RP492

# COLUMN CURVES AND STRESS-STRAIN DIAGRAMS

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ABSTRACT

The only important column formula which rests on a strictly theoretical basis is the Considère-Engesser formula, which includes the Euler formula as a special case. Attempts have been made to underpin by a general theoretical foundation other essentially empirical formulas, notably the Rankine formula; but these attempts have not been successful. It would be very satisfying, however, if these empirical formulas could be shown to be reasonable special forms of the Considère-Engesser formula. Any column formula for centrally loaded columns is a special case of the Considère-Engesser formula provided the compressive stress-strain diagram has a certain definite shape. The present paper examines a few of the commonest types of empirical formulas and determines the shape of the stress-strain diagram in each case which makes them compatible with the Considère-Engesser theory.


The only column formula which rests on a strictly theoretical basis and which at the same time is of practical importance, in the sense that tests confirm the theory, is the Considère-Engesser formula,[1] which includes the Euler formula as a special case. Attempts have been made to underpin by a general theoretical foundation other formulas, essentially empirical, notably the Rankine formula; but no successful attempt has yet been made. It would be very satisfying, however, if these empirical formulas could be shown to be reasonable special forms of the Considère-Engesser formula. Any column formula for centrally loaded columns is a special case of the Considère-Engesser formula, provided the compressive stress-strain diagram of the material has a certain definite shape. It is the purpose of the present paper to examine a few of the commonest empirical formulas and to determine the necessary shape of the stress-strain diagram in order that these formulas may be compatible with the Considère-Engesser theory. A somewhat similar investigation has been carried out by P. M. Frandsen,[2] but he refers the empirical formulas back to the Engesser formula.[3]

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[1] Developed in 1889 and the years following by A. Considère, Fr. Engesser, and F. Jasinski; re-presented independently by Theo. v. Kármán, Mitteilungen über Forschungsarbeiten, Verein deutscher Ingenieure, Heft 81, Berlin, Julius Springer, 1910, and by R. V. Southwell, Engineering, vol. 94, p. 249, London, Aug. 23, 1912.

[2] Den Teknisk Forenings Tidsskrift, Hæfte 19, p. 139, Copenhagen, Sept. 15, 1920.

[3] Zeitschrift des hannoverischen Architekten- und Ingenieur-Verein, vol. 35, p. 455, 1889.

The following notation will be used:

$P$ = the load on the column at failure by buckling.

$A$ = the cross-sectional area of the column.

$\sigma = \frac{P}{A}$ = the average normal stress on the cross section at failure.

$l$ = the "free length" of the column, the distance between two successive points of inflection of the center line.

$i$ = the least radius of inertia or radius of gyration of the cross section of the column, measured parallel to the plane of bending.

$E$ = the modulus of elasticity of the material of the column.

$E'$ = the "tangent modulus" at the stress $\sigma$; that is, $E'$ is the slope of the compressive stress-strain diagram at the stress $\sigma$.

$\overline{E} = \frac{E'I_1 + EI_2}{I}$, where $I_1$ is the moment of inertia about the axis of average stress of the part of the cross-sectional area which suffers an increase of stress at the instant of failure of the column, $I_2$ is the moment of inertia about the axis of average stress of the part of the cross-sectional area which suffers a decrease of stress at the instant of failure of the column, and $I = Ai^2$ is the moment of inertia of the total cross-sectional area of the column about the gravity axis perpendicular to the plane of bending; the position of the axis of average stress is defined by the relation $E'S_1 = ES_2$, where $S_1$ and $S_2$ are the statical moments about the axis of average stress, respectively, of the two parts of the cross-sectional area just mentioned in connection with $I_1$ and $I_2$.

$\epsilon$ = the strain due to the stress $\sigma$.

$\sigma_u$ = the short-column strength of the material, to be taken as the yield point in the case of ductile materials. (See Appendix II.)

$e = \frac{E\epsilon}{\sigma_u}$.

$s = \frac{\sigma}{\sigma_u}$.

$\sigma_p$ = the stress above which Euler's formula ceases to apply, strictly the proportional limit of the material, but practically likely to be considerably above the actual proportional limit.

$s_p = \frac{\sigma_p}{\sigma_u}$,

$\lambda = \frac{1}{\pi} \frac{l}{i} \sqrt{\frac{\sigma_u}{E}}$.

$C_n$, $C_R$, constants.

The assumptions underlying the Considère-Engesser theory are that the material is homogeneous, cross sections remain plane, the stress-strain relation for increasing strain is the same as that given by the stress-strain diagram, the stress-strain relation for decreasing strain is given by a line parallel to the tangent to the stress-strain diagram at the origin, the axis of the column is straight, the cross section of the column is uniform throughout the free length, and the loading is axial.

The Considère-Engesser formula may be written

$$\sigma = \frac{\pi^2 \bar{E}}{\left(\frac{l}{i}\right)^2} \tag{1}$$

In order that any empirical formula expressible as

$$\frac{l}{i} = f(\sigma) \tag{2}$$

give the same relation between $\sigma$ and $\frac{l}{i}$ as the Considère-Engesser formula, we must have

$$\bar{E} = \frac{\sigma}{\pi^2}\left(\frac{l}{i}\right)^2 \tag{3}$$

where $\frac{l}{i}$ is given by (2). The stress-strain diagram must satisfy this equation.

$\bar{E}$ is a function not only of the slope of the stress-strain diagram at $(\epsilon, \sigma)$ but also of the shape of the cross section, and no general reduction of equation (3) is possible. Fortunately, however, the effect of the shape of the cross section on the value of $\bar{E}$ does not vary greatly for cross sections ordinarily used which have an axis of symmetry perpendicular to the plane of bending.[4] It can be shown that $\bar{E}$ is smaller for the idealized H section (negligible web, flanges thin compared to distance between them) with the plane of bending normal to the flanges than for any other section with the symmetry just mentioned. Consequently, if $\bar{E}$ for the H section is used, equation (1) will give results on the safe side for all symmetrical sections, and equation (3) may be reduced with the assurance that the results obtained represent limiting values for such sections. The value of $\bar{E}$ for the idealized H section is [5]

$$\bar{E} = \frac{2\,EE'}{E+E'} \tag{4}$$

which, substituted in equation (3), with $E' = \frac{d\sigma}{d\epsilon}$, gives

$$\sigma = \frac{\pi^2}{\left(\frac{l}{i}\right)^2} \cdot \frac{2}{\frac{d\epsilon}{d\sigma} + \frac{1}{E}} \tag{5}$$

and this is the equation of the stress-strain curve if equation (2) is to represent a theoretically possible column formula.

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[4] See Appendix I.

[5] Kármán (see footnote 1, p. 571).

Instead of integrating equation (5) for various functions $\frac{l}{i}=f(\sigma)$, it is preferable for comparative purposes to introduce the nondimensional variables [6]

$$e=\frac{E\epsilon}{\sigma_u},\ s=\frac{\sigma}{\sigma_u},\ \lambda=\frac{1}{\pi}\frac{l}{i}\sqrt{\frac{\sigma_u}{E}} \tag{6}$$

In terms of these quantities equations (1) and (5) become

$$s=\frac{1}{\lambda^2}\frac{\bar{E}}{E} \tag{7}$$

and

$$s=\frac{1}{\lambda^2}\cdot\frac{2}{\frac{de}{ds}+1} \tag{8}$$

where in the latter equation $\lambda$ is a given function of $s$; (7) is the equation of a curve obtained from the column curve by dividing the average stresses by $\sigma_u$ and the slenderness ratios by $\pi\sqrt{\frac{E}{\sigma_u}}$, and (8) is the equation of a curve obtained from the stress-strain curve by dividing the stresses by $\sigma_u$ and the strains by $\frac{\sigma_u}{E}$.

In the Euler range, for which $\bar{E}=E$, equation (7) reduces to

$$s=\frac{1}{\lambda^2} \tag{9}$$

and equation (8) becomes, with the use of equations (6)

$$s=e \tag{10}$$

For values of $s$ above $s_p$; that is, above the Euler range, for a continuous stress-strain diagram the solution of (8) becomes

$$e=2\int_{s_p}^{s}\frac{ds}{s\lambda^2}-s+2\,s_p \tag{11}$$

The reduced stress-strain diagram represented by equations (10) and (11) will now be considered for some empirical column formulas.

*The parabolic or hyperbolic type of formula.*—A common type of formula is one of the form

$$\sigma=\sigma_u\left[1-C_n\left(\frac{l}{i}\right)^n\right] \tag{12}$$

If

$$C_n=\frac{2}{n\pi^n}\left(\frac{n}{n+2}\right)^{1+\frac{n}{2}}\left(\frac{\sigma_u}{E}\right)^{\frac{n}{2}}$$

the curve represented by (12) is tangent to the Euler curve at $\sigma=\frac{n}{n+2}\sigma_u$.

[6] The first of these has been used by K. Hohenemser (Zeitschrift für angewandte Mathematik und Mechanik, vol. 11, p. 15, February, 1931), the second has been used several times before and is obvious, and the third has been used by L. B. Tuckerman, the late S. N. Petrenko, and C. D. Johnson (National Advisory Committee for Aeronautics, Technical Note No. 307, Washington, D. C.).

With this value of $C_n$, in terms of the variables defined by (6), equation (12) becomes

$$s=1-\frac{2}{n}\left(\frac{n}{n+2}\right)^{1+\frac{n}{2}}\lambda^n \tag{13}$$

which applies for $\frac{n}{n+2} \leqq s \leqq 1$. Figure 1 shows the curves

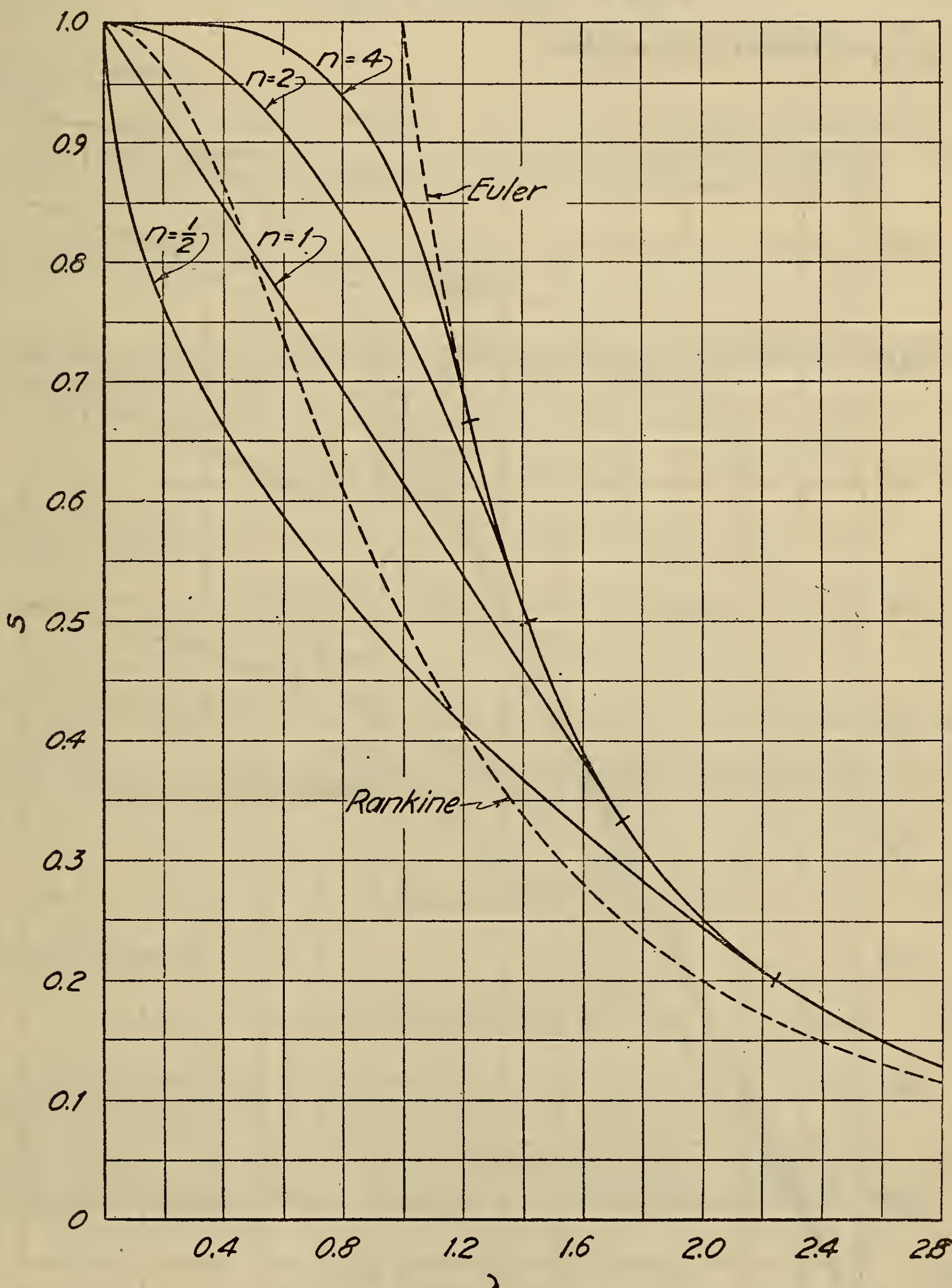


FIGURE 1.—*Reduced column curves of several types*

represented by equation (9) (marked Euler) and equation (13) for different values of $n$. Equation (13) solved for $\lambda$ gives

$$\lambda=\left(\frac{n}{2}\right)^{\frac{1}{n}}\left(\frac{n+2}{n}\right)^{\frac{1}{n}+\frac{1}{2}}(1-s)^{\frac{1}{n}} \tag{14}$$

and substitution of this expression in equation (11) gives finally

$$e=2\left(\frac{2}{n}\right)^{\frac{2}{n}}\left(\frac{n}{n+2}\right)^{1+\frac{2}{n}}\int_{s_p}^{s}\frac{ds}{s(1-s)^{\frac{2}{n}}}-s+2s_p \qquad (15)$$

for values of $s$ from $\frac{n}{n+2}$ to 1, inclusive. For $s$ less than or equal to $s_p=\frac{n}{n+2}$ equation (10) applies.

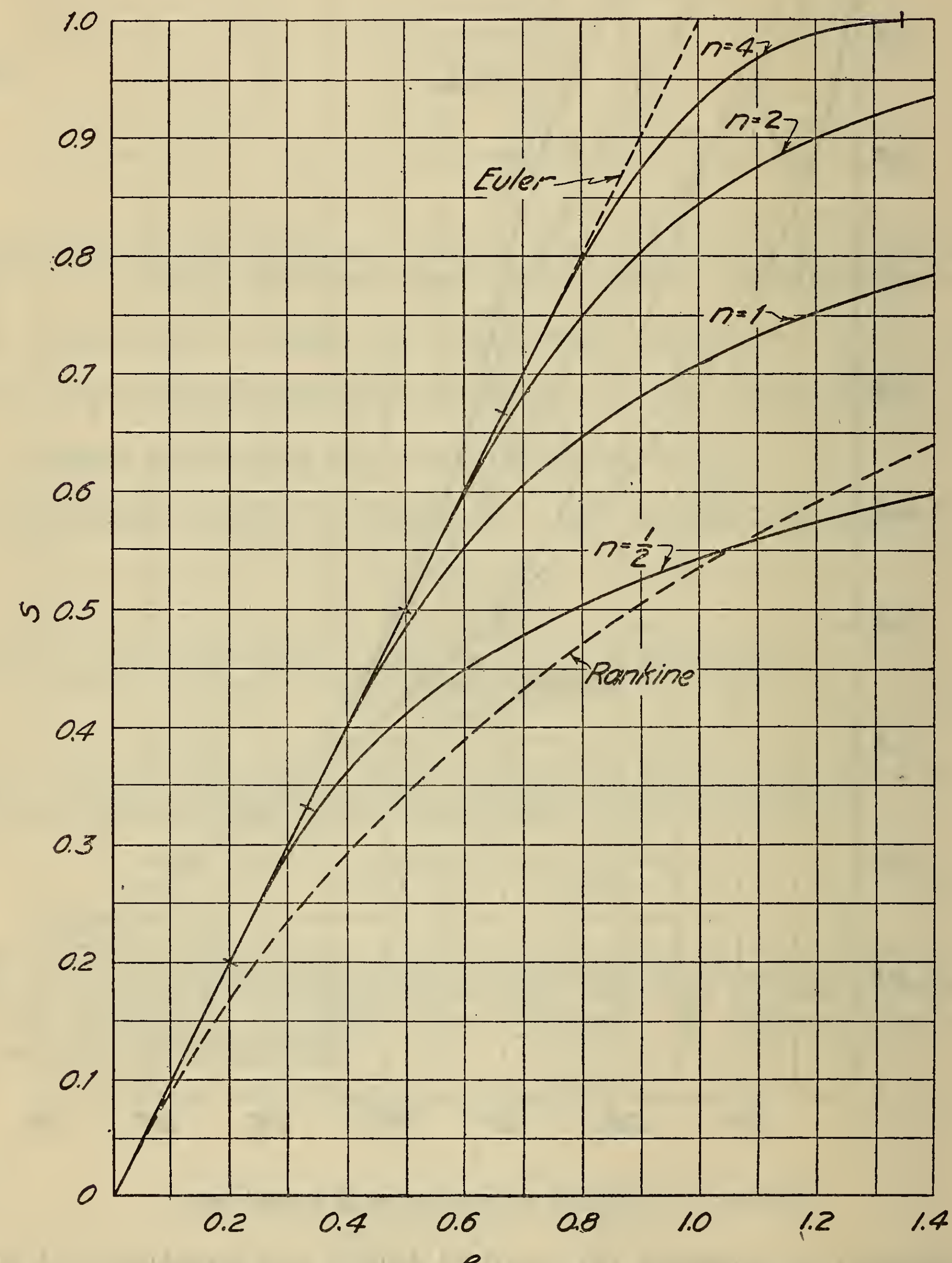


Figure 2.—*Reduced compressive stress-strain diagrams corresponding to the column curves of Figure 1*

Figure 2 shows the curves represented by equation (10) (marked Euler) and equation (15) for different values of $n$.

The case $n=\frac{1}{2}$ represents a column formula which experience has shown to be suitable for brittle materials like cast iron. (See fig. 1.) Equation (15) becomes

$$e=\frac{512}{3125}\int_{1/5}^{s}\frac{ds}{s(1-s)^4}-s+\frac{2}{5} \tag{16}$$

which reduces to

$$e=0.1638\left[\frac{1}{3(1-s)^3}+\frac{1}{2(1-s)^2}+\frac{1}{1-s}+\log\frac{s}{1-s}\right]-s+0.1877 \tag{17}$$

applicable for $\frac{1}{5}\leqq s\leqq 1$. (See fig. 2.)

The case $n=1$ represents a column formula suitable for common grade wood. (See fig. 1.) Equation (15) becomes

$$e=\frac{8}{27}\int_{1/3}^{s}\frac{ds}{s(1-s)^2}-s+\frac{2}{3} \tag{18}$$

which reduces to

$$e=\frac{8}{27}\left(\frac{1}{1-s}+\log\frac{s}{1-s}\right)-s+0.4276 \tag{19}$$

applicable for $\frac{1}{3}\leqq s\leqq 1$. (See fig. 2.)

The case $n=2$ represents a column formula suitable for ductile materials like mild steel. (See fig. 1.) Equation (15) becomes

$$e=\frac{1}{2}\int_{1/2}^{s}\frac{ds}{s(1-s)}-s+1 \tag{20}$$

which reduces to

$$e=\frac{1}{2}\log\frac{s}{1-s}-s+1 \tag{21}$$

applicable for $\frac{1}{2}\leqq s\leqq 1$. (See fig. 2.)

The case $n=4$ represents a column formula suitable for hard materials like hard steel. (See fig. 1.) Equation (15) becomes

$$e=\frac{4\sqrt{3}}{9}\int_{2/3}^{s}\frac{ds}{s(1-s)^{1/2}}-s+\frac{4}{3} \tag{22}$$

which reduces to

$$e=\frac{4\sqrt{3}}{9}\log\frac{1-\sqrt{1-s}}{1+\sqrt{1-s}}-s+2.347 \tag{23}$$

applicable for $\frac{2}{3}\leqq s\leqq 1$. (See fig. 2.)

*The Rankine formula.*—The Rankine formula

$$\sigma=\frac{\sigma_u}{1+C_R\left(\frac{l}{i}\right)^2} \tag{24}$$

has been used for all values of $\frac{l}{i}$ with $C_R=\frac{\sigma_u}{\pi^2 E}$, in which case the curve represented by (24) approaches tangency to the Euler curve at infinity. With this value of $C_R$, in terms of the variables defined by (6), equation (24) becomes

$$s=\frac{1}{1+\lambda^2} \tag{25}$$

which solved for $\lambda$ gives

$$\lambda=\sqrt{\frac{1-s}{s}} \tag{26}$$

and substitution of this expression in equation (11) gives

$$e=2\int_0^s \frac{ds}{1-s}-s$$

or

$$e=-2\log(1-s)-s \tag{27}$$

for values of $s$ from 0 to 1, inclusive.

The curves represented by equations (25) and (27) are shown in Figures 1 and 2, respectively.

Appendix III shows an American Bridge Co. reduced straight-line column curve and the corresponding reduced stress-strain diagram.

All of the curves in Figure 2 except the curve marked $n=4$ (equation (23)) approach the horizontal line $s=1$ asymptotically. The curve marked $n=4$ has a horizontal tangent at $e=1.347$, $s=1$. A comparison of the curves of Figure 1 with the corresponding curves of Figure 2 shows the type of compressive stress-strain diagram which is necessary in any particular case in order that a given column curve may represent accurately the strength of columns. The shapes of the reduced stress-strain diagrams are all reasonable except for high values of $s$ and possibly in the case of the Rankine formula. The shape of the diagram for high values of $s$ is relatively unimportant, however, since the corresponding portion of the column curve lies in the region of "short columns," and such columns do not usually fail by buckling. It is doubtful whether any structural material would show a reduced compressive stress-strain diagram like that required for the Rankine formula.

Viewed in another way, any (reduced) compressive stress-strain diagram in Figure 2 will yield the (reduced) column curve corresponding to it in Figure 1. If, therefore, the short-column strength, $\sigma_u$, of the material is such that, or can be defined (Appendix III) so that, the (reduced) compressive stress-strain diagram fits a curve of Figure 2, the column strength of the material will be given by the corresponding curve of Figure 1. It will frequently be possible by a suitable choice of $\sigma_u$ to obtain good agreement between the actual (reduced) compressive stress-strain diagram and one of the curves of Figure 2 up to some limiting value of $s$, beyond which the agreement will not be good. In that case the corresponding curve of Figure 1 will represent the column strength up to the same value of $s$. Thus, if representative compressive stress-strain diagrams of a material are available, it may be possible to draw the corresponding column curve without tests. This method is not to be recommended to the exclusion of tests, however, particularly not if the agreement between the actual (reduced) compressive stress-strain diagram and a curve of Figure 2 is not extremely good.

The author wishes to express his indebtedness to Dr. Walter Ramberg, of the engineering mechanics section, Bureau of Standards, for checking the equations.

## APPENDIX I

Figure 3 shows the variation of $\frac{\bar{E}}{E}$ with $\frac{E'}{E}$ for a number of different cross sections, as listed on the figure. The curves for the thin circular-ring section (thickness negligible in comparison with the diameter) and the solid circular section are plotted from formulas by R. V. Southwell.[7]

The formula for the rectangular section is given by T. v. Kármán,[8] and the six points for the I section are taken from a graph by W. Gehler.[9] The two curves for the thin T section (thickness of the web and the flange negligible in comparison with their lengths) have been worked out by the author.

It may be noted that the curve for an unsymmetrical section may lie considerably below the curve for the idealized H section, and the lower limit for such sections is the curve $\frac{\bar{E}}{E}=\frac{E'}{E}$. (Fig. 3.) Consequently, Frandsen's treatment [10] may be considered as applying unmodified to unsymmetrical sections.

## APPENDIX II

In interpreting column tests of ductile materials it is of some importance how $\sigma_u$ is defined. If $\sigma_u$ is defined as the stress determined by the intersection of the stress-strain curve with a line through the origin having a slope $\alpha E$ $(0<\alpha<1)$, the experimental points when plotted as a $\lambda$, $s$-diagram are likely to lie more nearly on a smooth curve than otherwise. Under ideal conditions of test (perfectly straight specimens, perfect centering, etc.) if different materials having affine stress-strain diagrams are tested, and if $\sigma_u$ is determined as outlined, all points $\lambda,s$ will lie on one and the same curve. In tests made at the Bureau of Standards the values $\alpha=\frac{5}{9}$ and $\alpha=\frac{2}{3}$ have

[7] Southwell (see footnote 1, p. 571.)
[8] Kármán (see footnote 1, p. 571.)
[9] Proceedings of the Second International Congress for Applied Mechanics, p. 366, Zurich, Sept. 12–17, 1926.
[10] See foot note 2, p. 571.

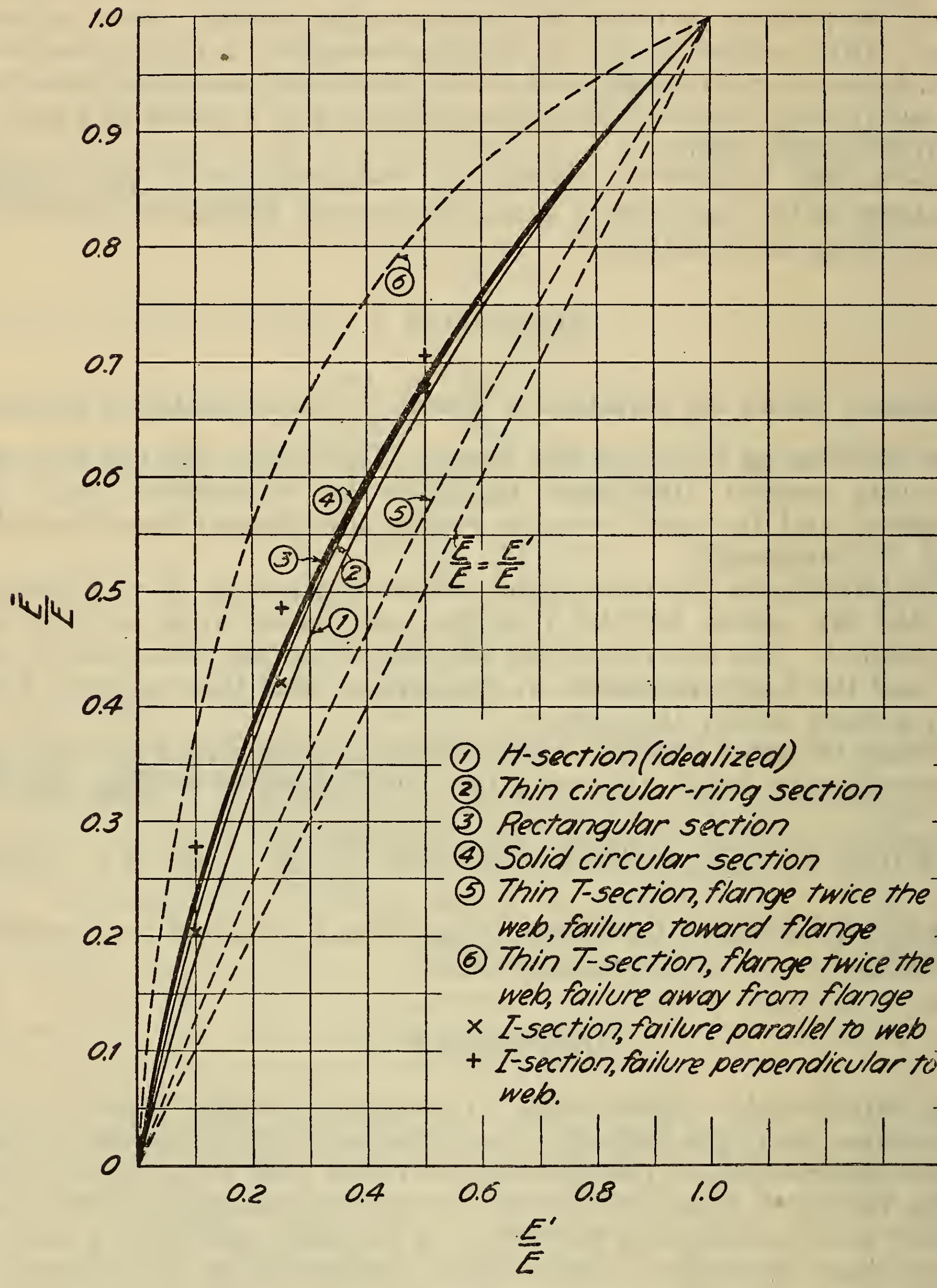


FIGURE 3.—*Variation of* $\frac{\bar{E}}{E}$ *with* $\frac{E'}{E}$ *for various shapes of cross section*

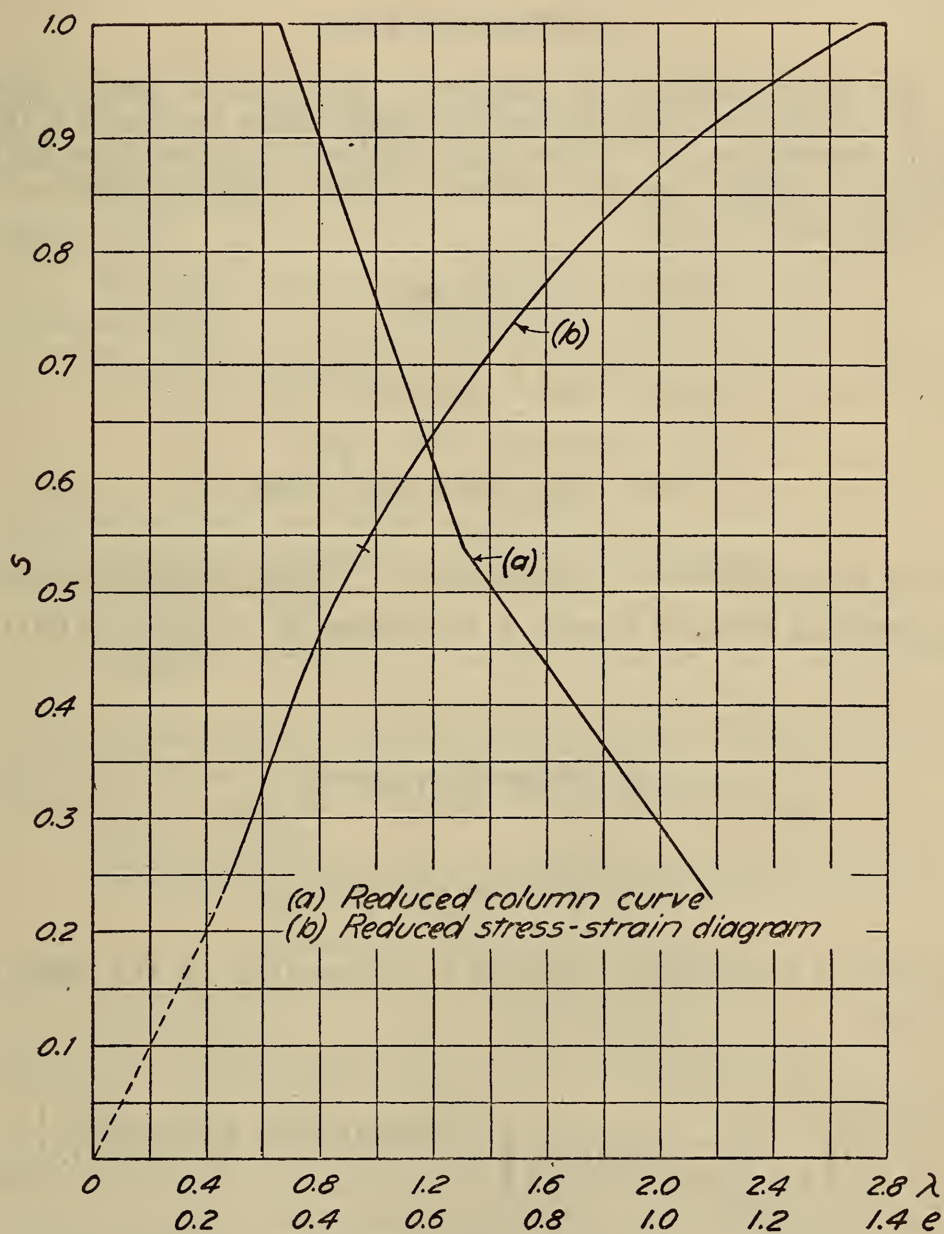


FIGURE 4.—*Reduced American Bridge Co. column curve and corresponding reduced compressive stress-strain diagram*

been found to give satisfactory results for chromium-molybdenum steel and duralumin columns, respectively.

By adjusting the value of $\alpha$, it may be possible to fit the actual (reduced) compressive stress-strain diagram rather closely to one of the curves of Figure 2 up to a sufficiently high value of $s$ to include all except short columns.

## APPENDIX III

It may be of interest to compare a specification column curve with the stress-strain diagram which would make the curve a theoretically correct one. The American Bridge Co. curve given by the following equations has been chosen for this purpose (units are in pounds per square inch):

$$\left.\begin{aligned}\sigma_s&=13{,}000 \quad \text{for } 0\leqq\frac{l}{i}\leqq 60,\\ \sigma_s&=19{,}000-100\,\frac{l}{i} \quad \text{for } 60\leqq\frac{l}{i}\leqq 120,\\ \sigma_s&=13{,}000-50\,\frac{l}{i} \quad \text{for } 120\leqq\frac{l}{i}\leqq 200,\end{aligned}\right\} \tag{28}$$

where $\sigma_s$ is the allowable average stress. These equations become, respectively, in terms of $\lambda$ and $s$, if we assume $\frac{\sigma_u}{E}\left(=\frac{\pi^2}{8450}\right)=0.001168$,

$$\left.\begin{aligned}s&=1,\\ s&=\frac{19}{13}-\frac{\sqrt{2}}{2}\,\lambda \quad \text{for } 1\geqq s\geqq\frac{7}{13}\\ s&=1-\frac{\sqrt{2}}{4}\,\lambda \quad \text{for } \frac{7}{13}\geqq s\geqq\frac{3}{13}\end{aligned}\right\} \tag{29}$$

The reduced stress-strain diagram is represented by the following equations:

$$\left.\begin{aligned}s&=1\\ e&=\frac{169}{361}\left(\frac{1}{1-\frac{13}{19}s}+\log\frac{s}{\frac{19}{13}-s}\right)-s+0.0913+C \quad \text{for } 1\geqq s\geqq\frac{7}{13}\\ e&=\frac{1}{4}\left(\frac{1}{1-s}+\log\frac{s}{1-s}\right)-s+C \quad \text{for } \frac{7}{13}\geqq s\geqq\frac{3}{13}\end{aligned}\right\} \tag{30}$$

where the constant of integration $C$ is arbitrary.

Equations (29) and (30) are represented graphically in Figure 4.

WASHINGTON, July 14, 1932.

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