tangent modulus, p to denote stress, and e to denote strain; the definition of tangent modulus may be expressed as

$$E' = \frac{dp}{de}$$

in which dp and de are corresponding increments of stress and of strain, observed when the stress has the value to which E' relates.

2. ENGESSER'S THEORY OF COLUMN STRENGTH

Because Euler's formula contains Young's modulus as a factor, it can not be used to estimate the strength of a column if this strength exceeds the proportional limit of the material. Engesser [2] appears to have been the first to suggest that this difficulty may be obviated by the simple expedient of using the tangent modulus in Euler's formula in place of Young's modulus. Although, as a result of the work of Considere,[3] Engesser subsequently modified his views upon column strength, and while there may be theoretical objections to the validity of this use of the tangent modulus, it seems desirable to see how this formula works when applied to the tests under examination, and it seems proper to give it Engesser's name both to distinguish it from the well-known formula of Euler and to indicate its first advocate.

3. ASSUMPTIONS MADE IN APPLICATION

Engesser in his formula used the tangent modulus of the material of which the column was fabricated. In the series of tests discussed here stress-strain data for the material itself is not available.

As in theoretical discussions of columns it is customary to make a number of assumptions as to the character of the material as to the form of the column and as to the manner of loading, and as such assumptions are necessary for the logical development of a theory, it may be assumed that the stress-strain data taken for the columns of small slenderness ratio (50 L/R) very nearly represent the actual stress-strain data of the material of which they are composed. These columns may actually have had different kinds of steel in different parts of their sections; most of them had been punched and riveted; some of them may have been straightened after fabrication; they were not precisely straight; and they were loaded with more or less eccentricity.

Tangent moduli obtained from specimens of the column material might have very different values. Nevertheless, in the absence of the stress-strain data for the material itself, the assumption, that the data derived from the tests of the short columns may logically be considered as representative of the material, may not be seriously in error. In this discussion the tangent moduli used (which are derived from the test data for short columns) will, therefore, be spoken of as that of the material.

Our problem then is to interpret data taken from actual tests of actual columns from the standpoint of the tangent modulus (E'). The tangent modulus values that appear to be needed for

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[2] Engesser, Zeitschrift des Hannov. Ing. und Arch.-Ver., p. 455, 1889; Schweizerische Bauzeitung, **26**, p. 24, 1895; Zeitschrift des Vereines deutscher Ingenieure, p. 927, 1898.

[3] Resistance des Pièces comprimées Congrès International des Procédés de Construction, Annexe à comptes Rendus, p. 382, 1891.

this study are those that characterize the columns in their tests—average values for their entire sections and for as much of their lengths as possible.

4. PRACTICAL METHOD FOR OBTAINING TANGENT MODULI

In a column test the stress-strain curve is not obtained directly, strains are observed at certain applied stresses, corresponding stress and strain values are plotted as coordinates of points on a diagram, and the stress-strain curve is then drawn as an interpretation of the data. In Figure 1 the curve *KLM* has the same form as the curve *OABC* explained above, but on the curve *KLM* points are shown at which corresponding stresses and strains were normally observed in the column tests, and through which the curve *KLM* has been drawn. The coordinates of such points are fundamental data; the stress-strain curve is a matter of interpretation.

The points on curve *KLM*, Figure 1, are connected not only by the smooth stress-strain curve, but by a sort of stress-strain staircase in which the risers represent increments of stress while the treads represent increments of strain. Thus, as the stress was increased from 29,000 to 30,000 lbs./in.^{2}, the increment of stress is shown in the figure as 1,000 lbs./in.^{2}, while the increment of strain is given as 0.0106 per cent. As a rough approximation, these increments may be identified with dp and de, respectively, to give a tangent modulus of

$$\frac{1{,}000 \text{ lbs./in.}^2}{0.000106} = 9{,}400{,}000 \text{ lbs./in.}^2$$

which characterizes the column material at some stress between 29,000 and 30,000 lbs./in.^{2}; and the stress appropriate to such a modulus value will be taken as half-way between the limiting stresses used in making the estimate; that is, in this case, 29,500 lbs./in.^{2} This method of estimating values of the tangent modulus is applicable to every one of the steps of the stress-strain stairway (fig. 1); the stress-strain curve itself is not needed in this method of determining tangent moduli; the values obtained depend upon the data directly, and not upon a smoothed curve drawn from the data.

5. APPLICATION TO THE DATA OF A COLUMN TEST

In order to illustrate the process as applied to the data of an actual column test, we may refer to Table 1 and to Figure 2, both of which relate to the test of column No. 176, type 1A, 20 L/R. In the table the first column lists the stresses at which gauge-line measurements were made; the second, third, fourth, and fifth columns contain the decrease in length which was observed at each stress for each of the four gauge lines whose numbers (1, 3, 5, and 7) appear at the head of these columns and whose locations

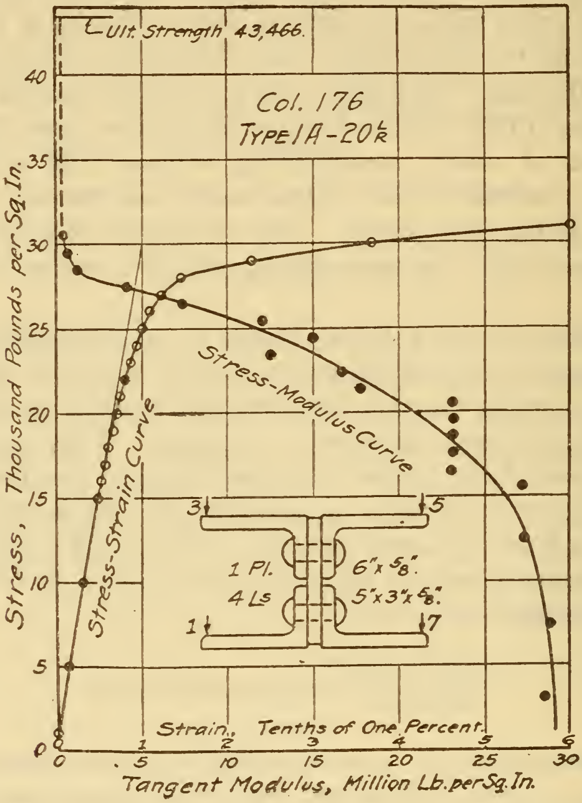

FIG. 2.—*Comparison of stress-strain curve and stress-modulus curve for column No. 176, type 1A, 20 L/R*

The arrowheads numbered 1, 3, 5, and 7 indicate locations of gauge lines on the section

on the column section are indicated by arrow heads in Figure 2. In the sixth column of the table there is shown at each stress the average compression or shortening of the four gauge lines, the mean of the values given in the four preceding columns for the same stress. Since the gauge lines were 30 inches long, the strain at any stress is found by dividing this mean compression by 30 inches; and the stress-strain curve of Figure 2 has been plotted to such values of strain and to the values of stress that are listed in the first column of the table.

TABLE 1.—Computation of Tangent Modulus for Column No. 176, Type 1A, 20 L/R

[Gauge lines 30 inches long]

Stress (in pounds per square inch)	Compression					Mean increment of compression	Tangent modulus E′
	Gauge line No. 1	Gauge line No. 3	Gauge line No. 5	Gauge line No. 7	Mean		
	Inch	Inch	Inch	Inch	Inch	Inch	Million lbs./in.²
1,000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0042	28.6
5,000	.0050	.0045	.0041	.0033	.0042	.0052	28.8
10,000	.0100	.0100	.0093	.0082	.0094	.0055	27.3
15,000	.0150	.0155	.0157	.0133	.0149	.0011	27.3
16,000	.0165	.0165	.0168	.0143	.0160	.0013	23.1
17,000	.0178	.0178	.0180	.0158	.0173	.0013	23.1
18,000	.0192	.0190	.0193	.0170	.0186	.0013	23.1
19,000	.0208	.0202	.0208	.0180	.0199	.0013	23.1
20,000	.0221	.0215	.0220	.0191	.0212	.0013	23.1
21,000	.0240	.0226	.0232	.0203	.0225	.0017	17.6
22,000	.0258	.0239	.0250	.0223	.0242	.0018	16.7
23,000	.0272	.0255	.0268	.0247	.0260	.0024	12.5
24,000	.0299	.0275	.0289	.0272	.0284	.0020	15.0
25,000	.0320	.0296	.0305	.0295	.0304	.0025	12.0
26,000	.0342	.0324	.0327	.0322	.0329	.0040	7.5
27,000	.0390	.0366	.0360	.0361	.0369	.0073	4.1
28,000	.0472	.0465	.0410	.0420	.0442	.0245	1.2
29,000	.0750	.0810	.0600	.0590	.0687	.0418	.7
30,000	.1250	.1350	.0970	.0850	.1105	.0700	.4
31,000	.2000	.2300	.1520	.1400	.1805		
43,466 [1]							

[1] Ultimate strength.

The values given in the seventh column of Table 1 are the differences of the successive values found in the preceding column; these differences when divided by 30 inches, the length of the gauge lines, are increments of strain, the treads of the stress-strain stairway, while the risers of this stairway are the increments of stress whose values are evident from the first column of the table. The last column of the table contains values of the tangent modulus, E', expressed in million pounds per square inch. Any one of these values is found by multiplying the increment of stress by the ratio of 30 inches to the increment of compression found in the next to the last column of the table; thus the first value of the tangent modulus is

$$\frac{4{,}000 \text{ lbs./in.}^2 \times 30 \text{ in.}}{0.0042 \text{ in.}} = 28{,}600{,}000 \text{ lbs./in.}^2$$

which value of E' is assigned to the stress 3,000 lbs./in.², while the last value of the tangent modulus given in the table is

$$\frac{1{,}000 \text{ lbs./in.}^2 \times 30 \text{ in.}}{0.0700 \text{ in.}} = 400{,}000 \text{ lbs./in.}$$

which value is given to the stress 30,500 lbs./in.²

6. STRESS-MODULUS CURVE

In Figure 2 the points represented by solid, small circles have ordinates which represent stresses that lie midway between successive stresses at which gauge-line measurements were made, as indicated in Table 1, while the abscissas of these points represent tangent moduli as given in the last column of Table 1, plotted to the lower scale along the base line of Figure 2. The curve marked stress-modulus curve has been drawn as a smooth curve to represent a possible interpretation of the manner in which the tangent modulus for this column gradually varied as the stress was increased. Since the ultimate strength of the column was 43,466 lbs./in.^{2}, and since strain-gauge readings were not made above 31,000 lbs./in.^{2}, the upper part of the stress-modulus curve is shown as dotted to indicate that its exact location is not determined.

7. CHECKING THE LOCATION OF A STRESS-MODULUS CURVE

One may check the location of a stress-modulus curve by a process which is practically the reverse of that used above in finding values of the tangent modulus. This is based upon the evident relation

$$e=\int_{po}^{p}\frac{dp}{E'}$$

Within the accuracy here obtainable the integral may be replaced by the summation

$$e=\sum_{po}^{p}\frac{\Delta p}{E'}$$

One reads *from the stress-modulus curve* the values of the tangent modulus at the stresses which were used in plotting the points; knowing the increment of stress which corresponds to each modulus value, one can calculate the increment of strain which the curve provides for each stress increment. The summation of these values from the initial stress up to any observed stress gives the total strain which the curve provides for that stress, and these values may be compared with the strain data given by the test.

Table 2 shows such a check calculation for the location of the stress-modulus curve of Figure 2. The last column of this table shows zero values at the beginning and at the end of the comparison; the intermediate values are nearly all negative, whereas they should be about equally distributed between positive and negative values and have a total which is approximately zero. As the difference between the integral and the summation could nowhere be as great as 0.001 inch, this check calculation shows that the modulus values were chosen slightly too small near the end of the comparison, but the error is so slight that it has not been thought worth while to make any change in the curve shown in Figure 2.

It will be noted that the stress-strain curve of Figure 2 has not been used in finding the stress-modulus curve of the same figure. The stress-strain curve has been shown in order to give the reader an opportunity to compare these two types of curves, both of which are based upon the same test data. The stress-modulus curve emphasizes the departure of the stress-strain curve from the straight line relation at rather small stresses.

TABLE 2.—Calculation for Check on Location of Stress-Modulus Curve Shown in Figure 2

Stress (in pounds per square inch)	Tangent modulus read from curve	Calculated increment of compression	Summation of increments of compression	Actual compression given in Table 1	Difference
	Million lbs./in.^{2}	Inch	Inch	Inch	Inch
1,000			0.0000	0.0000	0.0000
3,000	29.1	0.00412			
5,000			.0041	.0042	−.0001
7,500	28.7	.00521			
10,000			.0093	.0094	−.0001
12,500	27.5	.00545			
15,000			.0148	.0149	−.0001
15,500	26.0	.00115			
16,000			.0159	.0160	−.0001
16,500	25.1	.00120			
17,000			.0171	.0173	−.0002
17,500	24.1	.00125			
18,000			.0184	.0186	−.0002
18,500	23.0	.00130			
19,000			.0197	.0199	−.0002
19,500	21.7	.00138			
20,000			.0211	.0212	−.0001
20,500	20.2	.00148			
21,000			.0225	.0225	.0000
21,500	18.7	.00160			
22,000			.0241	.0242	−.0001
22,500	17.0	.00176			
23,000			.0259	.0260	−.0001
23,500	15.2	.00197			
24,000			.0279	.0284	−.0005
24,500	13.0	.00231			
25,000			.0302	.0304	−.0002
25,500	10.6	.00283			
26,000			.0330	.0329	.0001
26,500	7.7	.00390			
27,000			.0369	.0369	.0000

8. END CONDITIONS

A column is said to have fixed ends if its axis at both ends remains tangent to the same straight line throughout the period under discussion. For the purpose of estimating the strength of a column, this period under discussion extends from the beginning of the loading to the instant of maximum load. We are not interested in what happens to the column after it has passed this maximum. The columns under consideration were tested with "flat ends," but as a matter of fact, which will be discussed later, these columns in their tests up to maximum load acted in almost the same manner as they would have acted if their ends had been fixed. It is doubtful if their action would have been improved if their ends had been securely riveted or welded to the heads of the testing machine. We shall, therefore, treat these tests as if the columns had fixed ends.

9. ENGESSER'S FORMULA

Engesser's formula for the strength of a column with fixed ends may be written as

$$p=\frac{4\pi^2 E'}{\left(\frac{L}{R}\right)^2}$$

in which p is the ultimate strength per unit of sectional area, E' is the tangent modulus of the fabricated material of the column when the average stress on the column is p and applied for the first time, L is the length of the column between its flat ends, and R is the least radius of gyration of its section.

10. GRAPHICAL SOLUTION

Engesser's formula can not be solved algebraically to give the ultimate strength p of a column, unless the general relation between p and E', as shown by the stress-modulus curve, can be expressed in algebraic form. We shall resort to a graphical solution. We may write Engesser's formula in the form

$$\frac{p}{E'}=\frac{4\pi^2}{\left(\frac{L}{R}\right)^2}$$

The first member of this equation is simply the ratio of the ultimate strength of the column to the value which the tangent modulus has at this average stress. For a column of any particular slenderness ratio, the right member of this equation is a constant whose value can be calculated without knowing anything about the column except (1) that it has fixed ends and (2) what is the value of its slenderness ratio. Thus, if the slenderness ratio is 100, the value of this constant is 0.00395, while it is 0.00175 if the slenderness ratio is 150.

Figure 3 A shows a diagram in which the ordinates and abscissas are practically the same as those seen in Figure 2 for the stress-modulus curve there shown; but in Figure 3 the stresses refer to ultimate strengths of columns, while the moduli refer to

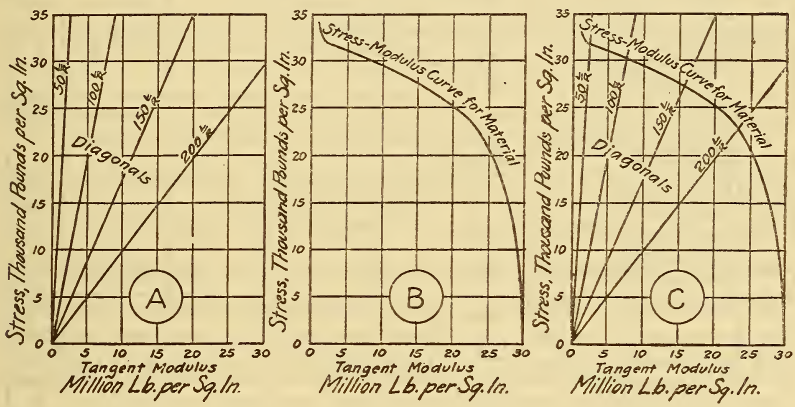

FIG. 3.—*Graphical method for estimating column strengths by means of the stress-modulus curve in combination with diagonals for slenderness ratio*

the tangent moduli at which columns fail. The diagonal straight lines marked "100 L/R" and "150 L/R" represent the constant ratios, pointed out above, of strength to tangent modulus at failure for columns of slenderness ratios 100 and 150, respectively. The other diagonals are appropriate for the slenderness ratios marked thereon.

Figure 3 B shows a diagram in which the ordinates and abscissas are marked in the same way as in Figure 3 A, but these stresses and moduli do not necessarily refer to failing conditions. Figure 3 B shows a curve marked "stress-modulus curve for material." To obtain a true curve of this type, the specimen must be relatively short or provided with lateral support, so that the location of the curve will depend upon characteristics of the material alone and be entirely independent of slenderness ratio. This curve shows just what values of stress and modulus will appear simultaneously in any test of a column of this material.

If Engesser's formula is correct, failure will occur for any column when the simultaneously appearing values of stress and modulus have the particular ratio which characterizes the slenderness ratio of that column; that is, failure will happen where the

TABLE 3. Properties of Column Sections

Type	Make-up of section	Rivets	Height	Width	Sketch	Area	Moment of inertia about H-axis	Moment of inertia about V-axis	Radius of gyration about H-axis	Radius of gyration about V-axis	Section modulus about H-axis	Section modulus about V-axis	Type
		In.	In.	In.		In.²	In.⁴	In.⁴	In.	In.	In.³	In.³	
1	1 Pl.-6" × 5/16" 4 Ls-5" × 3" × 5/16"	3/4	6 1/4	10 5/16		11.47	70.2	57.6	2.47	2.24	22.4	11.2	1
1A	1 Pl.-6" × 5/8" 4 Ls-5" × 3" × 5/8"	3/4	6 1/4	10 5/8		22.19	122.9	127.7	2.36	2.40	39.4	24.0	1A
2	2-6"-[-10 1/2# 2 Pls.-8" × 1/4"	1/2	6 1/2	8		10.18	69.2	54.4	2.61	2.32	21.3	13.6	2
2A	2-6"-[-15 1/2# 2 Pls.-8" × 1/2"	1/2	7	8		17.12	123.5	93.6	2.69	2.33	35.3	23.4	2A
3	2-5"-[-6 1/2# 2 Pls.-9 1/2" × 1/4"	1/2	5 1/2	9 1/2		8.65	47.4	71.6	2.34	2.88	17.2	15.1	3
3A	2-5"-[-11 1/2# 2 Pls.-9 1/2" × 1/2"	1/2	6	9 1/2		16.26	92.7	134.3	2.39	2.87	30.9	28.3	3A
4	1-8"-I-18# 2-8"-[-11 1/4#	5/8	8.44	8		12.03	148.3	68.4	3.51	2.38	35.1	17.1	4
4A	2-8"-I-20 1/2# 2-8"-[-18 3/4#	5/8	8.98	8		17.05	234.0	91.7	3.70	2.32	52.1	22.9	4A
5	Beth-8"-H-32#		7 7/8	8		9.17	105.7	35.8	3.40	1.98	26.9	8.9	5
5A	Beth-8"-H-62#		8 3/4	8.24		18.27	240.2	80.0	3.63	2.09	54.9	19.4	5A
5B	Beth-8"-H-91#		9 1/2	8.47		26.64	385.3	125.1	3.80	2.17	81.1	29.6	5B
6	1-10"-I-25# 2 Pls.-11" × 5/16"	5/8	10 5/8	11		14.25	304.6	76.2	4.62	2.31	57.5	13.8	6
6A	1-10"-I-35# 2 Pls.-11" × 5/8"	5/8	11 1/4	11		24.04	534.8	147.1	4.72	2.47	95.2	26.7	6A
7	1 Pl.-6" × 5/16 4 Bulb Ls-5" × 10.1#	5/8	6 1/4	10 5/16		13.76	85.9	123.4	2.51	3.01	27.5	24.0	7
8	1 Pl.-7" × 1/4" 4-4"-Z-1/4"	3/4	8 1/4	12 5/8		11.39	68.5	134.0	2.46	3.42	16.6	21.3	8
8A	1 Pl.-7" × 5/8" 4-4"-Z-5/8"	3/4	8 5/8	11 7/8		26.55	167.1	278.7	2.51	3.24	38.8	47.0	8A
10	2 Pls.-5 1/4" × 1/4" 2 Pls.-9" × 1/4" 4-Ls-2" × 2" × 1/4"	1/2	6	9		10.88	66.1	59.3	2.39	2.33	22.0	13.2	10
10A	2 Pls.-5 1/4" × 7/16" 2 Pls.-9" × 7/16" 4-Ls-2" × 2" × 7/16"	1/2	6 3/8	9		18.71	109.8	102.4	2.42	2.34	34.4	22.8	10A
Area Light	2-8"-[-16 1/4#	3/4	8	10 1/4		9.56	79.8	202.8	2.89	4.55	20.0	39.5	Area Light
Heavy	2-8"-[-21 1/4#	3/4	8	10 1/4		12.50	95.6	261.6	2.77	4.54	23.8	51.1	Heavy

stress-modulus curve for the material (fig. 3 B) crosses the appropriate diagonal (fig. 3 A) for the slenderness ratio of the column.

In Figure 3 C we have the two preceding figures superposed. Engesser's formula requires that columns whose material is properly represented by the stress-modulus curve shown, which are tested with fixed ends, and have slenderness ratios 50, 100, 150, and 200, shall have strengths given by the ordinates of the points of intersection of the stress-modulus curve and their corresponding diagonals; thus for a column of 100 L/R the strength should be about 30,100 lbs./in.^{2}, while it should be about 27,300 lbs./in.^{2} if its slenderness ratio is 150.

11. SERIES OF COLUMNS TESTED

Table 3 shows the properties of the sections of 20 different types of steel columns whose tests were completed in 1916 by the Bureau of Standards in collaboration with the American Society of Civil Engineers and the American Railway Engineering Association. Table 4 shows the identifying numbers or names which were assigned to the individual columns as they were tested, so arranged in this table as to indicate their types and slenderness ratios.

In Table 4 it will be seen that three columns were tested of each type in each of the three slenderness ratios, 50, 85, and 120. These columns comprised the series of tests as initially planned. The columns that were tested in slenderness ratios 20 and 155 were decided upon at a later date than the initital series, and these additional columns were ordered and fabricated as a separate lot. In the two American Railway Engineering Association groups the columns whose numbers lie between 172 and 204, inclusive, embraced minor variations from the initial American Railway Engineering Association type, such as the omission of bases and the use of lighter lattice bars with smaller rivets; but, as these particular variations appeared to have no effect upon their test strength, their values have been included in this study as belonging to the initial American Railway Engineering Association types.

12. APPLICATION OF ENGESSER'S FORMULA

From Table 4 one sees that the initial series of column tests provided three test columns for each type in each of three slenderness ratios. The attempt was seriously made to have all columns of the same type constructed of the same grade of steel and fabricated in precisely the same manner. This series then obviously provides an excellent opportunity to test the value of Engesser's formula in a practical way. One can use the tests that were made on columns of 50 L/R to determine the stress-modulus curve for each type of column. He can use this curve to estimate, by the method already explained, what the strength of longer columns of the same type should be in accordance with Engesser's formula. He can then compare these estimates of strength of the longer columns with the actual strengths found by the bureau in testing such longer columns of the same types. It is clear that such estimates should be totally independent of the test results, since all the estimates are based on short columns, while the strengths to be checked are obtained from tests of longer columns.

TABLE 4.—Index to Column Numbers

Type	20/L R			50/L R			85/L R			120/L R			155/L R		
1	178	179	180	8	38	77	18	84	85	30	56	69	207	218	222
1A	175	176	177	93	96	97	91	94	95	92	98	99	205	206	208
2	…	…	…	11	41	43	21	59	65	32	54	70	215	217	221
2A	…	…	…	101	124	165	145	146	151	135	136	137	214	216	219
3	…	…	…	1	2	42	14	87	88	24	35	47	…	…	…
3A	…	…	…	105	168	171	147	148	150	138	139	143	…	…	…
4	…	…	…	6	36	75	13	79	80	26	45	49	211	212	220
4A	…	…	…	102	166	169	152	153	154	132	133	134	209	210	213
5	107X	108X	114X	107	122	123	114	116	119	108	110	111	…	…	…
5A	106X	112X	117X	106	120	121	115	117	118	109	112	113	…	…	…
5B	181CX	182AX	183BX	182	225	226	183	227	228	181	223	224	…	…	…
6	…	…	…	10	40	44	22	63	64	31	52	71	…	…	…
6A	…	…	…	103	125	128	155	158	159	129	130	131	…	…	…
7	…	…	…	5	37	76	15	89	90	28	48	51	…	…	…
8	…	…	…	9	39	78	16	81	82	27	46	50	…	…	…
8A	…	…	…	100	126	127	156	160	161	162	163	164	…	…	…
10	…	…	…	3	4	53	19	83	86	23	33	34	…	…	…
10A	…	…	…	104	167	170	144	149	157	140	141	142	…	…	…
A. R. E. A. light.	…	…	…	12	57	74	17	58	61	29	55	66	…	…	…
				…	…	…	199	200	204	172	173	174	…	…	…
				…	…	…	191	192	197	…	…	…	…	…	…
A. R. E. A. heavy.	…	…	…	7	72	73	20	60	62	25	67	68	…	…	…
				…	…	…	201	202	203	…	…	…	…	…	…

In a few cases, after a column had been tested, one or two short straight lengths were cut from it, and these portions were again tested. Such a retest column carried its original number with a suffix A or B added. The following numbers relate to such retests: 84A and 84B, type 1, 20 L/R; 85A, type 1, 20 L/R; 98A and 98B, type 1A, 20 L/R; 99A, type 1A, 20 L/R; 160A, type 8A, 12 L/R; 160B, type 8A, 20 L/R; 161A and 161B, type 8A, 20 L/R.

13. TYPICAL STRESS-MODULUS CURVES

A stress-modulus curve was drawn for each test column of 50 L/R by the method explained in detail for No. 176, type 1A, 20 L/R (Table 1 and fig. 2). Figure 4 shows the individual stress-modulus curves for columns numbered 93, 96, and 97, which are seen in Table 4 as the three columns of type 1A, 50 L/R. While these curves are very similar to one another, indicating similar column material, they are not identical. In estimating column strength for each column type, we need a single curve; it is necessary to obtain in some manner a composite curve from these three somewhat different separate curves. To form this composite curve, the following method was used. The stress was noted at which each curve of Figure 4 crosses each modulus value indicated in this figure; the mean stress was then found for each modulus value, and these mean stresses were used as ordinates for the mean curve desired. Thus, in Figure 4 for a tangent modulus of

20,000,000 lbs./in.[2] the curve for column 93 shows a stress of 21,800 lbs./in.[2], that for column 96 shows 21,200 lbs./in.[2], while that for column 97 shows 22,400 lbs./in.[2]; the mean of these three stresses is 21,800 lbs./in.[2], a value used in drawing the mean curve. A mean or composite curve obtained in this manner from tests of

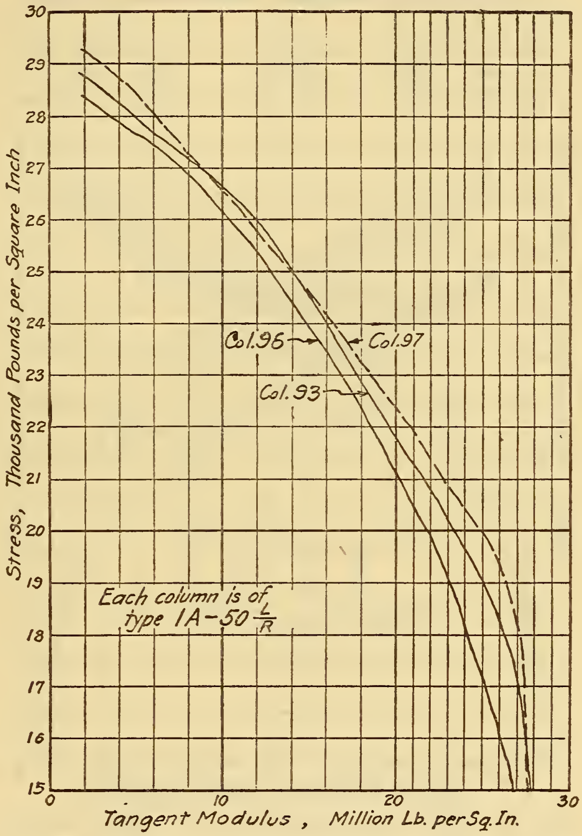

FIG. 4.—*Individual stress-modulus curves for the three columns that were tested of type 1A, 50 L/R*

The typical curve for this type, shown in Figure 5, is drawn through points whose ordinates are the averages of the ordinates of these three curves

columns of 50 L/R will be called a *typical* curve as characterizing the *type* of columns for which it is drawn.

Figures 5 to 13, inclusive, show in heavy lines typical stress-modulus curves, obtained in this manner, for the 20 types of columns, referred to in Tables 3 and 4. Wherever dashes occur in these curves, this indicates that such a portion of a curve has not been well established. Above and below these heavy curves in Figures 5 to 13 are shown lighter curves with crosshatching between them; these lighter curves indicate the upper and lower stresses of the three curves whose ordinates have been averaged to obtain those of the heavy typical curve. The width of this area within which the typical curve is drawn, gives some indication

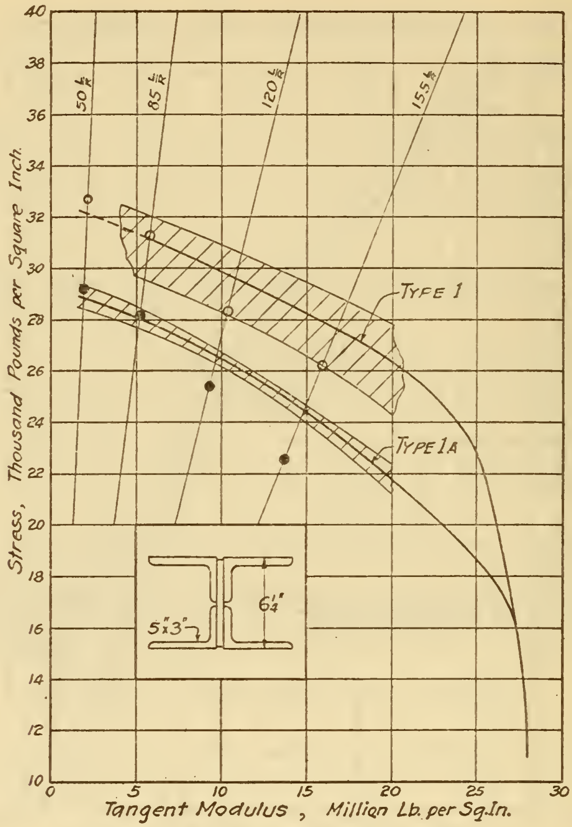

FIG. 5.—*Typical stress-modulus curves for types 1 and 1A*

The curves are based upon tests of columns of 50 L/R. The ordinates of the points marked by small circles show the average strengths found by testing columns of these types and whose slenderness ratios were those marked on the diagonals of this figure, open circles relating to type 1 and solid circles to type 1A

of the uniformity of the material of the different columns of 50 L/R; thus, columns of types 1A and 4A appear to have been of more uniform material than those of types 1 and 4.

14. ESTIMATES OF COLUMN STRENGTH

Diagonals corresponding to those shown in Figure 3 will be found in each of the diagrams of Figures 5 to 13, each marked with the slenderness ratio to which it refers. As explained above, the typical stress-modulus curves taken in connection with such diagonals, may be used in estimating the strengths of columns of similar material, but of greater slenderness ratios than that of the columns used in determining the stress-modulus curves. Thus, in Figure 5 one sees that a column whose slenderness ratio is 85

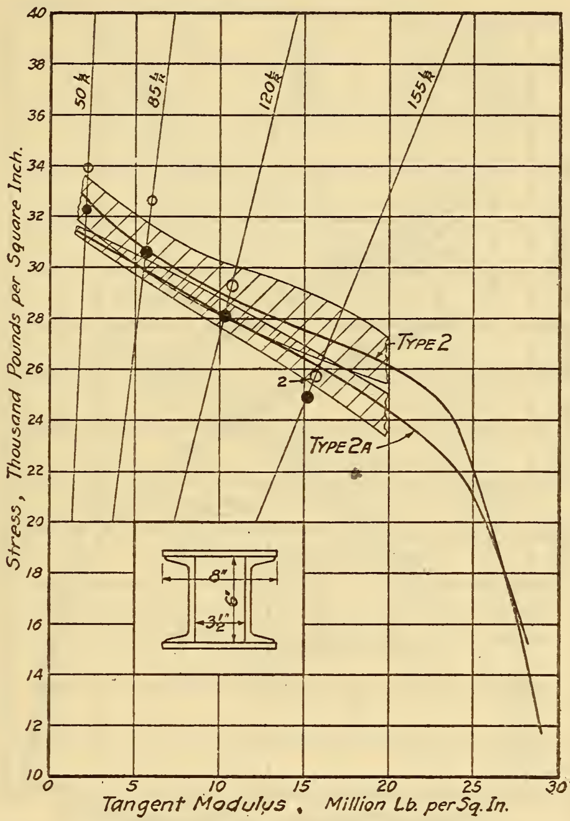

FIG. 6.—*Typical stress-modulus curves for types 2 and 2A*

The curves are based upon tests of columns of 50 L/R. The ordinates of the points marked by small circles show the average strengths found by testing columns of these types and whose slenderness ratios were those marked on the diagonals of this figure, open circles relating to type 2 and solid circles to type 2A

should have, according to Engesser's formula, a strength of about 28,000 lbs./in.[2] if it is of type 1A, and about 31,100 lbs./in.[2] if it is of type 1. Similarly, if the column has a slenderness ratio of 120, one sees from Figure 5 that its strength should be about 26,600 lbs./in.[2] for type 1A and about 29,600 lbs./in.[2] for type 1.

15. TEST STRENGTHS ON DIAGRAMS

To facilitate the comparison of estimated strength with actual strength in any particular case, the average strengths that were found by testing three columns of each slenderness ratio for each type are shown on the diagrams of Figures 5 to 13. These average test strengths are represented by the ordinates of small circles to be found on the diagonals of these figures. Thus in Figure 5 one sees on the diagnonal for 85 L/R a small open circle at ordinate

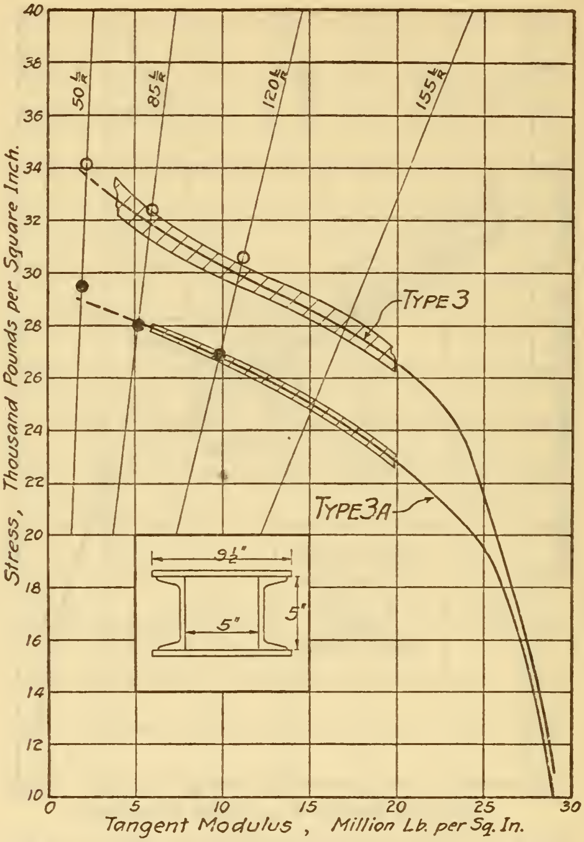

FIG. 7.—*Typical stress-modulus curves for types 3 and 3A*

The curves are based upon tests of columns of 50 L/R. The ordinates of the points marked by small circles show the average strengths found by testing columns of these types and whose slenderness ratios were those marked on the diagonals of this figure, open circles relating to type 3 and solid circles to type 3A

31,200 lbs./in.^{2} and a small solid circle at ordinate 28,100 lbs/in.^{2}; the former indicates the average strength of columns of type 1, 85 L/R, while the latter indicates the average strength of columns of 1A, 85 L/R.

16. COLUMNS OF 50 L/R

It will be noted that Figures 5 to 13 show diagonals for slenderness ratio 50 and that these diagonals carry the small circles whose ordinates indicate the test strengths of these columns. It will be noted further that the stress-modulus curves intersect the diagonals for 50 L/R below these circles, and this may seem peculiar considering that the stress-modulus curves were drawn from the same tests as those for which the circles are plotted. In tests in which strain-gauge readings were taken at the utimate strength,

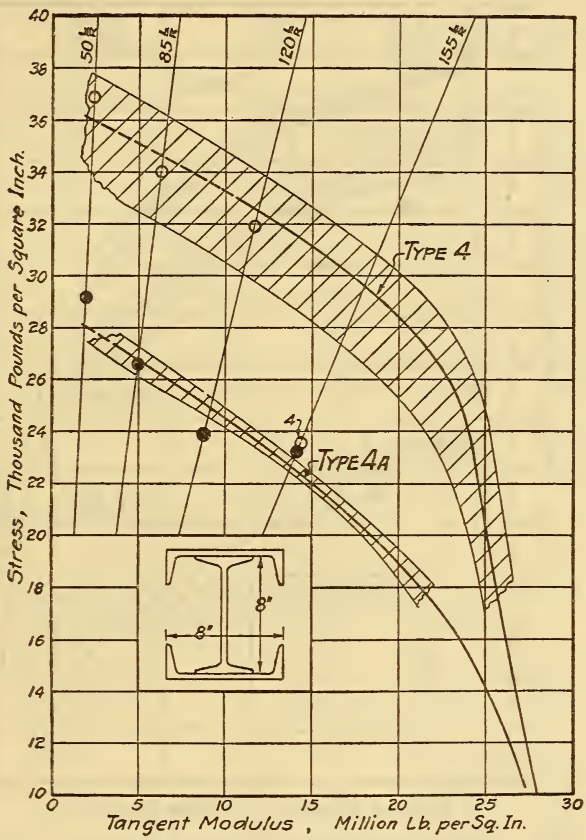

FIG. 8.—*Typical stress-modulus curves for types 4 and 4A*

The curves are based upon tests of columns of 50 L/R. The ordinates of the points marked by small circles show the average strengths found by testing columns of these types and whose slenderness ratios were those marked on the diagonals of this figure, open circles relating to type 4 and solid circles to type 4A

the last plotted point of the stress-modulus curve would normally have an ordinate about 500 lbs./in.^{2} below the ultimate strength. Reference to Figure 2 will suggest that the stress-modulus curve for steel of the grade used in these columns may take a rather sudden upward turn near the point where it would cross the diagonal for 50 L/R. In tests of columns of 50 L/R, this upward turn is not clearly shown by the stress-strain data for any type except type 10A, whose curve is shown in Figure 12. In some cases readings for strain were not obtained at the maximum load, the tests being conducted somewhat as that of column No. 176, type 1A, 20 L/R, whose stress-modulus curve is seen in Figure 2. In the case of this column strain readings were discontinued at

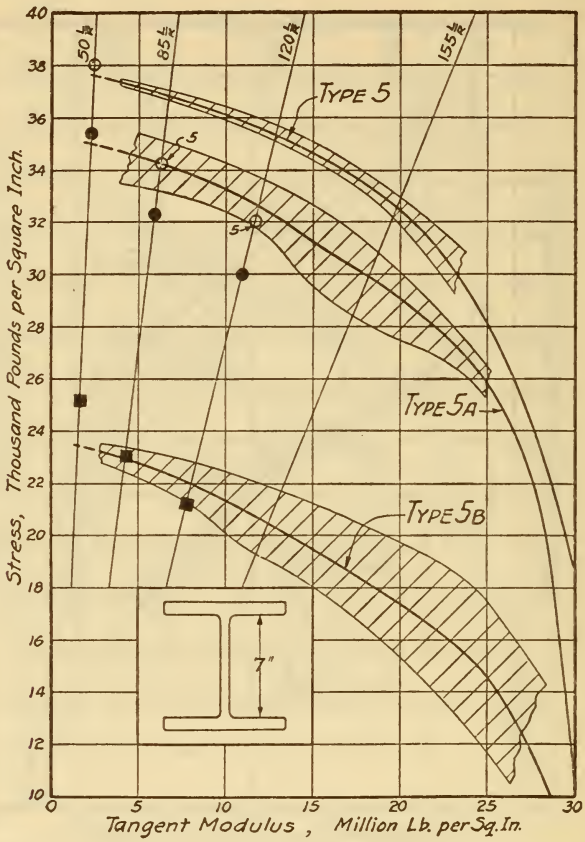

FIG. 9.—*Typical stress-modulus curves for types 5, 5A, and 5B*

The curves are based upon tests of columns of 50 L/R. The ordinates of the points marked by small circles or squares show the average strengths found by testing columns of these types and whose slenderness ratios were those marked on the diagonals of this figure, open circles relating to type 5, solid circles to type 5A, and small squares to type 5B

31,000 lbs./in.^{2} and the loading in the subsequent part of the test was conducted more rapidly than in the preceding part. These three circumstances, viz, (1) the method used in plotting points on the stress-modulus curve, (2) the upward turn of the stress-modulus curve at the left side of the diagrams, and (3) the lack of readings for strain at the ultimate strength in the case of certain tests, are believed amply to account for the fact that the stress-modulus curves intersect the 50 L/R diagonals somewhat below the failing strengths of these columns of 50 L/R.

TABLE 5.—Comparison of Actual Strengths of Columns of 85 L/R with Estimates Made From Curves for Columns of 50 L/R

Column type	Strengths found in tests	Estimates from curves	Excess of estimates
	Lbs./in.²	Lbs./in.²	Lbs./in.²
1	31,200	31,100	−100
1A	28,100	28,000	−100
2	32,600	30,700	−1,900
2A	30,600	29,900	−700
3	32,400	31,900	−500
3A	28,000	28,100	100
4	34,000	34,600	600
4A	26,600	26,700	100
5	34,300	36,800	2,500
5A	32,300	34,200	1,900
5B	23,000	22,900	−100
6	29,100	29,100	0
6A	26,800	26,300	−500
7	31,600	32,300	700
8	32,800	33,900	1,100
8A	31,400	31,600	200
10	32,100	31,900	−200
10A	28,300	28,900	600
A. R. E. A., light	33,900	34,100	200
A. R. E. A., heavy	33,100	32,800	−300
Average excess of estimates			180
Average deviation between strengths and estimates			620

TABLE 6.—Comparison of Actual Strengths of Colums of 120 L/R with Estimates Made From Curves for Columns of 50 L/R

Column type	Strengths found in tests	Estimates from curves	Excess of estimates
	Lbs./in.²	Lbs./in.²	Lbs./in.²
1	28,300	29,600	1,300
1A	25,400	26,600	1,200
2	29,300	28,800	−500
2A	28,100	28,100	0
3	30,600	29,900	−700
3A	26,900	26,800	−100
4	31,900	32,500	600
4A	23,900	24,900	1,000
5	32,000	35,300	3,300
5A	30,000	32,600	2,600
5B	21,200	22,000	800
6	27,200	27,000	−200
6A	24,800	24,600	−200
7	28,100	31,400	3,300
8	29,700	32,200	2,500
8A	27,300	30,200	2,900
10	28,400	28,900	500
10A	26,300	27,400	1,100
A. R. E. A., light	33,000	33,100	100
A. R. E. A., heavy	30,800	31,700	900
Average excess of estimates			1,020
Average deviation between strengths and estimates			1,190

TABLE 7.—Comparison of Actual Strengths of Columns of 155 L/R with Estimates Made From Curves for Columns of 50 L/R

Column type	Strengths found in tests	Estimates from curves	Excess of estimates
	Lbs./in.²	Lbs./in.²	Lbs./in.²
1	26,200	27,600	1,400
1A	22,600	24,500	1,900
2	25,700	27,100	1,400
2A	24,900	26,100	1,200
4	23,600	29,600	6,000
4A	23,300	22,600	−700
Average excess of estimates			1,870
Average deviation between strengths and estimates			2,100

17. TABULAR COMPARISON OF STRENGTHS AND ESTIMATES

Tables 5, 6, and 7 have been prepared to facilitate the comparison of the average test strengths of columns of slenderness ratios 85, 120, and 155, and the estimates of their strengths made by means of the stress-modulus curves obtained from the tests of columns of 50 L/R and used in accordance with Engesser's formula. Each table gives for each type of column, the average strength, the estimated strength, and the excess of the estimate, while at the bottom of the table are given the average excess of estimates and the average deviation between strengths and estimates without regard to sign.

18. PROVISIONAL CONCLUSIONS

The above application of Engesser's formula to the tests in question shows remarkably good results for columns of slenderness ratio 85 as seen in Table 5, but the progressive increase in the excess of estimated strength with slenderness ratio, viz, 180 for 85 L/R, 1,020 for 120 L/R, and 1,870 for 155 L/R, suggests that some other consideration than tangent modulus as here found may enter the determination of column strength. Furthermore, the study thus far made may be looked upon as too much detached from the tests of the longer columns; the exhibit may be said to be incomplete without a demonstration that the ordinary test of a long column approximately follows the typical stress-modulus curve for its type up to about its intersection with its appropriate diagonal and then fails, which action is involved as a fundamental assumption in Engesser's formula. While Engesser's formula is an interesting suggestion in considering the laws of column strength, in that *it is based upon characterictics of the column material as loaded in a column test,* it is not evident that it takes into consideration a sufficient number of these characteristics to make its use reliable.

19. TECHNICAL OBJECTIONS TO THE ABOVE TREATMENTS

It may be noted that while the stress-modulus curve shown in Figure 2 is based upon gauge measurements made upon four gauge lines, arranged with respect to the section in a nearly symmetrical manner, this column test, as well as nearly all the tests of the 20

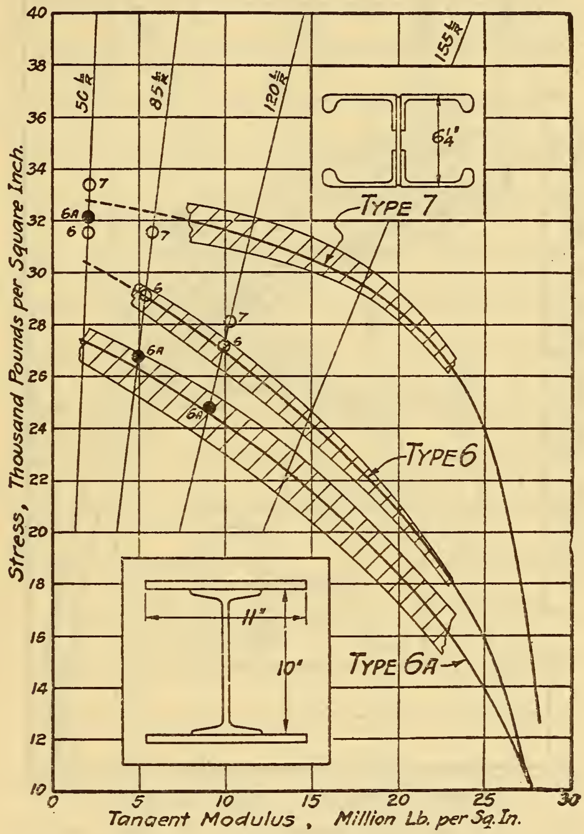

FIG. 10.—*Typical stress-modulus curves for types 6, 6A, and 7*

The curves are based upon tests of columns of 50 L/R. The ordinates of the points marked by small circles show the average strengths found by testing columns of these types and whose slenderness ratios were those marked on the diagonals of this figure, open circles relating to type 6 or type 7, as indicated by numerals, and solid circles relating to type 6A

different types of columns, showed an additional gauge line along the middle of the upper column face, midway between lines 3 and 5, Figure 2. The data used in drawing the typical stress-modulus curves of Figures 5 to 13 included compression values for this additional line. This additional line upset the symmetry of the arrangement of the gauge lines so that if the columns deflected vertically, the average compression for the five lines would not normally be the same as the average for the section, and this appeared to involve a possibility that the typical stress-modulus curves might misrepresent the properties of the column material.

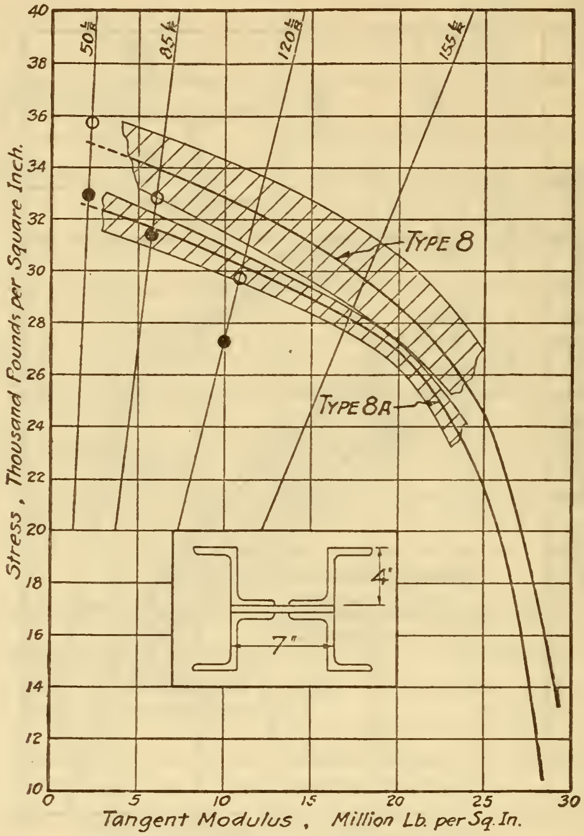

FIG. 11.—*Typical stress-modulus curves for types 8 and 8A*

The curves are based upon tests of columns of 50 L/R. The ordinates of the points marked by small circles show the average strengths found by testing columns of these types and whose slenderness ratios were those marked on the diagonals of this figure, open circles relating to type 8 and solid circles to type 8A

The method used for finding a composite curve based upon three individual stress-modulus curves has been questioned, apparently because the natural method seems at first to be that of averaging the modulus values at the several stresses used in the tests. Since the portion of the curves that is of particular interest lies in the left half of each figure (see figs. 5 to 13), an examination of the figures will show that a composite obtained in the manner used will give values reaching to smaller moduli than if one averages the moduli at the various stresses. Thus in Figure 4, if one averages the stresses at given modulus values, he obtains a value at 2,000,000 lbs./in.², while if he averages the moduli at given stresses. the smallest average modulus value will be

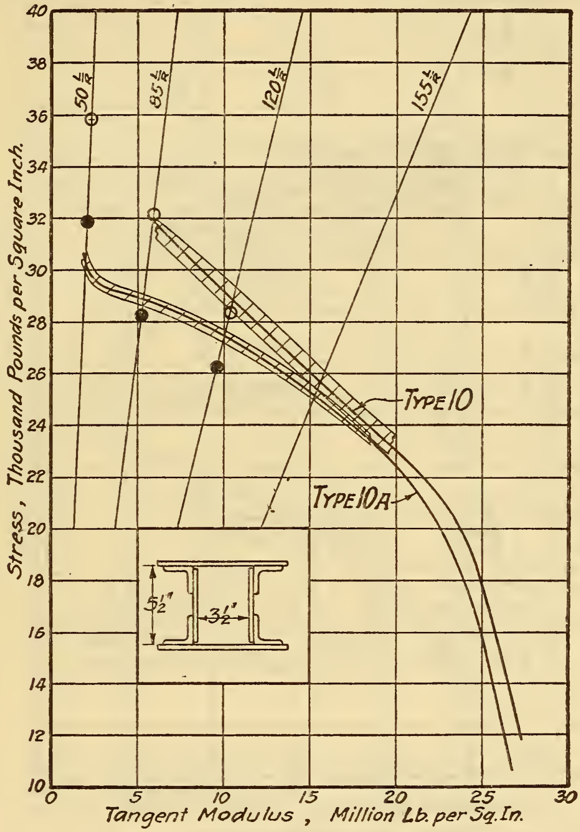

FIG. 12.—*Typical stress-modulus curves for types 10 and 10A*

The curves are based upon tests of columns of 50 L/R. The ordinates of the points marked by small circles show the average strengths found by testing columns of these types and whose slenderness ratios were those marked on the diagonals of this figure, open circles relating to type 10 and solid circles to type 10A

about 4,900,000 lbs./in.², and this difference is more marked in cases where the individual curves are farther apart.

20. STRESS-MODULUS CURVES FOR COLUMNS OF ALL LENGTHS

In order to throw additional light upon the relation between the tangent modulus and the action of columns in their tests, and to satisfy the above technical objections, nearly the entire set of data was gone over in determining stress-modulus curves for columns of all lengths in each type of the American Society of Civil Engineers series; that is, the first 18 types of Tables 3 and 4. In this part of the study, the compression data for the additional unsymmetrically located gauge line were excluded; and in

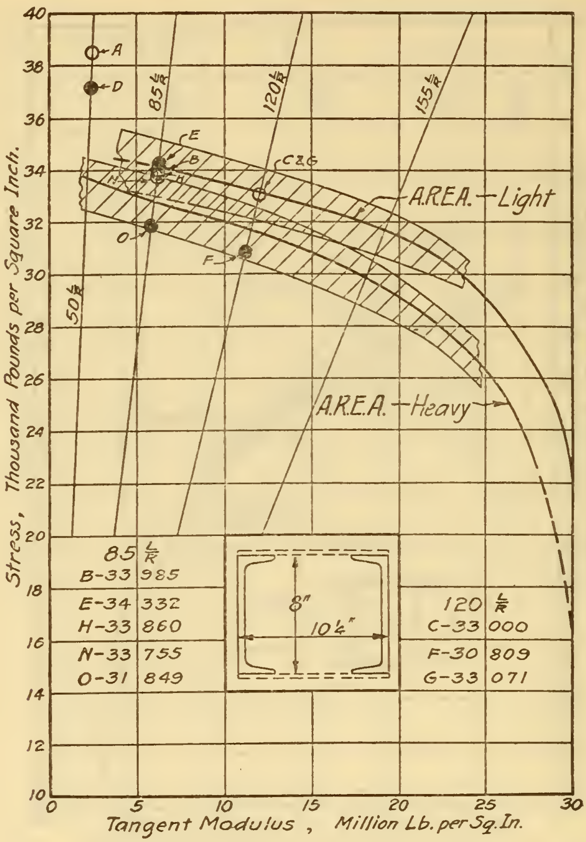

FIG. 13.—*Typical stress-modulus curves for A. R. E. A. columns of light and of heavy section*

The curves are based upon tests of columns of 50 L/R. The ordinates of the points marked by small circles show the average strengths found by testing columns of these types and whose slenderness ratios were those marked on the diagonals of this figure, open circles relating to light sections and solid circles to heavy sections. The average strengths of all the groups of longer columns are given in the lower part of the figure

obtaining each composite curve, the moduli were averaged at the several stresses. This method of averaging the moduli at the several stresses is much more expeditious than the one previously used because it does not generally involve drawing the curves for the individual tests. The curves obtained in this manner are shown in Figures 14 to 21, inclusive.

21. TESTS THAT HAVE BEEN OMITTED

In drawing the typical stress-modulus curves (figs. 5 to 13) it was thought necessary to base each curve upon three tests in order to represent the average properties of the columns, and this was done in spite of the fact that the data of some of the tests were

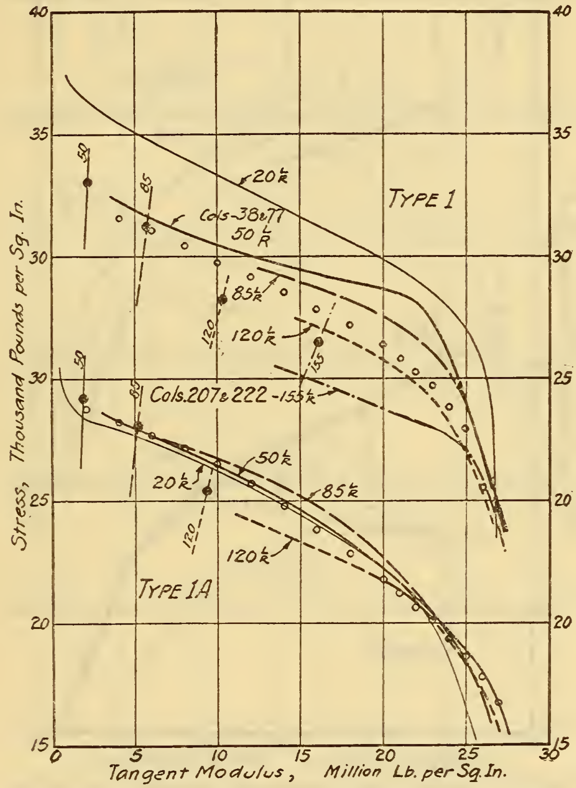

FIG. 14.—*Stress-modulus curves for columns of all lengths in types 1 and 1A*

Except as noted each curve represents three tests. No curve is given for type 1A, 155 L/R. Small open circles show locations of typical stress-modulus curves of Figure 5. Average column strengths for the several slenderness ratios are indicated by small solid circles

not wholly satisfactory. Some of the tests that were made at the beginning of the series, before the regular routine of testing that was later followed very strictly had been fully developed, naturally give data that are less satisfactory for the purposes in hand than are the data of subsequent tests, and for this reason some of these tests have been omitted in drawing the curves of Figures 14 to 21. Those that have been omitted for this reason are all of 50 L/R.

A few tests have been omitted because they give curves that appear to be anomalous. Columns numbered 206 and 208 of type 1A, 155 L/R, 218 of type 1, 155 L/R, and 214 and 219 of type 2A, 155 L/R show very high modulus values nearly up to their ultimate strengths, much as one would expect them to do if they

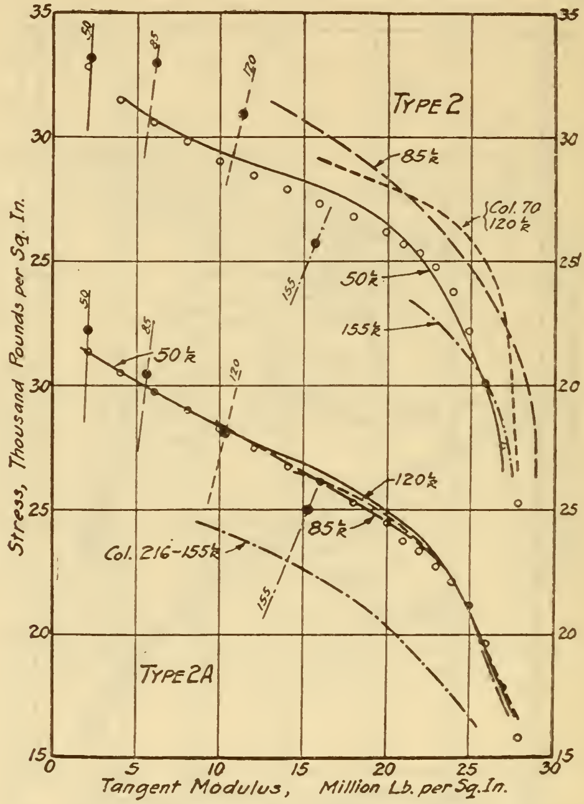

FIG. 15.—*Stress-modulus curves for columns of all lengths in types 2 and 2A*

Except as noted each curve represents three tests. Small open circles show locations of typical stress-modulus curves of Figure 6. Average column strengths for the several slenderness ratios are indicated by small solid circles. The scale of stress is offset between the upper and lower parts of the diagram

had been previously loaded to a somewhat smaller column load, as shown in Figure 36. Other isolated tests have been omitted because the data appeared to contain errors that could not be corrected with any degree of certainty, or because they appeared to show results that seemed improbable, such as increase in length on increase in column load.

22. CURVES FOR COLUMNS OF 50 L/R

In Figures 14 to 21 the stress-modulus curves for columns of 50 L/R are shown by solid heavy lines, while the former typical stress-modulus curves, based upon the same tests, are shown in these figures by small open circles. The two sets of curves agree

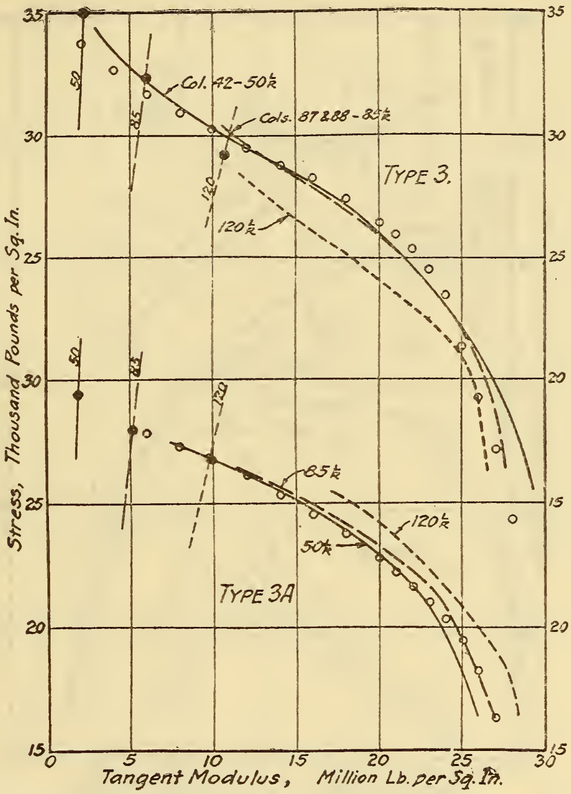

FIG. 16.—*Stress-modulus curves for columns of all lengths in types 3 and 3A*

Except as noted each curve represents three tests. Small open circles show locations of typical stress-modulus curves of Figure 7. Average column strenghts for the several slenderness ratios are indicated by small solid circles. The scale of stress is offset between the upper and lower parts of the diagram

closely enough for practical purposes except in certain cases where one or two tests were omitted, as explained above, in the determination of the solid curves. This indicates that the typical stress-modulus curves represent to a satisfactory degree of approximation the action of columns of such slenderness ratio in their tests.

23. CURVES FOR COLUMNS OF 20 L/R

Curves for columns of 20 L/R appear in Figures 14 and 18 only. Considering that these columns were ordered at a later date than those of 50 L/R, one sees a satisfactory general agreement between the curves for these two slenderness ratios. For four

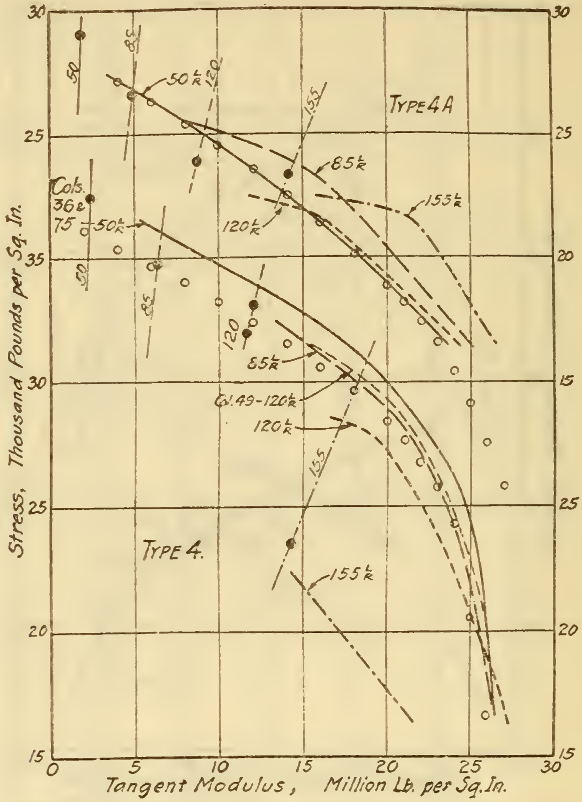

FIG. 17.—*Stress-modulus curves for columns of all lengths in types 4 and 4A*

Except as noted each curve represents three tests. Small open circles show locations of typical stress-modulus curves of Figure 8. Average column strengths for the several slenderness ratios are indicated by small solid circles. Of the two small solid circles that appear on the diagonal for type 4, 120 L/R, the upper one shows the strength of column No. 49 and the lower shows the average for the type. The scale of stress is offset between the upper and lower parts of the diagram

types out of five the curves for 20 L/R run through higher stresses than do those for 50 L/R. No reason is seen for this except difference in the material. The gauge lengths occupied practically the same proportion of the total column lengths in the two cases, 70 per cent for 20 L/R and 69 per cent for 50 L/R. The upward turn at small modulus values is more evident in the tests of columns of 20 L/R than in those of 50 L/R, appearing particularly in types 1A and 5B. This comparison of the curves for slenderness ratios 20 and 50 gives additional reason for thinking that the typical stress-modulus curves, already discussed at length, properly

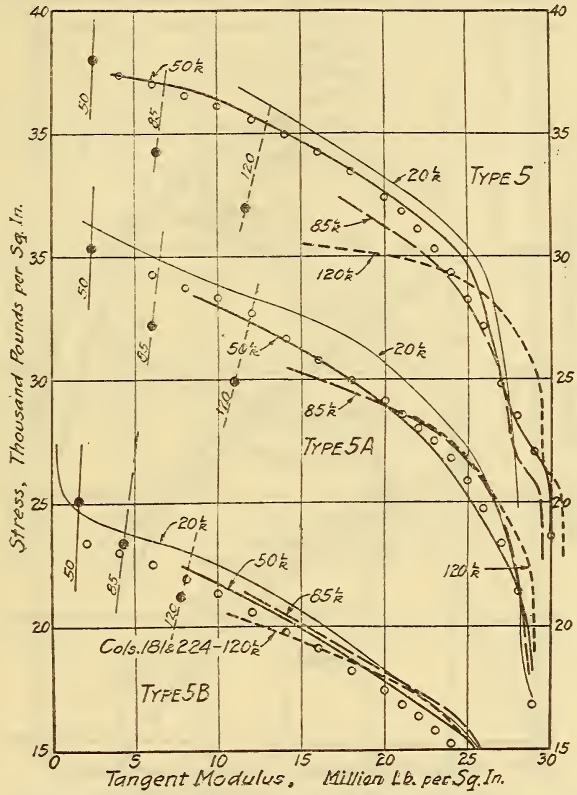

FIG. 18.—*Stress-modulus curves for columns of all lengths in types 5, 5A, and 5B*

Except as noted each curve represents three tests. Small open circles show locations of typical stress-modulus curves of Figure 9. Average column strengths for the several slenderness ratios are indicated by small solid circles. The scale of stress is offset for the upper one-third of the diagram

represent the properties of the material of the different types of columns for use in a practical way in applying Engesser's formula for estimating the strengths of columns of somewhat greater slenderness ratio than 50.

24. CURVES FOR COLUMNS OF 85 L/R

Curves for columns of 85 L/R, in Figures 14 to 21, agree fairly well with the typical stress-modulus curves shown by small open circles, except in the following cases: The curves for 85 L/R are relatively high in types 2 and 4A, and low in type 10A. The curve for column 160, type 8A, Figure 20, shows abnormal characteristics which may be due to very inferior material. In Figures 17 to 20, inclusive, columns of this slenderness ratio of types 4A,

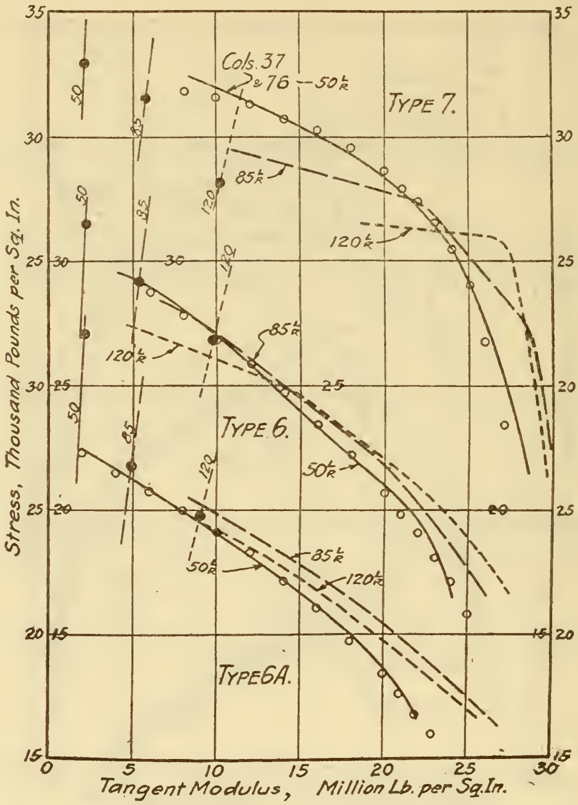

FIG. 19.—*Stress-modulus curves for columns of all lengths in types 6, 6A, and 7*

Except as noted each curve represents three tests. Small open circles show locations of typical stress-modulus curves of Figure 10. Average column strengths for the several slenderness ratios are indicated by small solid circles. The scale of stress is offset between successive parts of the diagram

5A, 7, and 8A, are seen to terminate in parts that are nearly straight and that have a smaller slope than the corresponding parts of the typical stress-modulus curves with upturned ends. Such portions of these curves will be called *drooping ends*. For columns of 85 L/R the gauge lengths occupied an average of 51 per cent of the column lengths.

25. CURVES FOR COLUMNS OF 120 L/R

Curves for columns of 120 L/R, in Figures 14 to 21, agree fairly well with the typical stress-modulus curves shown by small open circles, except in the following cases: The curves for 120 L/R are relatively high in types 2 and 3A and low in types 3, 4, and 10A.

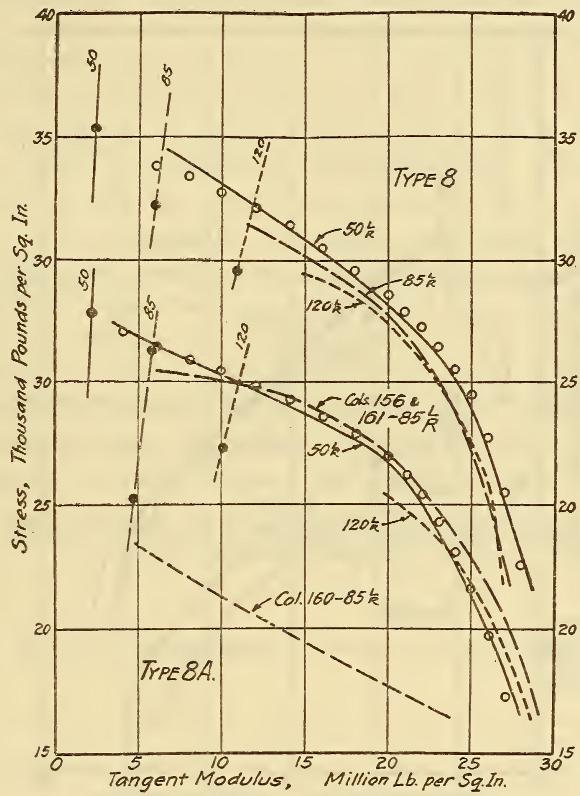

FIG. 20.—*Stress-modulus curves for columns of all lengths in types 8 and 8A*

Except as noted each curve represents three tests. Small open circles show locations of typical stress-modulus curves of Figure 11. Average column strengths for the several slenderness ratios are indicated by small solid circles. Two of these small solid circles appear on the diagonal for type 8A, 85 L/R, the upper one referring to columns 155 and 161 and the lower one to column 160. The scale of stress is offset between the upper and lower parts of the diagram

The differences may probably be attributed to variations in the material. About 60 per cent of these curves for 120 L/R will be seen to terminate in drooping ends, mentioned above; this characteristic being particularly marked in types 5 and 7. For columns of this slenderness ratio the gauge lengths occupied about

82 per cent of the column lengths in types 5 and 5A, and about 53 per cent in the other types.

26. CURVES FOR COLUMNS OF 155 L/R

Curves for columns of 155 L/R are shown in Figures 14, 15, and 17 only. Except in type 2 the curves for this slenderness

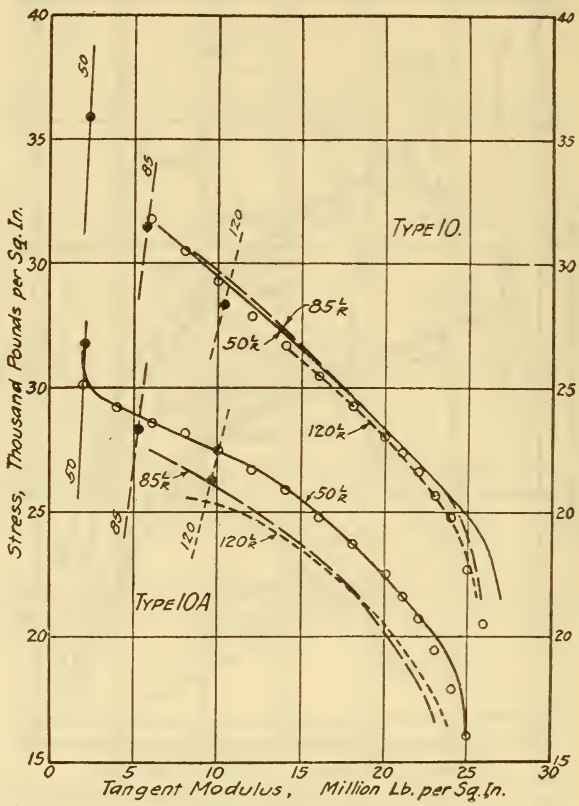

Fig. 21.—*Stress-modulus curves for columns of all lengths in types 10 and 10A*

Each curve represents three tests. Small open circles show locations of typical stress-modulus curves of Figure 12. Average column strengths for the several slenderness ratios are indicated by small solid circles. The scale of stress is offset between the upper and lower parts of the diagram

ratio differ rather widely from the typical stress-modulus curves shown by small open circles. This may also be attributed to variations in material, these columns having been ordered at a later date than those of 50 L/R. Curves for six columns of this slenderness ratio have been omitted. The gauge lines for these particularly long columns occupied about 55 per cent of the column lengths.

27. COLUMN STRENGTHS AND OWN STRESS-MODULUS CURVES

It is noted above that Engesser's formula involves the assumption that the ordinary long column in its test follows the stress-modulus curve for its material up to about the point where it would intersect the diagonal for its slenderness ratio and then fails by buckling. Of the various stress-modulus curves shown in Figures 14 to 21, about one-sixth either intersect their diagonals or approach very close to them. If one extends the other stress-modulus curves until they intersect their diagonals, he can obtain strength estimates for these columns from their own stress-modulus curves in accordance with Engesser's formula. This has been done and the results compare with the test strengths as given in Table 8. While this method of making the estimates of strengths gives results somewhat closer the actual strengths than does the method wherein the typical stress-modulus curves are used, it is subject to a large probable error, due to the arbitrary extension of the curves, and it can not be said to confirm Engesser's formula in any satisfactory manner. As a fact, many of the individual tests act almost precisely as would be expected from Engesser's formula while many of them do not.

TABLE 8.—Comparison of Actual Strengths of Columns with Estimates Made From Their Own Stress-Modulus Curves

Slenderness ratio	Average excess of estimates above strengths	Average deviation between strengths and estimates
	Lbs./in.²	Lbs./in.²
50	−600	900
85	60	600
120	−440	690
155	−1,180	1,180

28. KARMAN'S THEORY

Karman's theory [4] purports to be an improvement upon that involved in Engesser's formula. It is a simple straightforward theory for ideal conditions, based upon Considere's [5] suggestion, which takes into account the gradual variation in the tangent modulus with increase in stress, as well as the well-known fact that if a tensile or compressive specimen is loaded until its tangent modulus has become small for increase of stress, and if its load is

[4] *Untersuchungen über Knickfestigkeit*, by Theodor von Karman, Mitteilungen über Forschungsarbeiten auf dem Gebiete des Ingenieurwesens, 81, Berlin; 1910.

[5] Considere, loc. cit.

suddenly removed, its change of length for decreasing stress takes place approximately in accordance with the full value of Young's modulus for elastic conditions, and not in accordance with the small value of the tangent modulus last reached in the loading. In Figure 22, the shaded area represents the section of a rectangular specimen subjected to compressive stress until its tangent modulus has the value E'; if the specimen is now suddenly bent by the application of a small external moment, the axis of curvature being somewhere to the left of the section and the total load being unchanged, the left portion of the section will be subjected to increase of compressive stress in accordance with the small modulus E', the right portion will suffer decrease of compressive stress in accordance with the full modulus E; and since the moduli are unequal the line 2–2 which separates these two portions of the section will not pass through the center of gravity, but will lie to the right of it by some distance u.

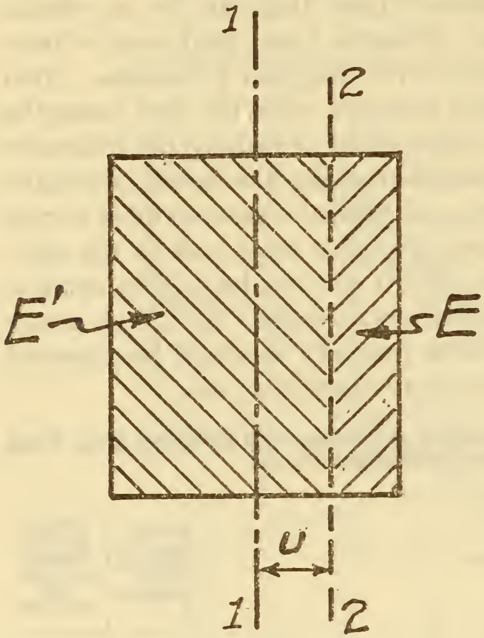

FIG. 22.—*Section of rectangular bar used as a column axially loaded and buckling in accordance with Karman's theory*

In buckling, the portion to the left of 2–2 is subject to increased compressive stress while the remaining portion of the section has its stress reduced. Karman assumes that increase in stress occurs in accordance with the tangent modulus while its decrease occurs in accordance with Young's modulus

If we let p_E represent the buckling strength of a specimen of rectangular section axially loaded under elastic conditions, and if p_K represents the buckling strength of the same specimen as given by Karman's theory with tangent modulus E', it is easily shown that

$$\frac{p_K}{p_E}=\left(\frac{2}{1-\sqrt{\frac{E}{E'}}}\right)^2$$

Since p_E may be calculated by Euler's formula for any given column under elastic conditions, the above formula enables one to draw curves which resemble the diagonals of Figure 3 A; such Karman diagonals are not straight lines but curved concave to the right, as seen in Figure 23.

29. KARMAN'S TESTS

Karman made tests upon small columns of rectangular section accurately mounted on knife edges to give axial loading. His

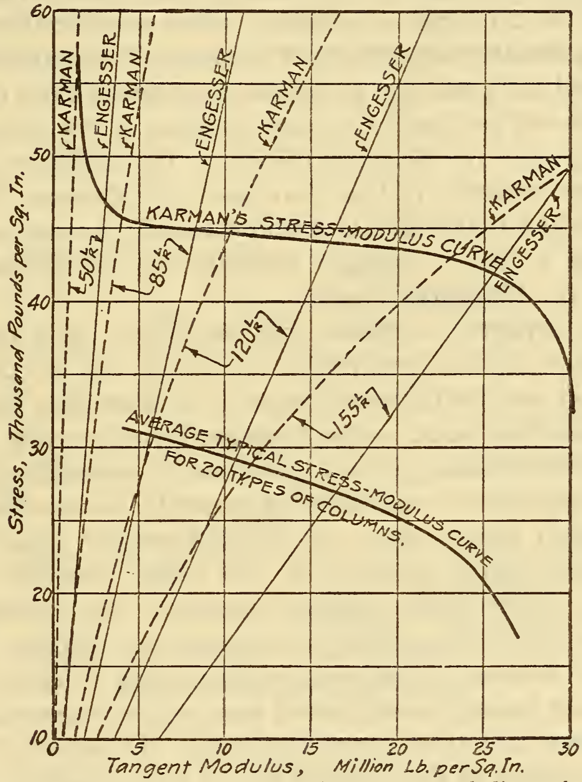

FIG. 23.—*Karman's stress-modulus curve and diagonals for rectangular sections compared with average typical stress-modulus curve for steel columns and Engesser diagonals*

largest section measured about 0.99 by 1.58 inches. He used an unusually hard steel whose stress-modulus curve was approximately that shown in Figure 23. At the middle of this figure, for tangent modulus 15,000,000 lbs./in.^{2}, the stress for the Karman curve is about 60 per cent greater than that for the average typical stress-modulus curve for 20 types of columns.

For Karman's tests in which the slenderness ratio exceeded 38.2 (corresponding to 76.4 L/R for fixed ends), his actual strengths as found by tests are about as close to estimates made on the basis of Engesser's formula as they are to estimates made according to the Karman theory; but for tests in which the slenderness ratio was smaller than the above value, the test results are given much better by the Karman theory than by Engesser's formula.

In Figure 23 it will be noted that between moduli 5,000,000 and 25,000,000 lbs./in.^{2} the Karman stress-modulus curve is much more nearly horizontal than is the average typical stress-modulus curve for the 20 types of columns under consideration. If one wishes to estimate the strength of a column of rectangular section, having fixed ends and any particular slenderness ratio between 85 and 155, he will get about the same estimate of strength no matter whether he uses the Karman theory or the Engesser formula if the column material is like that used by Karman, but if the material is like that which is actually used in American columns, he will get a higher strength estimate on the Karman theory than with the Engesser formula.

Column strength estimates that are based upon the portion of the stress-modulus curve that is turned up at the left of the diagram may be regarded properly as interesting peculiarities of little practical value. The Engesser formula when used with typical stress-modulus curves to estimate the strengths of columns of greater slenderness ratio than 50, generally gives estimates that are somewhat higher than the actual strengths found in tests. The Karman theory would give still higher strength estimates than those given by the Engesser formula. For columns of sections other than rectangular or circular the Karman theory is much more difficult of application than is the Engesser formula. The Karman theory is not looked upon as an improvement over the Engesser formula for practical use. It takes into account one characteristic of the material that is not considered in Engesser's formula, and this chacteristic increases the strength estimates. Column material doubtless has additional characteristics that are not considered in the Karman theory, and some of these additional characteristics act to decrease the column strength. To be a practical improvement upon Engesser's formula, a simple theory needs to be devised that will consider more characteristics of the material, some that increase strength estimates and some that decrease them.

30. KARMAN'S DIAGONALS FOR I SECTIONS

As noted above, Karman's theory is not easily applied in a strict manner to such sections as are commonly used in columns.

For most purposes it is sufficiently accurate to use his diagonals for rectangular sections. An item of special interest attaches to his diagonals for columns of I section in which the radius of gyration about an axis perpendicular to the web is slightly larger than the radius about the axis parallel to the web.

Figure 24 shows curves from which correcting factors may be read, these factors to be applied to the ordinates of Karman diag-

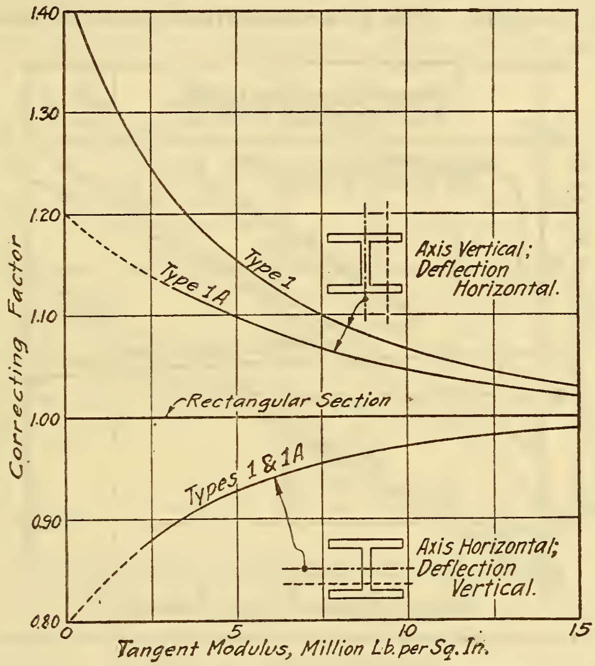

FIG. 24.—*Diagram giving correcting factors that should be applied to the ordinates of Karman diagonals for rectangular sections to make them appropriate for types 1 and 1A*

onals for rectangular sections, to transform them into diagonals appropriate to columns of the same slenderness ratio and of types 1 or 1A. In using such a diagram for a column of definite slenderness ratios with respect to the two axes, one perpendicular to the web and one parallel to it, one calulates the Karman diagonals assuming the section to be rectangular. The ordinates of the curve so obtained and whose slenderness ratio is based upon a radius of gyration about an axis parallel to the web, must be increased in accordance with the upper curves of Figure 24, while the ordinates of the other curve must be decreased in accordance with the lower curve of that figure. If the larger slenderness ratio of the column is based upon a radius of gyration which is about an axis parallel to the web, it may happen that the resulting curves cross, so that the strength of the column, according to the Karman theory, may depend upon the smaller slenderness ratio, if failure is expected at a tangent modulus that is smaller than that of the point of intersection of the curves.

Such a case is illustrated in Figure 25, which refers to columns of type 1, 120 L/R. The two slenderness ratios of this column

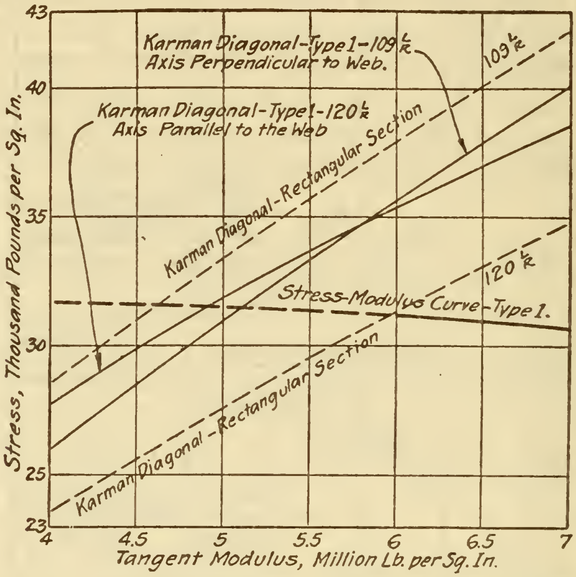

FIG. 25.—*Diagram showing intersection of the two Karman diagonals for type 1, 120 L/R*

For most columns of type 1, Karman's theory makes the strength depend upon the smaller slenderness ratio

are 120 and 109, the radius of gyration about an axis parallel to the web being the smaller. The Karman diagonals for rectangular section are shown as dotted. The ordinates of the lower curve are increased in accordance with Figure 24 and the ordinates of the upper curve are decreased in accordance with that figure. The resulting curves cross at a tangent modulus of about 5,800,000 lbs./in.[2] The curve for 120 L/R would be used for strength estimates for columns whose stress-modulus curves cut it at a greater tangent modulus, while the curve for 109 L/R governs for other cases. It is evident from this figure that Karman's theory makes all columns of type 1, and of slenderness ratio 120 or smaller, weaker in the plane parallel to the web than in a plane perpendicular to it.

IV. RELATED PROBLEMS

1. DIRECTION OF DEFLECTION

Figures 26 to 34, inclusive, show deflection paths for individual columns of slenderness ratios 85 and 120, and for most of the types tested. Columns of type 5B are not included, and of the American Railway Engineers Association latticed channel columns, those only are included whose numbers do not exceed 68, as seen in Table 4. Each path is marked with the number of the column to which it refers; and to distinguish columns of different slender-

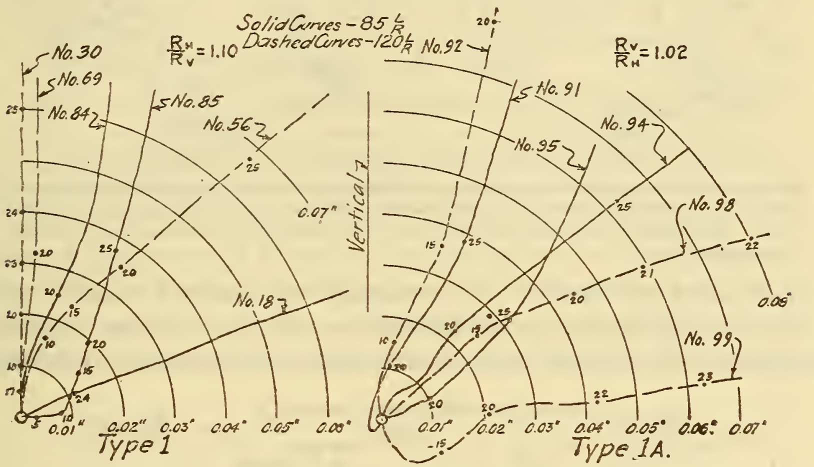

FIG. 26.—*Deflection paths for columns of 85 and 120 L/R in types 1 and 1A*

The path of each column is traced on a plane perpendicular to its axis. The arcs are drawn from a center representing the initial position of the column and at intervals representing 0.01 inch deflection. The numerals found along the paths indicate the corresponding stresses in thousand pounds per square inch

ness ratio, paths for columns of 85 L/R are drawn solid, while those for columns of 120 L/R are shown in dashed lines.

In the testing machine each of these columns occupied a horizontal position and it was counterweighted at its middle by an upward force equal to half the weight of the column. The section of each column was oriented with respect to horizontal and vertical axes as indicated in the sketches shown in Table 3 and also in Figures 5 to 13. At certain loads, measurements were taken upon the horizontal and vertical components of deflection at the middle of each column. These horizontal and vertical components of deflection have been plotted as abscissas and ordinates of points on the deflection paths (figs. 26 to 34). Any point on one of these paths shows the displacement of the axis of the column at mid-length with respect to its initial position, the plane of the paths being perpendicular to the axis of the column. The stresses at which the readings were taken, in thousand pounds per square inch, are indicated by numerals

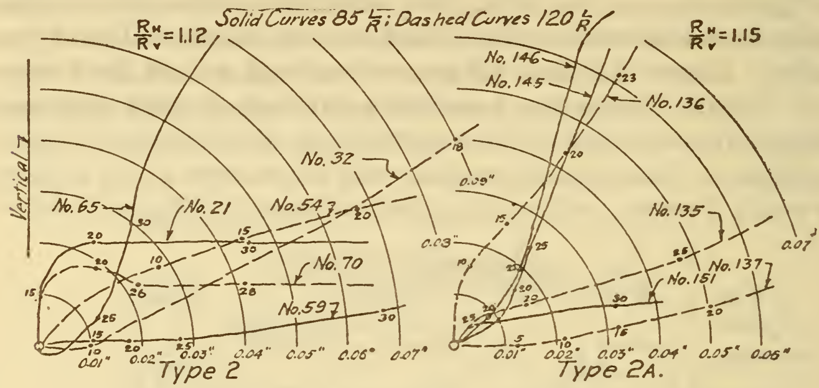

FIG. 27.—*Deflection paths for columns of 85 and 120 L/R in types 2 and 2A*

The path of each column is traced on a plane perpendicular to its axis. The arcs are drawn from a center representing the initial position of the column and at intervals representing 0.01 inch deflection. The numerals found along the paths indicate the corresponding stresses in thousand pounds per square inch

found along the paths. In arranging the paths for each type into a compact group, no distinction has been made between upward or downward general deflection, or between northward

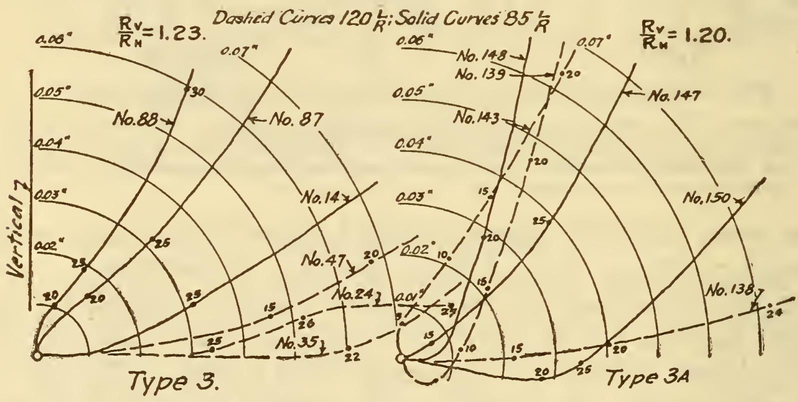

FIG. 28.—*Deflection paths for columns of 85 and 120 L/R in ty es 3 and 3A*

The path of each column is traced on a plane perpendicular to its axis. The arcs are drawn from a center representing the initial position of the column and at intervals representing 0.01 inch deflection. The numerals found along the paths indicate the corresponding stresses in thousand pounds per square inch

or southward general deflection; but these distinctions have been made in drawing different parts of the same path. Circular arcs, seen in each figure, are drawn from the origin of deflection, the interval between successive arcs representing an increment of 0.010 inch in radial deflection.

In connection with the group of paths for each type of columns, there is given the ratio of the principal radii of gyration of the column section, as designed. The numerator is the maximum

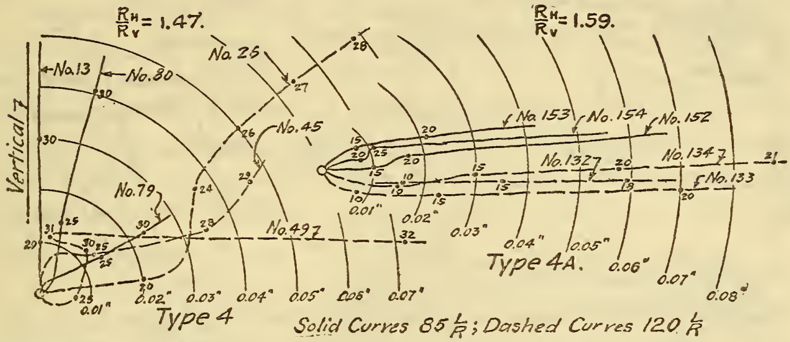

FIG. 29.—*Deflection paths for columns of 85 and 120 L/R in types 4 and 4A*

The path of each column is traced on a plane perpendicular to its axis. The arcs are drawn from a center representing the initial position of the column and at intervals representing 0.01 inch deflection. The numerals found along the paths indicate the corresponding stresses in thousand pounds per square inch

radius of gyration and it carries a subscript V or H indicating the axis to which it refers; the denominator is the minimum radius of gyration and also carries a distinguishing subscript. If the direction of final deflection were determined solely by the relative

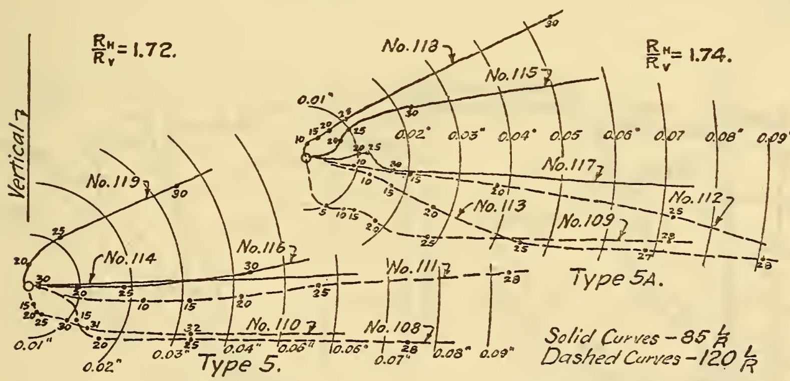

FIG. 30.—*Deflection paths for columns of 85 and 120 L/R in types 5 and 5A*

The path of each column is traced on a plane perpendicular to its axis. The arcs are drawn from a center representing the initial position of the column and at intervals representing 0.01 inch deflection. The numerals found along the paths indicate the corresponding stresses in thousand pounds per square inch

magnitudes of the radii of gyration, as is assumed by Engesser's formula, one would expect a column to show no deflection until the load approaches its maximum value, and then the deflection should develop rapidly in the plane which includes the axis of

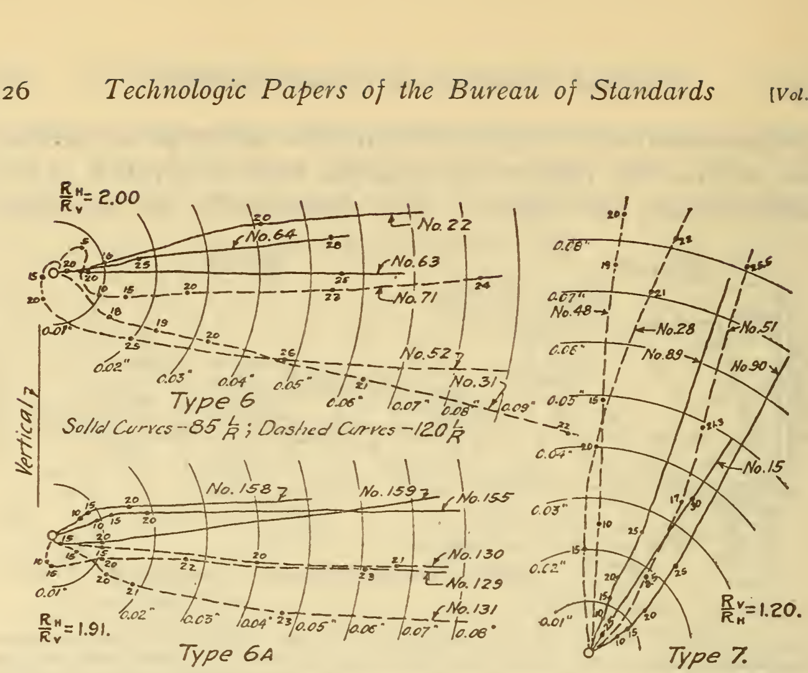

FIG. 31.—*Deflection paths for columns of 85 and 120 L/R in types 6, 6A, and 7*

The path of each column is traced on a plane perpendicular to its axis. The arcs are drawn from a center representing the initial position of the column and at intervals representing 0.01 inch deflection. The numerals found along the paths indicate the corresponding stresses in thousand pounds per square inch

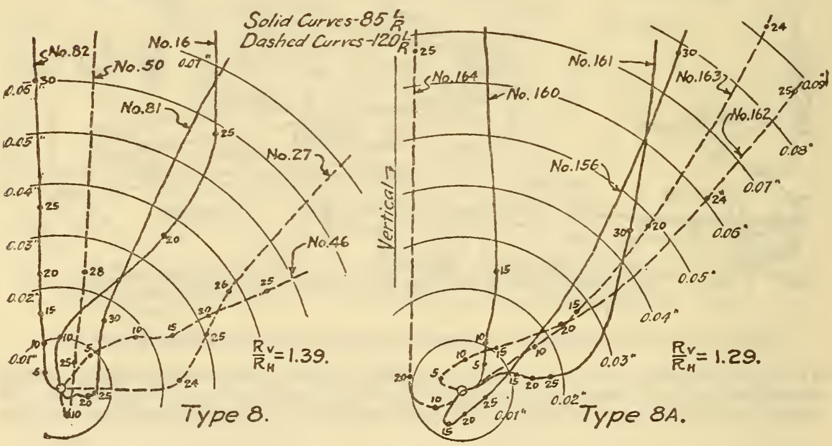

FIG. 32.—*Deflection paths for columns of 85 and 120 L/R in types 8 and 8A*

The path of each column is traced on a plane perpendicular to its axis. The arcs are drawn from a center representing the initial position of the column and at intervals representing 0.01 inch deflection. The numerals found along the paths indicate the corresponding stresses in thousand pounds per square inch major moment of inertia; that is, its direction in the figures should be that of the subscript of the *numerator* of the ratio of radii of gyration.

But as actual columns always involve more or less eccentricity of loading, no matter how accurately they may be fabricated and adjusted to the testing machine, we ought to expect a column to show small deflections at moderate loads; the deflection path should have any initial direction as determined purely by accident; but as the loading continues the direction of this path should change so as to become nearly parallel to the axis of major moment of inertia.

In order to have some roughly quantitative means of expressing the agreement which these deflection paths appear to indicate with the action which might be expected of such columns, as stated in the above paragraph, Table 9 has been prepared as a rough interpretation of these diagrams (figs. 26 to 34).

TABLE 9.—Comparison of Directions in Which Failure Occurs in Columns of 85 L/R and 120 L/R, with Directions in Which Failure is Expected

Column type	Ratio of radii of gyration	Per cent of failures in direction parallel to axis of maximum inertia
1A	1.02	50
10	1.02	67
10A	1.03	67
1	1.10	17
2	1.12	83
2A	1.15	50
3A	1.20	83
7	1.20	100
3	1.23	50
8A	1.29	100
8	1.39	83
4	1.47	67
4A	1.59	100
A. R. E. A., light	1.59	83
A. R. E. A., heavy	1.65	50
5	1.72	100
5A	1.74	100
6A	1.91	100
6	2.00	100

The peculiar action of the columns of type 1 is believed to be due to the wide and thin flanges attached to the webs by closely spaced rivets, rather than to the effect discussed in connection with Karman's theory. If this were attributed entirely to the Karman effect, it would be hard to explain why the Karman effect did not make all the columns of type 1A deflect vertically. The aimless wandering character of some of the paths for the

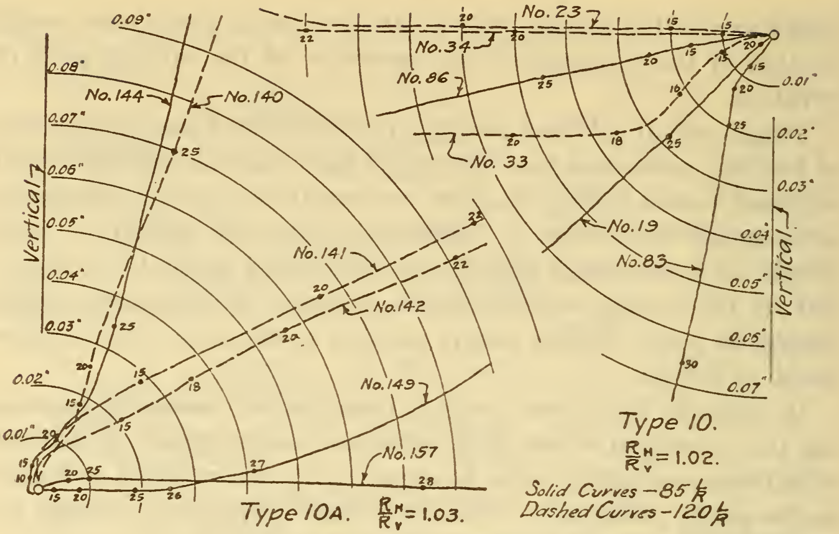

FIG. 33.—*Deflection paths for columns of 85 and 120 L/R in types 10 and 10A*

The path of each column is traced on a plane perpendicular to its axis. The arcs are drawn from a center representing the initial position of the column and at intervals representing 0.01 inch deflection. The numerals found along the paths indicate the corresponding stresses in thousand pounds per square inch

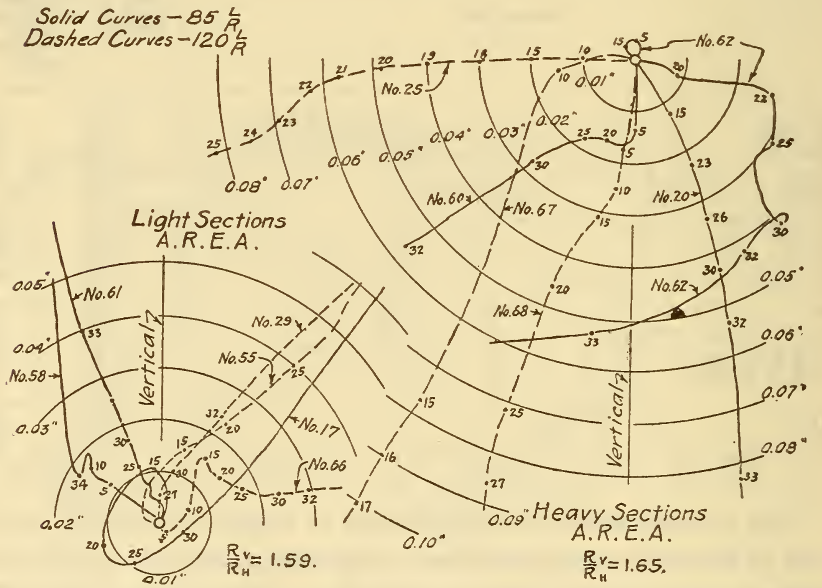

FIG. 34.—*Deflection paths for columns of 85 and 120 L/R in light and heavy A. R. E. A. types*

The path of each column is traced on a plane perpendicular to its axis. The arcs are drawn from a center representing the initial position of the column and at intervals representing 0.01 inch deflection. The numerals found along the paths indicate the corresponding stresses in thousand pounds per square inch latticed channel A. R. E. A. columns, shown in Figure 34, is probably related to the lack of continuity in the section of these columns. If Table 9 is a fair interpretation of the action of these columns whose paths are shown, one may conclude that a column of *continuous section* may fail in a test in some direction other than parallel to the axis of maximum inertia unless one radius of gyration is about 60 per cent larger than the other.

2. ECCENTRICITY OF LOADING

Assuming elastic conditions, one can readily estimate the eccentricity of loading (after the initial loading) of a column, using the gauge-line readings, the locations of the gauge lines on the section, and the properties of the section. This has been done for most of the columns of slenderness ratio 85 at the stress 16,000 lbs./in.[2] The average values of the eccentricity components for the different types of columns are given in Table 10. The values are so small that many of them may be attributed to slight errors in the determination of the gauge readings rather than to real eccentricities of loading. The large value for type 6 is not believed to be genuine; first, because these columns showed almost no deflection at this stress, and second, because these columns had already passed their proportional limits at this stress, as may be seen from Figure 19 and made probable by Figure 10.

TABLE 10.—Components of Eccentricity of Loading for Columns of 85 L/R Estimated From Gauge-Line Readings at 16,000 lbs./in.[2] on the Assumption of Elastic Conditions

Column type	Horizontal component	Vertical component
	Inch	Inch
1	0.023	0.030
1A	.010	.021
2	.027	.017
2A	.019	.032
3	.033	.022
3A	.056	.024
4	.032	.031
4A	.030	.043
5	.017	.017
5A	.011	.021
5B	.024	.032
6	.014	.127
6A	.021	.036
7	.020	.005
8	.059	.025
8A	.016	.011
10	.038	.038
10A	.013	.020
Mean	.026	.031

One may estimate the eccentricities of loading from the observed deflection components, assuming elastic conditions and assuming that the axis of the column takes the form of a sinusoid symmetrically arranged. Such estimates have been made in this manner for the columns used in the preparation of Table 10. These eccentricity components show a moderate degree of correlation with those computed from the gauge readings, but they are smaller by about 50 per cent. From this, one may suspect that the elastic curve of the column axis at this stress is not a symmetrically arranged sinusoid, but has some accidental form determined by such circumstances as slight initial crookedness of the column or variation in properties in different parts of the column length.

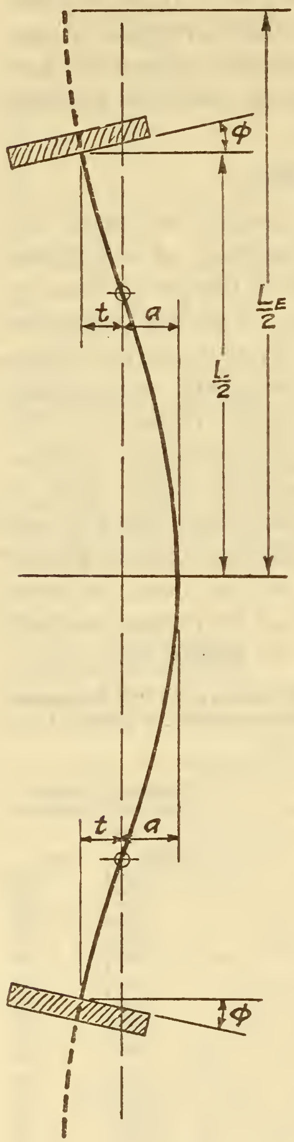

FIG. 35.—*Diagrammatic sketch of axis of ideal column deflecting under maximum load and the inclined heads of the testing machine*

3. END RESTRAINT IN COLUMN TESTS

Attention has been called to the fact that all the column tests considered have been treated as if the columns had fixed ends, and the meaning of this term "fixed ends" has been stated. If the ends of a column are not completely restrained in a test the compression heads of the testing machine becomes slightly inclined as the column deflects, and the load at each end remains more nearly axial than would be the case if the ends were fixed. In order to estimate the amount by which the restraint of a column fails to be complete, one needs observations upon the angular deflection of the heads of the testing machine with respect to a line joining their centers. Such observations were not made in the tests studied, but one can form a rough estimate of the lower limit of this restraint from considerations of the properties of the columns and of the testing machine.

Figure 35 represents the axis of an ideal column with incomplete end restraint, deflecting under its maximum load, the curve of the axis being assumed to be sinusoidal. While this form of the curve does not appear to characterize such a column under moderate loads, the assumption is probably much more nearly justified at or near maximum load. In Figure 35 the length of the actual column is represented as L, while L_E represents the greater length of a similar column of the same strength but having fixed ends. The points of inflection are marked on the column axis by small circles; the line of force passes through these points of inflection, giving eccentricity a at the middle of the column, while at the column ends the eccentricity is smaller and is represented by t. The load acting upon the heads of the machine at eccentricity t has caused the heads to become inclined by the small angle ϕ from their normal positions.

The equation of the axis of the column gives the value of t as

$$t = -a \cos \frac{\pi L}{L_E}$$

and the value of ϕ as

$$\phi = \frac{2\pi a}{L_E} \sin \frac{\pi L}{L_E}$$

Let the straining bars of the testing machine have a moment of inertia N times that of the column and neglect all deformations of the heads of the machine. Assuming their length to be the same as that of the column, and that they are subject to uniform bending moment Pt, one may write another expression for the small angle ϕ as

$$\phi = \frac{LPt}{2\,EIN}$$

in which P is the maximum load of the column, and may be written as

$$P = \frac{4\pi^2 E'I}{L^2_E}$$

If the two expressions for ϕ are set equal, the resulting equation can be reduced to

$$N\frac{E}{E'} = -\frac{\dfrac{\pi L}{L_E}}{\tan \dfrac{\pi L}{L_E}}$$

This is an angle, slightly smaller than 180°, expressed in radians. So long as L is not smaller than about 0.90 L_E, the following equation gives practically the same results as the one derived above, and is much simpler to use:

$$\frac{L}{L_E} = \frac{1}{1 + \dfrac{E'}{NE}}$$

and it seems appropriate to call L/L_E "degree of restraint."

The moment of inertia of the straining bars of the machine used in these tests is about 1,675 (in.)^4 with respect to a vertical axis, and its value is very much greater about a horizontal axis. One sees from Table 3 that columns of type 6A had a moment of inertia about a vertical axis of 147 (in.)^4, and that of all the types this is the largest value of minimum inertia about this axis. This type should, therefore, show the smallest value for degree of restraint. These inertia values make N in the above formula equal to 11; if we take E' as half of E, the value of E'/NE becomes 0.045, and the minimum value of the degree of restraint is

$$\frac{L}{L_E} = 0.95$$

If this minimum value of degree of restraint were characteristic of all the columns, this could have been taken account of in our diagrams by drawing the diagonals to represent slenderness ratios about 5 per cent in excess of the actual slenderness ratios of the columns. This would have rotated these diagonals to the right by a small amount and given slightly smaller strength estimates. This has not been done for several reasons: the whole matter is uncertain and complicated and the modification would not have changed the strength estimates by amounts that exceed the probable errors involved in determining the stress-modulus curves.

It will be noted that the degree of restraint depends upon the value of the tangent modulus at which the column reaches its maximum load. Since short columns reach their maximum loads at smaller moduli than long columns do, the degree of restraint for a short column at maximum load may be considerably greater than that of a corresponding long column; or, we may say that the degree of restraint of a short column may increase as its load increases and as its modulus falls off. It was in view of this circumstance that E' was not chosen larger than one-half of E in the above estimate of the minimum value of degree of restraint.

It was noted above that curves for certain columns were omitted in Figures 14 and 15 because these columns showed very high modulus values nearly up to their ultimate strengths. In such a case one might take E' equal to E and thus get a small value for degree of restraint. Making this assumption, one still gets a degree of restraint of 0.95 for column No. 214, type 2A, 155 L/R, which is the only one of these columns which failed by buckling in a horizontal plane.

4. EFFECT OF LOADING ON STRESS-MODULUS CURVE

It is well known that if a steel specimen is tested in tension after it has been subjected to a tensile load, its stress-strain curve will be found to follow Young's modulus up to a stress that somewhat exceeds the stress that has been previously applied. This is a matter of interest and importance in connection with column strength and it is well illustrated in Figure 36. The lower portion of this figure shows the stress-modulus curve for column No. 98, type 1A, 120 L/R, which failed at 25,457 lbs./in.^{2} in a manner that is in good agreement with the Engesser formula. After this test was completed, two short, straight portions were cut from the column and called 98A and 98B. Column No. 98A was tested 17 days after the test on column No. 98 and showed the stress-modulus curve given in the upper part of Figure 36. The test on column 98B was made one day later and showed practically the same effect. Similar results were obtained from tests of short columns cut from each of the following columns after the long columns had been tested: 84 and 85 of type 1, 85 L/R; 99 of type 1A, 120 L/R; 160 and 161 of type 8A, 85 L/R. These tests suggest that if a column has once carried a certain load without undue deflection, this column will safely carry the same load at any subsequent date.

5. TIME EFFECT

In the ordinary routine of testing, some time effect occurs in the interval needed for taking the gauge-line readings; that is, the readings change slightly with duration of loading. This effect is small so long as the modulus values are high, but it becomes marked as the modulus values decrease. One might, therefore, suspect that this effect introduces a large error into the determination of the modulus values at high stresses. This is not necessarily the case, because for low modulus values the increment of compression on change of load is large, so that a considerable change in a reading due to time effect may not seriously change the modulus value.

At stresses that are multiples of 5,000 lbs./in.^{2} it was customary in these column tests to release the load and take set readings, after which readings were repeated at the highest load already reached. These repeat readings generally differed somewhat from the first readings at the same load, and, in general, the repeat readings showed an increase of compression as compared with the first. This effect may be attributed partly to time effect and partly to changes in temperature or some other circumstance. In these cases the modulus values were usually high. For the next increment of stress, the increment of compression is generally found to be slightly large if based upon the first readings at the previous stress, but much too small if based upon the repeat readings.

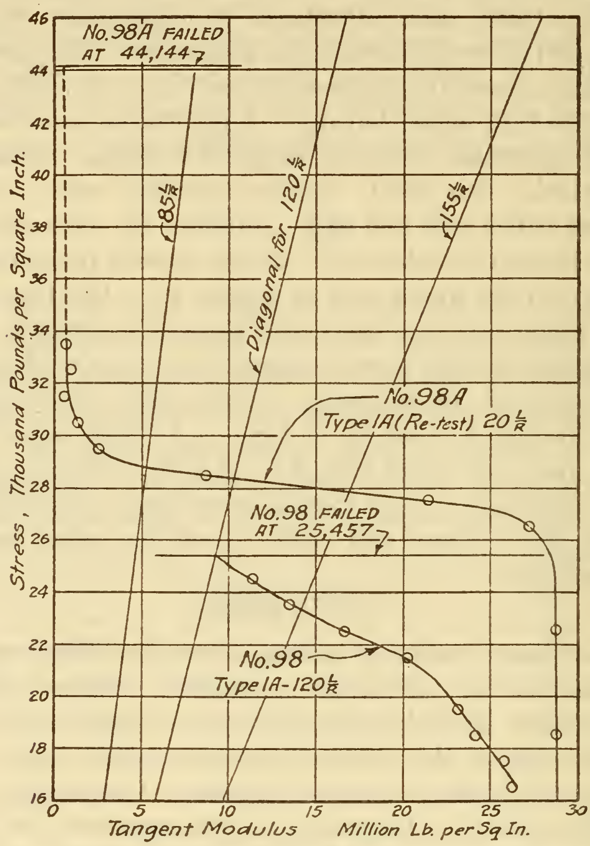

FIG. 36.—*Stress-modulus curves of individual columns 98 ana 98A*

After column No. 98 had been tested, column No. 98A was cut therefrom. The loading of column No. 98 so changed the properties of its material as to give a very different curve on second test

In nearly all cases, therefore, modulus values for such cases have been based upon the first set of readings at the previous stress, but in drawing the curves these particular modulus values have been given less weight than the others.

In the tests of columns 47, type 3, 120 L/R, and 54, type 2, 120 L/R, the regular routine of loading was altered in order to find the effect of duration of loading upon the action of columns at rather high stresses. Most of the results obtained in this way are shown in Figure 37 in which the abscissas represent duration of loading in minutes while ordinates represent increase in the average compression as given by five gauge lines 150 inches long, both ordinates and abscissas being plotted to logarithmic coordinates. The curves are roughly represented by the following equations, in which i

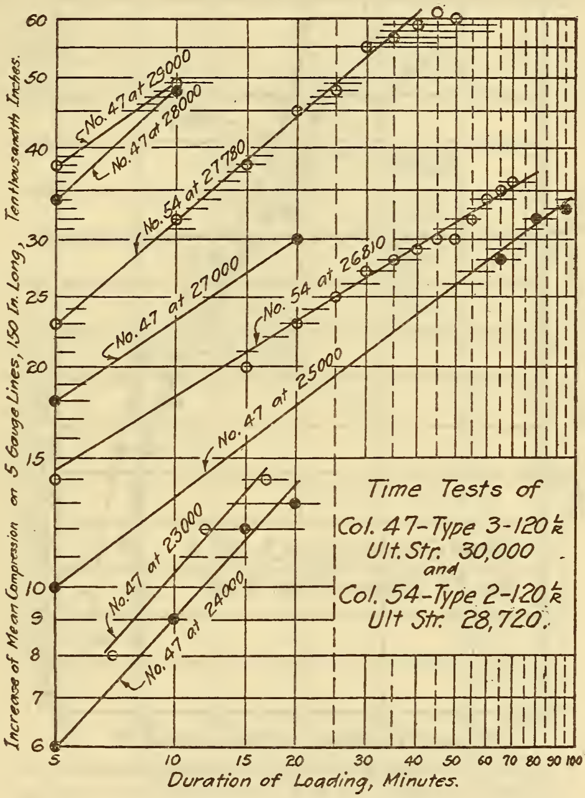

FIG. 37.—*Diagram showing increase in compression with duration of loading in tests of columns 47, type 3, 120 L/R, and 54, type 2, 120 L/R.*

represents increase in compression of these gauge lines in inches and M represents duration of loading in minutes:

$i = 0.00084\ M^{0.34}$ for column 54 at 26,680 lbs./in.^{2}
$i = 0.00102\ M^{0.47}$ for column 54 at 27,780 lbs./in.^{2}
$i = 0.00052\ M^{0.41}$ for column 47 at 25,000 lbs./in.^{2}

In Figures 38 and 39 stress-modulus curves are shown for each of these columns along with the curves for the two other columns that were tested as duplicates of each of the two which were tested for time effect. In Figure 38 the curve for column No. 47 is seen at the right of the curves for the duplicate columns and which were tested in the regular manner, and in Figure 39 the curve for column No. 54 is seen to occupy a position between those for its duplicates which were tested in the regular manner. One may conclude from this that duration of loading has not had an in-

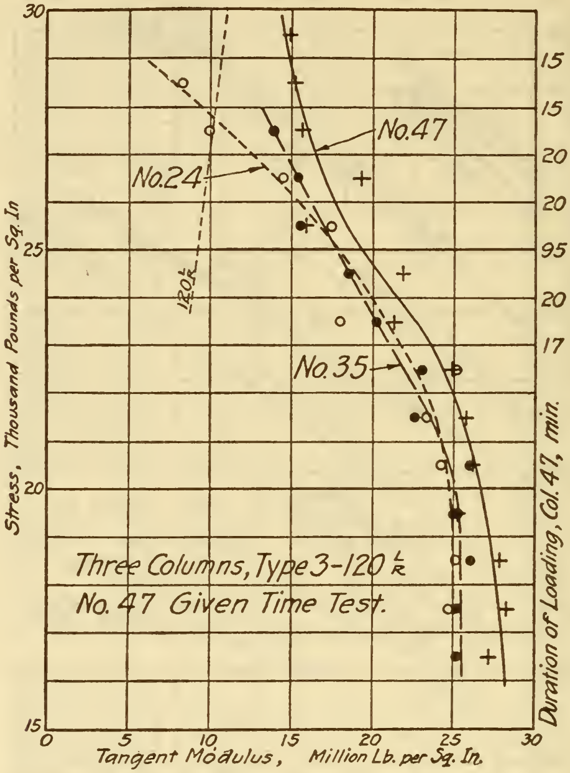

FIG. 38.—*Stress-modulus curves for individual columns 24, 35, and 47 of type 3, 120 L/R, of which column No. 47 was tested for duration of loading*

jurious effect upon the material of the columns as represented by their stress-modulus curves determined from such time tests.

As column No. 54 was standing under its two largest loads, which were maintained constant to observe the effect of time, its increase of compression was not uniform over the section. Referring to Figure 40, which shows the section of this column, the compression values obtained for gauge lines 1, 3, 5, and 7 while the stress was maintained at 27,780 lbs./in.[2] have been plotted vertically from their gauge-line positions as origin in each case. Lines 1 and 2 showed in each case a decrease of compression, while lines 5 and 7 show large increases of compression. By connecting in pairs the points so plotted in Figure 40, one obtains two points on the axis whose fibers did not change in length provided the column maintained plane sections; and the axis for 27,780 lbs./in.^{2} has been drawn through these two points. The axis shown for 26,810 lbs./in.^{2} was obtained in a similar manner. It will be noted that the axis for the larger stress is nearer the center of gravity of the section than is the other.

The action of column No. 47, when standing at steady loads, was somewhat similar to that shown in Figure 40 for column No. 54,

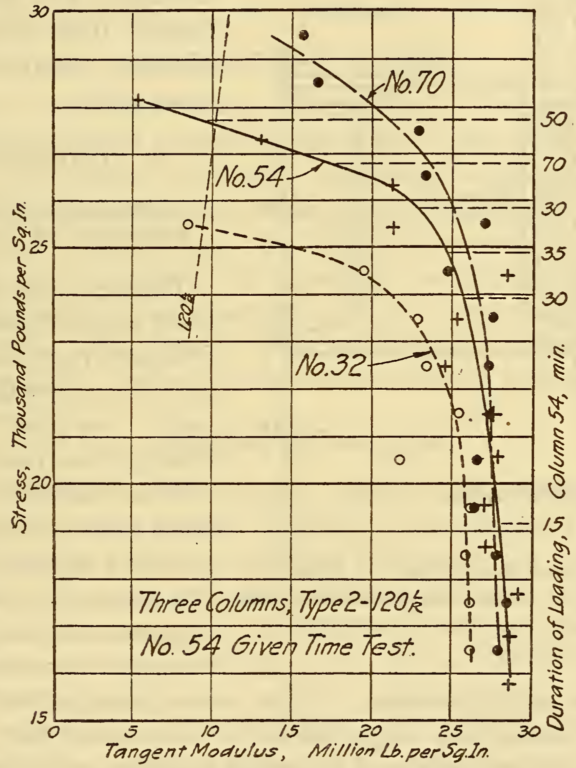

FIG. 39.—*Stress-modulus curves for individual columns 32, 54, and 70 of type 2, 120 L/R, of which column No. 54 was tested for duration of loading*

but for the five largest loads used on column No. 47, it is in the case of the largest one only that the axis touches the section.

Figure 40 resembles Figure 22 in that each shows an axis which separates the two parts of the section which experience elongation on one side and compression on the other. But Karman's theory assumes a sudden deflection at maximum load, while Figure 40 refers to a slowly increasing deflection at a load that is below the maximum. In Karman's theory, the axis considered should pass through the center of gravity for elastic conditions and depart from that center gradually as the tangent modulus decreases, never passing completely off the section. In these columns under time tests, the axis appears to be off the section at small stresses and to approach its center of gravity as the stress increases. It seems, therefore, that the distribution of strains as indicated in Figure 40 resembles the Karman distribution for some other reason than that a real Karman distribution has been found.

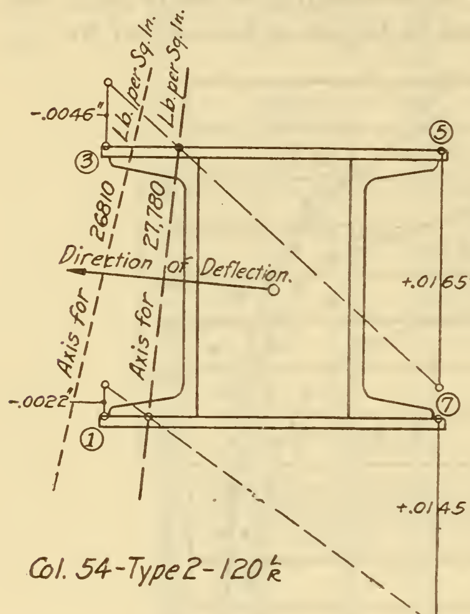

FIG. 40.—*Diagram showing how the increase in compression for column No. 54 was distributed over its section as it stood for some time at a stress of 27,780 lbs./in.*[2]

V. SUGGESTIONS

1. CHARACTER OF COMPRESSION IN COLUMNS

Figures 2, 36, 38, and 39 show series of points whose ordinates represent stresses and whose abscissas represent tangent moduli. Among these points in each case a smooth curve has been drawn to represent a probable interpretation of the manner in which the tangent modulus gradually decreases in value as the stress is gradually increased. Is the assumption justified that the stress-modulus curve is a smooth curve? In a rough way, one may say that the irregularities of the stress-modulus curve decrease as the gauge line is increased in length. That the estimates of column strength made by means of such curves have given such good results indicates that these curves closely represent real characteristics of the column material and it probably indicates that they are comparatively smooth.

Under elastic conditions and axial load, we are safe in saying that the compression that is found in any particular length of a column, will also be found in any other length of it under the same load. Within what limitations is a similar statement true of a column after its load has increased to such an extent that its material is in the semielastic condition, which is characterized by decreased values of the tangent modulus? Both the Karman theory and the Engesser formula assume the semielastic column to have the same strain in all parts of the column length at any one stress. That the stress-modulus curve generally becomes more nearly smooth as the gauge lines are increased in length, may imply that semielastic strain is more or less local, developing first in one part of a column length and then in another part. This suspected characteristic is believed to be the greatest obstacle in devising a wholly satisfactory theory for the strength of actual columns failing under semielastic conditions.

2. STABILITY OF SECTION

What are the conditions for the stability of the column section under the semielastic conditions which are characterized by small values of the tangent modulus? The major errors found in the strength estimates made in accordance with Engesser's formula for slenderness ratios 85 and 120 L/R are in types having rather wide outstanding flanges and the same characteristic marks the types that develop the drooping ends of stress-modulus curves. Some of these types showed local buckling of the flanges in columns of 50 L/R. Is the stability of the section a function of the column length?

3. FAILURE BY TWISTING

In the discussions of this paper, columns of type 8 having light *Z* sections have been treated as if they were expected to fail by buckling, just as has been expected of columns of other types. As a matter of fact, of the nine columns which were tested of this type, the first column to be tested (No. 9, 50 L/R) failed by buckling, while each of the remaining eight columns failed by twisting about its longitudinal axis. This is apparently due to a lack of torsional stiffness in columns of this type, and almost nothing is known about the torsional stiffness of sections other than those which are cylindrical or rectangular. Would it not be worth while to investigate the conditions which determine whether a column will fail by twisting or by buckling?

4. EFFECTS OF FABRICATION

How is the stress-modulus curve of a fabricated column related to the stress-modulus curves of its parts before fabrication? When the writer began this study he had the impression that the fabrication process has a very injurious effect upon the material as indicated by its stress-modulus curve, and hence upon its column strength. If one looks for this effect in the typical stress-modulus curves, he sees that the curve for type 10 having eight rows of rivets, is unusually steep, and that the riveted columns, except the A. R. E. A. types, show a much lower proportional limit than do types 5 and 5A. But this is not regarded as a satisfactory answer to the question. If the properties of columns are to be judged by the properties of their component parts, tested before fabrication, definite information should be known upon the effects of fabrication.

5. TIME EFFECT

Columns are employed to carry loads for indefinite periods of time, but as yet very little is known regarding the effects of time upon the action of a column under load. This appears to be a matter that calls for thorough investigation.

6. IMPROVEMENTS IN COLUMN TESTING

Comparatively recent contact with the Bureau of Standards shows that in column testing, the initial load is placed on the column with practically zero eccentricity and that observations are now made on the heads of the testing machine to detect angular changes in their positions throughout the test. The need for these improvements in method were evident in the tests under review.

Would it not be advisable to adopt standard speeds of loading for use as a column approaches its maximum load, and to make these speeds rather small and proportional to the column length?

7. VARIATION IN MATERIAL

In Figure 41 the various typical stress-modulus curves that were shown in small groups in Figures 5 to 13 have been collected for the purpose of facilitating comparison one with another, and to show the wide variation of the properties of the fabricated material, all of which was obtained under the same specification. This specification provided that the tensile yield stress of the material should be 38,000 lbs./in.^{2}, with an allowable variation of 1,000 lbs./in.^{2} either way. In accepting the material these limits were not strictly adhered to. Efforts were made by the Bureau of Standards and by the column committees of the two cooperating engineering societies to secure the most uniform material practicable, and there is every reason to think that the steel mills undertook to satisfy these three interested parties; but the result is column material, which (if we interpret the tangent moduli obtained from the 50 L/R tests as representing the properties of the material) at a modulus value of 15,000,000 lbs./in.^{2} ranges in average stress from 34,600 to 15,500 lbs./in.^{2}

Is it not a fair conclusion from the above that ordinary steel columns may show a still wider range of properties than that found here for this group of columns made under unusually rigid specification and with the apparent cooperation of the steel mills in an

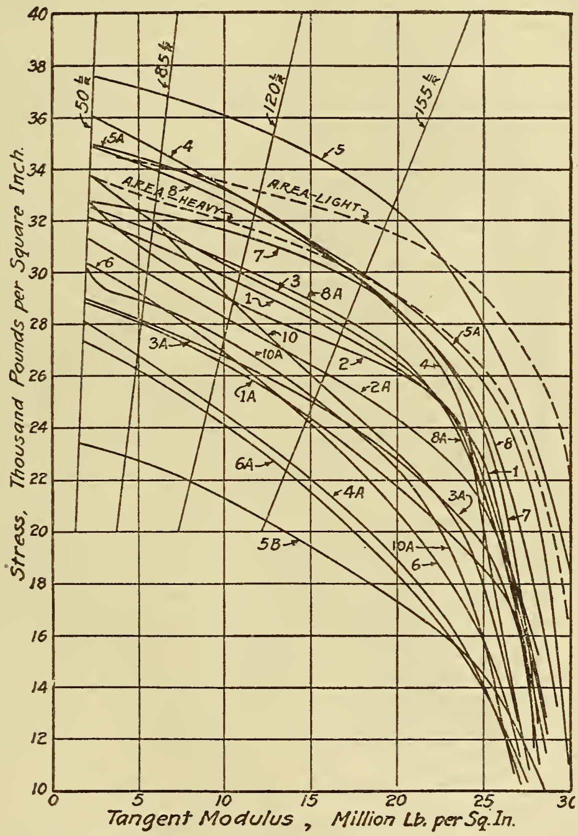

Fig. 41.—*Collection of typical stress-modulus curves previously shown in Figures 5 to 13*

effort to make these columns the best that could be produced? Is it not evident that there is something seriously wrong in judging the quality of steel for use in columns by the ordinary tensile specimen tests?[6] Is it not probable that a compressive buckling test can be devised which will operate at a satisfactory speed and give reliable information regarding the suitability of material for use in columns?

[6] For the opinion of the A. S. C. E. column committee see Trans. A. S. C. E., 83, p. 1612; 1919–20.

8. AVERAGE CURVES

In Figure 42 the curve marked "All sections" is a composite of all the typical stress-modulus curves shown in Figure 41, formed by averaging the stresses at each of various moduli. The upper curve in Figure 42 is a similar composite for all the light sections, types 1, 2, 3, etc., whose average thickness of material is about 0.32 inch. The curve marked "Heavy sections" is a composite for types 1A, 2A, 3A, etc., whose average thickness of material is about 0.50 inch. The lower curve represents the single type, 5B, whose metal averaged about 1.12 inches thick. The diagonals

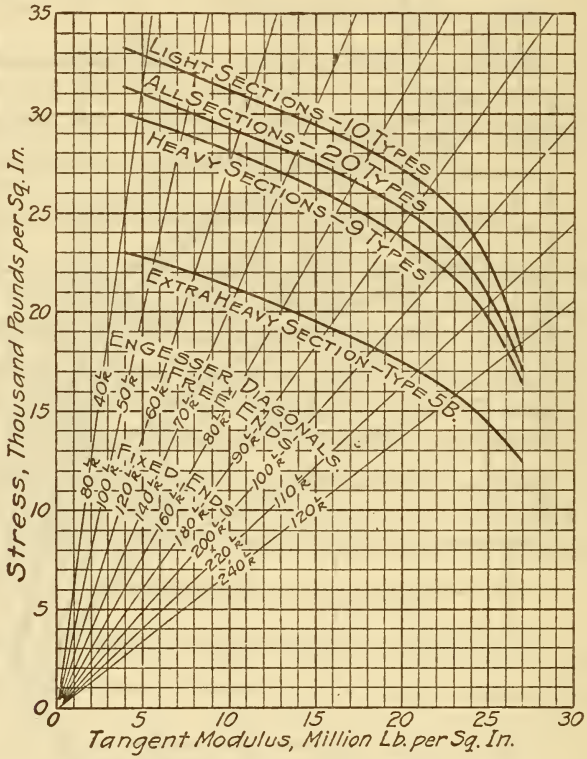

FIG. 42.—*Average typical stress-modulus curves*

shown in this figure are marked with a double set of slenderness ratios, one corresponding to fixed ends and the other to free ends; they are located in accordance with the Engesser formula, which gives somewhat too large estimates of strength for columns of certain types in large slenderness ratios.

WASHINGTON, April 30, 1924.