Spectral Measures and Separation of Variables*

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This article gives an expression for the spectral measure corresponding to a self-adjoint operator for which separation of variables is possible. The construction makes use of the amalgamation theorem for normal operators in a natural way to obtain the required measure as a tensor convolution of the spectral measures of the part operators.

Key words: Convolution; Hilbert space; separation of variables; spectral measure; tensor products.

Let $A$ be a self-adjoint operator in a complex Hilbert space $\mathcal{H}((\cdot,\cdot))$. A separation of variables consists of a description of $\mathcal{H}$ as a Hilbert space tensor product

$$\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$$

of two spaces $\mathcal{H}_1, ((\cdot, \cdot)_1)$ and $\mathcal{H}_2, ((\cdot, \cdot)_2)$ together with a decomposition of $A$. The requirements of the decomposition are (1) that there exist self-adjoint operators $A_1$ in $\mathcal{H}_1$ and $A_2$ in $\mathcal{H}_2$ such that on the elementary products $u_1 \otimes u_2$, $u_1$ in a core $\mathcal{D}_1$ of $A_1$ and $u_2$ in a core $\mathcal{D}_2$ of $A_2$, $A$ has the expression

$$A(u_1 \otimes u_2) = A_1 u_1 \otimes u_2 + u_1 \otimes A_2 u_2;$$

and (2) that the linear hull $\mathcal{D}$ of such products be a core of $A$. The operator $A$ is said to be separated with $A_1$ and $A_2$ part operators, and the decomposition is written

$$A = A_1 \otimes I_2 + I_1 \otimes A_2.$$

Denote by $E_1$, $E'_1$ and $E_2$ the spectral measures corresponding to $A$, $A_1$ and $A_2$, respectively. The goal is to give meaning to and to justify the relation

$$E = E_1 \otimes E_2.$$

The first steps are to use the amalgamation theorem to define a tensor product spectral measure $E_1 \otimes E_2$ analogous to the product measure for complex measures. Then the tensor convolution $E_1 \otimes E_2$ has a natural definition. Finally, $E_1 \otimes E_2$ is identified with $E$.

Denote by $\mathcal{B}$ the family of Borel sets of the reals $R$, and by $\mathcal{B}^2$ the Borel sets of $R \times R$. A spectral measure defined over $\mathcal{B}$ is said to be real. Here all spectral measures are normalized. The generality needed here is given by the following version of the

**AMALGAMATION THEOREM:** If $\hat{E}_1$ and $\hat{E}_2$ are commuting real spectral measures in a Hilbert space $\mathcal{H}$, then there exists one and only one spectral measure $\hat{E}$ in $\mathcal{H}$ over $\mathcal{B}^2$ such that

$$\hat{E}(B \times B') = \hat{E}_1(B) \hat{E}_2(B'), \quad \text{for all } B, B' \in \mathcal{B}.$$

That $\hat{E}_1$ and $\hat{E}_2$ commute means

$$\hat{E}_1(B) \hat{E}_2(B') = \hat{E}_2(B') \hat{E}_1(B)$$

for all $B, B' \in \mathcal{B}$.  

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The problem at hand is to use the amalgamation theorem to construct the spectral measure $E$ for a separating self-adjoint operator $A$. The first step is

**Lemma 1:** If $E_1$ and $E_2$ are real spectral measures in $\mathcal{S}_1$ and $\mathcal{S}_2$, respectively, then $E_1 \otimes I_2$ and $I_1 \otimes E_2$ are commuting real spectral measures in $\mathcal{S}_1 \otimes \mathcal{S}_2$.

**Proof:** Clearly $E_1 \otimes I_2$ defined by

$$(E_1 \otimes I_2)(B) = E_1(B) \otimes I_2, \quad \text{for all } B \in \mathcal{B},$$

is an $\mathcal{S}_1 \otimes \mathcal{S}_2$-projection-valued function defined on $\mathcal{B}$, and

$$(E_1 \otimes I_2)(B) = I_1 \otimes I_2 = I.$$

The countable additivity follows from that of $E_1$ and the fact that strong convergence of factor operators implies strong convergence of their tensor product. Thus $E_1 \otimes I_2$ is a real spectral measure; and by a parallel argument, so is $I_1 \otimes E_2$. Finally

$$[(E_1 \otimes I_2)(B)] [I_1 \otimes E_2](B') = E_1(B) \otimes E_2(B') = E_1(B) \otimes E_2(B') = E_1(B) [I_1 \otimes E_2](B'),$$

for all $B, B' \in \mathcal{B}$, by elementary computations.

The next step is to establish a product spectral measure analogous to a product measure derived from ordinary measures. As usual, the product is defined on rectangles and then extended. This matter is taken care of by

**Lemma 2:** The $\mathcal{S}_1 \otimes \mathcal{S}_2$-projection-valued set function $E_1 \otimes E_2$ defined on rectangles $B \times B' \in \mathcal{B}^2$ by

$$(E_1 \otimes E_2)(B \times B') = E_1(B) \otimes E_2(B')$$

has an unique extension as a spectral measure in $\mathcal{S}_1 \otimes \mathcal{S}_2$ over $\mathcal{B}^2$.

**Proof:** This is a direct application of the amalgamation theorem in which

$$\hat{E}_1 = E_1 \otimes I_2, \quad \hat{E}_2 = I_1 \otimes E_2,$$

and

$$\hat{E} = E_1 \otimes E_2.$$

The relation

$$\hat{E}(B \times B') = \hat{E}_1(B) \hat{E}_2(B')$$

can be read off from the last lines of the proof of Lemma 1.

It is quite natural to call $E_1 \otimes E_2$ the tensor product spectral measure of $E_1$ with $E_2$.

The third step is to define the tensor convolution of $E_1$ with $E_2$ as for convolutions of complex measures. In preparation we need

**Lemma 3:** Let $E$ be a spectral measure on $\mathcal{R}^2$ and for each $B \in \mathcal{B}$ let

$$B^2(B) = \{(x, y) \in \mathcal{R}^2 | x+y \in B\},$$

then $E_\ast$ defined on $\mathcal{B}$ by

$$E_\ast(B) = E[B^2(B)]$$

is a real spectral measure.

**Proof:** Since

$$B^2(B) \in \mathcal{B}^2, \quad \text{for all } B \in \mathcal{B},$$

$E_\ast$ is a projection-valued set function on $\mathcal{B}$; and clearly

$$E_\ast(B) = I.$$
Further if

\[ B \cap B' = \emptyset, \]

then

\[ B^2(B) \cap B^2(B') = \emptyset, \]

and if

\[ B = \bigcup B_i, \]

then

\[ B^2(B) = \bigcup_i B^2(B_i). \]

These follow directly from the definition of \( B^2(B) \). Hence if

\[ B = \bigcup B_i \]

and

\[ B_i \cap B_j = \emptyset, \quad i \neq j, \]

then

\[ E_*(B) = E[B^2(B)] = E[\bigcup_i B^2(B_i)] = \sum_i E[B^2(B_i)] = \sum_i E_*(B_i), \]

so that \( E_* \) is countably additive.

Now it is natural to formulate the

**Definition:** Let \( E_1 \) and \( E_2 \) be real spectral measures in \( \mathfrak{S}_1 \) and \( \mathfrak{S}_2 \), respectively, and let \( E_1 \otimes E_2 \) be their tensor product. The tensor convolution of \( E_1 \) with \( E_2 \), designated \( E_1 \otimes E_2 \), is the real spectral measure in \( \mathfrak{S}_1 \otimes \mathfrak{S}_2 \) given by

\[ (E_1 \otimes E_2)(B) = (E_1 \otimes E_2)[B^2(B)], \quad \text{for all } B \in \mathfrak{B}, \]

where \( B^2(B) \) is as defined in Lemma 3.

The tensor convolution of \( E_1 \) with \( E_2 \) has a tidy relation to the convolution of the measures associated with \( E_1 \) and \( E_2 \) as given by

**Lemma 4:** Let \( E_* = E_1 \otimes E_2 \) where \( E_1 \) and \( E_2 \) are real spectral measures in \( \mathfrak{S}_1 \) and \( \mathfrak{S}_2 \), and let \( u = u_1 \otimes u_2 \) and \( v = v_1 \otimes v_2 \) be elementary tensor products, then for all such \( u \) and \( v \)

\[ (E_* u, v) = ((E_1 \otimes E_2) u_1, v_1)(E_2 u_2, v_2), \]

where \( \otimes \) indicates the convolution of measures.

**Proof:** Using the definition of \( E_1 \otimes E_2 \) it is evident that

\[ ((E_1 \otimes E_2)(B \times B') u, v) = (E_1(B) u_1, v_1)(E_2(B') u_2, v_2), \quad \text{for all } B, B' \in \mathfrak{B}. \]

Since the product measure

\[ (E_1 u_1, v_1)(E_2 u_2, v_2), \]

is the unique extension to \( \mathfrak{B}_2 \) of the right side of the preceding equation and \((E_1 \otimes E_2) u, v) \) is also an extension to \( \mathfrak{B}_2 \), the two extensions coincide, i.e.,

\[ ((E_1 \otimes E_2)(B^2) u, v) = ((E_1 u_1, v_1)(E_2 u_2, v_2)) (B^2), \quad \text{for all } B^2 \in \mathfrak{B}_2. \]

On specializing this to \( B^2(B) \) for any \( B \in \mathfrak{B} \) and invoking the definitions of the convolutions, the desired result follows.

Based on what has been done so far, it is now quite easy to show that

\[ E_* = E_1 \otimes E_2 \]

is, indeed, the spectral measure corresponding to \( A \). From Lemma 3 and the Definition, it is clear that \( E_* \) is a real spectral measure and consequently corresponds to some self-adjoint
operator $A_*$ in $\mathcal{S}$. The business at hand is to show that $A_*$ is equal to $A$. This will complete the construction and prove the

**Theorem:** Let $A$ be a separated self-adjoint operator with $A_1$ and $A_2$ part operators, and let $E_1$ and $E_2$ be the real spectral measures corresponding to $A_1$ and $A_2$, respectively; then the real spectral measure given by the tensor convolution $E_1 \ast E_2$ of $E_1$ with $E_2$ corresponds to $A$.

**Proof:** Let $A_*$ be the self-adjoint operator corresponding to $E_*$. We shall show that $A_*$ is defined and coincides with $A$ on the core $\mathcal{D}$ of $A$ made up of finite linear combinations of elementary products $u_1 \otimes u_2$, $u_1 \in \mathcal{D}_1$, $u_2 \in \mathcal{D}_2$.

Recall that the domain $\mathcal{D}_*$ of $A_*$ is given by

$$\mathcal{D}_* = \{ u \in \mathcal{S} | \int \lambda^2 d(E_* u, u) < \infty \}$$

and that

$$(A_* u, v) = \int \lambda d(E_* u, v)$$

for all $u \in \mathcal{D}_*$, $v \in \mathcal{S}$. Let $u = u_1 \otimes u_2$, $u_1 \in \mathcal{D}_1$, $u_2 \in \mathcal{D}_2$ and let $v = v_1 \otimes v_2$.

By separation of variables and the spectral theorem for $A_1$ and $A_2$ it follows that

$$||A u||^2 = \int_\mathbb{R} (\lambda_1 + \lambda_2)^2 d(E_1 u_1, u_1) d(E_2 u_2, u_2)$$

and

$$(A u, v) = \int_\mathbb{R} (\lambda_1 + \lambda_2) d(E_1 u_1, v_1) d(E_2 u_2, v_2).$$

By Fubini's theorem

$$||A u||^2 = \int_\mathbb{R} (\lambda_1 + \lambda_2)^2 d[(E_1 u_1, u_1) \times (E_2 u_2, u_2)]$$

and

$$(A u, v) = \int_\mathbb{R} (\lambda_1 + \lambda_2) d[(E_1 u_1, v_1) \times (E_2 u_2, v_2)].$$

Now by an immediate consequence of the definition of convolution of measures

$$||A u||^2 = \int_\mathbb{R} \lambda^2 d[(E_1 u_1, u_1) \otimes (E_2 u_2, u_2)]$$

and

$$(A u, v) = \int_\mathbb{R} \lambda d[(E_1 u_1, v_1) \otimes (E_2 u_2, v_2)].$$

According to Lemma 4, this is the same as

$$||A u||^2 = \int_\mathbb{R} \lambda^2 d(E_1 \otimes E_2 u, u)$$

and

$$(A u, v) = \int_\mathbb{R} \lambda d(E_1 \otimes E_2 u, v).$$

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Thus $A_u$ is defined on each such $u$, and by linearity on $\mathcal{D}$. Further, by the last equation

$$(Au, v) = (A_u u, v) \quad \text{for all } u \in \mathcal{D}, \quad v = v_1 \otimes v_2.$$ 

But since elementary products are total in $\mathcal{S}$,

$$Au = A_u u, \quad \text{for all } u \in \mathcal{D},$$

as was to be shown. 

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References


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