Automatic Computing Methods for Special Functions. Part III. The Sine, Cosine, Exponential Integrals, and Related Functions

Irene A. Stegun and Ruth Zucker

Institute for Basic Standards, National Bureau of Standards, Washington, D.C. 20234

(April 29, 1976)

Accurate, efficient, automatic methods for computing the sine, cosine, exponential integrals and hyperbolic sine and cosine integrals are detailed and implemented in an American National Standard FORTRAN program. The functions are also tabulated to 35 significant figures for arguments 0, \(10^0(10^j) 10^{j+1}\) with \(j = -2(1)2\).

Key words: Continued fraction; cosine integral; exponential integral; FORTRAN program; hyperbolic sine and cosine integrals; key values; recurrence relations.

1. Introduction

Since the sine, cosine, exponential integrals and hyperbolic sine, and cosine integrals are frequently encountered together in physical problems and their expansions have terms in common, we have incorporated these functions into Part III. (For Parts I and II, see 1).

While accuracy over the entire domain of definition remains our main concern, we have tended toward methods that also ensure efficiency, portability, and ease of programming and modification. The number of terms in series, the number of convergents in an iterative process, the starting arguments for different methods, are all determined by the program as a function of word length, arguments, accuracy desired, etc. More realistic results are returned when error conditions are encountered. The proper analytic behavior of the function will always be retained to further ensure correct limiting values, in particular of related functions and for purposes of differentiation and integration.

In Parts I and II in addition to the implementing ANS FORTRAN program, we had included a driver (test) program and its results. Since either of these driver programs can be readily modified to compute other functions, we have omitted the driver program and in place of its results have included a table of correct results to 35 significant figures covering essentially the functional range of present computers.

2. Mathematical Properties

Relevant formulas are collected here for completeness and ease of reference. In keeping with the convention of the Handbook [1], \(x\) here is a real variable.

---


2. Figures in brackets indicate the literature references on page 291.
A. Definitions

\[ Si(x) = \int_{0}^{x} \frac{\sin t}{t} \, dt \]

\[ Ci(x) = \gamma + \ln x + \int_{0}^{x} \frac{\cos t - 1}{t} \, dt \]

\[ Ei(x) = - \int_{-\infty}^{x} \frac{e^{-t}}{t} \, dt = \int_{-\infty}^{x} \frac{e^{t}}{t} \, dt \quad (x > 0) \]

(For \( \int_{-\infty}^{\infty} \frac{e^{-t}}{t} \, dt = E_{1}(x) \), often denoted by \(-E_{i}(-x)\), see Part II.)

\[ Shi(x) = \int_{0}^{x} \frac{\sinh t}{t} \, dt = \frac{Ei(x) + E_{1}(x)}{2} \]

\[ Chi(x) = \gamma + \ln x + \int_{0}^{x} \frac{\cosh t - 1}{t} \, dt = \frac{Ei(x) - E_{1}(x)}{2} \]

\( \gamma \) (Euler’s constant) = 0.57721 56649 . . . .

B. Series Expansions

\[ Si(x) = \sum_{k=0}^{\infty} \frac{(-1)^{k} x^{2k+1}}{(2k + 1)(2k + 1)!} \]

\[ Ci(x) = \gamma + \ln x + \sum_{k=1}^{\infty} \frac{(-1)^{k} x^{2k}}{(2k)(2k)!} \]

\[ Ei(x) = \gamma + \ln x + \sum_{k=1}^{\infty} \frac{x^{k}}{(k)(k)!} \quad (x > 0) \]

\[ Shi(x) = \sum_{k=0}^{\infty} \frac{x^{2k+1}}{(2k + 1)(2k + 1)!} \]

\[ Chi(x) = \gamma + \ln x + \sum_{k=1}^{\infty} \frac{x^{2k}}{(2k)(2k)!} \]

C. Continued Fraction

\[ -Ci(x) + i[Si(x) - \pi/2] = e^{-ix} \left[ \frac{1}{ix + 1} + \frac{1}{ix + 1} + \frac{2}{ix + 1} + \frac{2}{ix + \ldots} \right] \quad (0 < x) \]

\[ = E_{1}(ix) \]

292
D. Asymptotic Expansions

\[ Si(x) = \pi/2 - f(x) \cos x - g(x) \sin x \]

\[ Ci(x) = f(x) \sin x - g(x) \cos x \]

where \( f(x) \sim \frac{1}{x} \sum_{k=0}^{\infty} \frac{(-1)^k (2k)!}{x^{2k}} \)

and \( g(x) \sim \frac{1}{x} \sum_{k=0}^{\infty} \frac{(-1)^k (2k+1)!}{x^{2k+1}} \)

\[ Ei(x) \sim \frac{e^x}{x} \sum_{k=0}^{\infty} \frac{k!}{x^k} \quad (x > 0) \]

\[ Shi(x) = \frac{1}{x} [p(x) \cosh x + q(x) \sinh x] \]

\[ Chi(x) = \frac{1}{x} [p(x) \sinh x + q(x) \cosh x] \]

where \( p(x) \sim \sum_{k=0}^{\infty} \frac{(2k)!}{x^{2k}} \)

and \( q(x) \sim \sum_{k=0}^{\infty} \frac{(2k+1)!}{x^{2k+1}} \)

E. Special Values

\[ Si(0) = 0 \quad Ci(0) = -\infty \quad Ei(0) = -\infty \]
\[ Shi(0) = 0 \quad Chi(0) = -\infty \]

F. Symmetry Relations

\((x > 0)\)

\[ Si(-x) = -Si(x) \quad Ci(-x) = Ci(x) - i\pi \]
\[ Shi(-x) = Shi(x) \quad Chi(-x) = Chi(x) - i\pi \]

G. Interrelations

\[ Si(x) = \frac{1}{2i} \left[ E_i(\text{i}x) - E_i(-\text{i}x) \right] + \pi/2 \]

\[ Ci(x) = -\frac{1}{2} \left[ E_i(\text{i}x) + E_i(-\text{i}x) \right] \]

\[ Ei(x) = -\frac{1}{2} \left[ E_i(-x + i0) + E_i(-x - i0) \right] \quad (x > 0) \]

293
H. Value at Infinity

\[ \lim_{x \to \infty} Si(x) = \frac{\pi}{2} \]

I. Related Function

Logarithmic Integral

\[ li(x) = Ei(\ln(x)) \quad (x > 1) \]

3. Method

Evaluation of the integrals by means of quadrature formulas suited to the particular type of integrand tends to be inefficient and inaccurate. For \( Si(x) \) and \( Ci(x) \), the use of the asymptotic expansion is not valid for moderate values of \( x \), while the use of the continued fraction is inefficient and also inaccurate for small values of \( x \). An examination of the series expansion for the functions indicates several difficulties. Summation of the alternating series for \( Si(x) \) and \( Ci(x) \) will lead to greater round-off errors as \( x \) increases. The partial sum at a particular value of \( k \) may be zero. Additionally there may be cancellations in adding the logarithmic term and/or Euler’s constant for \( Ci(x) \), \( Ei(x) \) and \( Chi(x) \). The more rapidly accumulating round-off errors, in particular when summations are limited to a single register, eliminate the prolonged use of the series expansion. Since the maximum of \( Ci(x) \) occurs at \( \frac{\pi}{2} \) and \( Si(x) = \frac{\pi}{2} \) at \( x = 1.92 \), testing indicates \( x = 2(PSLSC) \) as a reasonable starting point for the use of the continued fraction. The starting point for the valid use of the asymptotic expansion for \( Si(x) \) and \( Ci(x) \) does not coincide with the starting point for \( Ei(x) \) (\( Shi(x) \) and \( Chi(x) \)). Testing also indicates that fewer terms are needed in the continued fraction than in the asymptotic expansion.

The following table gives an indication of the number of terms needed to obtain maximum machine accuracy for particular values of \( x \) with the various methods of computation. Throughout the paper, NBM is the maximum number of binary digits in the mantissa of a floating point number, and TOLER = \( 2^{-\text{NBM}} \).

<table>
<thead>
<tr>
<th>Method</th>
<th>Number of Terms</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>( x = 2 )</td>
</tr>
<tr>
<td></td>
<td>( NBM = 27 )</td>
</tr>
<tr>
<td>Power Series</td>
<td>7</td>
</tr>
<tr>
<td>Continued Fraction</td>
<td>24</td>
</tr>
<tr>
<td>(Even Form)</td>
<td>64</td>
</tr>
<tr>
<td>Numerical Integration</td>
<td></td>
</tr>
<tr>
<td>(Trapezoidal or Simpson’s Rule)</td>
<td></td>
</tr>
<tr>
<td>Power Series</td>
<td></td>
</tr>
<tr>
<td>Asymptotic Expansion</td>
<td></td>
</tr>
<tr>
<td>Continued Fraction</td>
<td>512</td>
</tr>
<tr>
<td>Numerical Integration</td>
<td></td>
</tr>
</tbody>
</table>

* We indicate the number of odd and even terms of the respective series.

The most accurate, efficient, automatic methods for \( Si(x) \) and \( Ci(x) \) then are the power series and the continued fraction; for \( Ei(x) \), \( Shi(x) \) and \( Chi(x) \) the power series and the asymptotic
expansion. The lower limit (AELL) for the use of the asymptotic expansion may be shown to approximate $|\ln \text{TOLER}| = \text{NBM}(\ln 2)$, where TOLER is the requested upper limit for the relative error. With this choice of the lower limit, one can also show that $\text{Shi}(x) = \text{Chi}(x) = \frac{1}{2} \text{Ei}(x)$. It is necessary then to consider only the asymptotic expansion for $\text{Ei}(x)$.

The series computations have been so arranged that the maximum number of functions may be obtained in a minimum of time. The even and odd terms of the series are summed independently both with and without the factor $(-1)^k$. Since $\text{Si}(x) - \pi/2$ and $\text{Ci}(x)$ are the imaginary and real parts respectively of the continued fraction expansion for $\text{Ei}(i x)$, there would be a saving in computing time with options on the functions to be computed. Invalid results are initially supplied for all functions. With the parameter $IC = 1$, $\text{Si}(x)$ and $\text{Ci}(x)$ only are computed; with $IC = 2$, $\text{Ei}(x)$ and $e^{-\text{Ei}(x)}$ only; with $IC = 3$, $\text{Ei}(x)$, $e^{-\text{Ei}(x)}$, $\text{Shi}(x)$ and $\text{Chi}(x)$ only and with $IC = 4$, all functions are computed.

The implementing program checks the input parameters. If $IC$ is outside the range 1–4, the working indicator IND is automatically set equal to 4. Since $\text{Ei}(x)$ is defined for positive $x$ only, if $IC = 2$ and $x < 0$, there is an error return and the indicator IERR is set equal to 1. If $x < 0$, $IC = 3$, $\text{Shi}(x)$ and $\text{Chi}(x)$ are computed; for $IC = 4$, $\text{Si}(x)$ and $\text{Ci}(x)$ are also computed, invalid results are returned for $\text{Ei}(x)$ and $e^{-\text{Ei}(x)}$ and the indicator IERR is set equal to 1. For $x > 0$, the indicator IERR is set equal to zero and valid results are returned only for the functions requested by the parameter $IC$ (or IND).

The computations are performed for positive $x$ ($=T$) only. Before the return, use is then made of the symmetry relations. Various cases are treated independently. Among these are $x = 0$ and $x$ equal to or greater than the supplied upper limit (ULSC) for the sine and cosine routine. The appropriate values of the functions are supplied. To avoid unnecessary computations of the exponential function and possible overflows and underflows in the final results, if $T$ is equal to or less than the upper limit for the relative error, the exponential of half the argument is set equal to 1. If $T$ is equal to or greater than the computed limiting argument (XMAXH) for $\text{Shi}(x)$ and $\text{Chi}(x)$, the maximum machine value (RINF) is supplied for the exponential of half the argument as well as for $\text{Shi}(x)$ and $\text{Chi}(x)$. The computed limiting argument (XMAXEI) for $\text{Ei}(x)$ can be shown to be approximately $\ln \text{RINF} + \ln[\ln \text{RINF} + \ln \ln \text{RINF}] - 1/\ln \text{RINF}$. The value of XMAXH is approximately XMAXEI + $\ln 2$. The computation of $e^{x/2}$ provides a slight improvement in accuracy and an extension of the range of $x$. Throughout the program, overflows are avoided as well as underflows affecting accuracy. Other underflows are assumed to be set equal to zero.

For $|x| \leq \text{PSLSC}$ ($=2$), all functions are computed by means of the power series. For $|x| > \text{PSLSC}$, $\text{Si}(x)$ and $\text{Ci}(x)$ are computed by means of the continued fraction. Only if $IC = 4$ is the working indicator IND set equal to 3. The functions $\text{Ei}(x)$, $e^{-\text{Ei}(x)}$, $\text{Shi}(x)$ and $\text{Chi}(x)$ are then computed by means of the power series or the asymptotic expansion depending on whether $|x| \leq \text{AELL}$ or $> \text{AELL}$ respectively. With NBM = 27, AELL = 18.7 and with NBM = 60, AELL = 41.6. To avoid underflow, $|x|$ is tested against a lower limit argument PSLL ($=2 \sqrt{\text{AMIN}}$). To simplify computation, AMIN, a minimum machine value is computed as the reciprocal of the maximum machine value (RINF). If $|x| \leq \text{PSLL}$, only the first term of the series of odd terms is used.

The following series definitions are in use
\begin{align*}
\text{Si}(x) &= \text{SI} = \text{SUMS} = \sum \text{SGN}(RK) \times \text{TM}(RK) \\
\text{Ci}(x) &= \text{CI} = \text{SUMC} + \text{XLOG} + \text{EULER} \\
\text{where SUMC} &= \sum \text{SGN}(RK) \times \text{TM}(RK) \\
\text{Ei}(x) &= \text{EI} = \text{SUMET} + \text{SUMOT} + \text{XLOG} + \text{EULER} \\
\text{Shi}(x) &= \text{SHI} = \text{SUMOT} + \sum \text{TM}(RK) \\
\text{Chi}(x) &= \text{CHI} = \text{SUMET} + \sum \text{TM}(RK) + \text{XLOG} + \text{EULER}
\end{align*}

with $\text{SGN}(1) = 1$, $\text{SGN}(RK + 1) = -\text{SGN}(RK)$ for $RK = 1, 3, \ldots$, and $\text{SGN}(RK + 1) = \text{SGN}(RK)$ for $RK = 2, 4, \ldots$. The term $\text{TM}(RK) = [T^k/k!] \times \text{PTM}(RK)/\text{RK}$, where $\text{PTM}(1) = T$ and $\text{PTM}(RK + 1) = \text{PTM}(RK)[T/(RK + 1)]$.

\[RK \geq 1.\]
The series of even and odd terms are always computed together. If the relative error RE computed as $\frac{TM}{|SUM|}$ is less than the prescribed tolerance both series are considered to have converged. If IND = 1 or 4, SUM is replaced by SUMS or SUMC; otherwise by SUMET or SUMOT. To avoid underflow, in generating the terms for $|x| \leq 2$, if $PTM \leq AMIN(RK)^{\frac{3}{2}}T$, the series are likewise considered to have converged. If the sum of terms is zero, the relative RE is automatically set equal to the maximum machine value. This condition is not encountered if the power series for $Si(x)$ and $Ci(x)$ is restricted to the region $|x| \leq 2$. It has been retained to permit the program’s use for experimental purposes.

To enable the continued fraction computations to be performed in double precision, since complex quantities are involved, the real notation only is used. Testing has also confirmed the improved accuracy and efficiency of this course. The continued fraction for $Si(x)$ and $Ci(x)$ in its “even” form

$$E_1(ix) = -Ci(x) + i[Si(x) - \pi/2] = e^{-ix} \left[ \frac{1}{1+i\bar{x}} - \frac{1}{3+i\bar{x}} - \frac{4}{5+i\bar{x}} - \ldots \right]$$

is evaluated in the forward direction. The first convergent $F_1/G_1 = A_1/B_1$ where $A_1 = 1$, $A_M = -(M - 1)^2$, $B_M = 2M - 1 + i\bar{x}$. If we define

$$F_{-1} = 1, F_0 = 0, G_{-1} = 0 \text{ and } G_0 = 1$$

then successive convergents $F_M/G_M$ for $M = 1, 2, \ldots$ may be obtained by the following recurrence relation

$$F_M = B_M F_{M-1} + A_M F_{M-2}$$
$$G_M = B_M G_{M-1} + A_M G_{M-2}$$

The continued fraction is considered to have converged either if in effect the relative error is equal to or less than the prescribed tolerance or the relative error increases.

Since the successive convergents are complex, $(RE)^2$ is compared with $(TOLER)^2$ where $(RE)^2 = \left[ \text{mod} (1 - \frac{F_{M-1}/G_{M-1}}{F_M/G_M}) \right]^2$. Throughout the computation, to avoid overflow, there is scaling by the absolute maximum (TMAX) of the real and imaginary parts of the numerator and denominator of the successive convergents. In addition, there is scaling by $|TMAX|$ if the product of the real part of $(B_M - A_M)$ and $|TMAX|$ is equal to or greater than 1/4 the maximum machine value.

The successive terms of the asymptotic expansion are likewise obtained by recurrence with $T_0 = 1$ and $T_K = [K/T]T_{K-1}$ for $K \geq 1$. Since the sum of terms for $Ei(x)$ is always greater than one, the term itself is a good approximation to the relative error. The summation is terminated when a term is less than the prescribed tolerance or the term is equal to or greater than the preceding term. In the latter case, the preceding term is subtracted from the summation to minimize the error.

4. Range

The range for $Si(x)$ and $Ci(x)$ (as well as the accuracy) is limited to the range (and accuracy) of the sine and cosine routine ($|x| < \text{ULSC}$). For the UNIVAC 1108, namely, $x < 2^{21}$ in single precision and $x < 2^{58}$ in double precision. For the function $Ei(x)$, the range of $x$ is essentially the range of the exponential routine. The function $Ei(x)$ is set equal to the machine maximum (RINF) for $x$ beyond XMAXEI, approximately 92.5 in single precision and 715.6 in double precision. For the function $e^{-x}Ei(x)$ beyond $x = \text{ULSC}$ only the first two terms of the asymptotic expansion are used. The functions $Shi(x)$ and $Chi(x)$ are set equal to the maximum machine value for $x$ beyond XMAXHF, approximately 93.2 in single precision and 716.3 in double precision.
5. Accuracy and Precision

Using the UNIVAC 1108 to compute the functions, the maximum relative error, except for regions in the immediate neighborhood of zeros, is 4.5 (-7) for single precision computations and 7.5 (-17) for double precision computations. Various auxiliary functions are available to greater accuracy at intermediate points in the subroutine. For example, since $Si(x) \to \pi/2$, $Si(x) - \pi/2$ should be taken as the imaginary part of the continued fraction. The functions $Ci(x)$, $Ei(x)$ and $Chi(x) - \gamma - \ln x$ are available from the sum of the appropriate series.

The precision may be set lower than the maximum by varying the value of NBM or deleting NBM and setting a precomputed value of TOLER. The above relative errors give an indication of the allowance for round-off errors.

6. Timing—UNIVAC 1108 Time/Sharing Executive System

(The time estimates given below are highly dependent on the operating system environment and consequently should not be relied on for critical timing measurements.)

<table>
<thead>
<tr>
<th>Single Precision</th>
<th>Double Precision</th>
</tr>
</thead>
<tbody>
<tr>
<td>$NBM = 27$</td>
<td>$NBM = 60$</td>
</tr>
<tr>
<td><strong>Region</strong></td>
<td><strong>Time (seconds)</strong></td>
</tr>
<tr>
<td>$0(0.01)2$</td>
<td>0.40</td>
</tr>
<tr>
<td>(201 values)</td>
<td>.56</td>
</tr>
<tr>
<td>$2.51(00)$</td>
<td>.0023</td>
</tr>
<tr>
<td>(197 values)</td>
<td>.0093</td>
</tr>
<tr>
<td>Maximum Time/Evaluation</td>
<td></td>
</tr>
<tr>
<td>$(x = 2)$</td>
<td></td>
</tr>
<tr>
<td>For $Si(x)$ and $Ci(x)$</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th><strong>For $Ei(x)$</strong></th>
<th><strong>Time (seconds)</strong></th>
<th><strong>Region</strong></th>
<th><strong>Time (seconds)</strong></th>
<th><strong>Method</strong></th>
</tr>
</thead>
<tbody>
<tr>
<td>$0(1.11)8$</td>
<td>0.54</td>
<td>$0(2.41)$</td>
<td>2.05</td>
<td>Power Series</td>
</tr>
<tr>
<td>(181 values)</td>
<td>.28</td>
<td>(206 values)</td>
<td></td>
<td>Continued Fraction</td>
</tr>
<tr>
<td>$18(1.41)$</td>
<td>.25</td>
<td>$41(25)100$</td>
<td>0.70</td>
<td>Asymptotic Expansion</td>
</tr>
<tr>
<td>(231 values)</td>
<td></td>
<td>(237 values)</td>
<td></td>
<td>Power Series</td>
</tr>
<tr>
<td>$41(25)100$</td>
<td>.0044</td>
<td>$(x = 41)$</td>
<td>.016</td>
<td>Asymptotic Expansion</td>
</tr>
<tr>
<td>(237 values)</td>
<td>.0015</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Maximum Time/Evaluation</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$(x = 18)$</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

7. Testing

The double precision results obtained were compared against available published values. Check values were obtained, where appropriate, by overlapping the power series with either the asymptotic expansion or the continued fraction. Various forms of the continued fraction were also employed as well as numerical integration. Two multi-precision packages were also utilized with varied precision. The single and double precision results agreed with the multi-precision results within the reported accuracy.

8. Many-Place Tables

In the appendix, we have included three tables; one for $Si(x)$ and $Ci(x)$, one for $Ei(x)$ and $e^{-x}Ei(x)$ and one for $Shi(x)$ and $Chi(x)$. The functions are tabulated to 35 significant figures for $x = 0, 10^j (10^j)^{10^{j+1}}$ with $J = -2(12)$.

---

1. Private Communication: Peavy, Bradley A., A Multi-Precision Arithmetic Package for Use with the UNIVAC 1108.

297
9. Special Values

Zeros

\[ Si(x_s) = \pi/2 \]
\[ x_0 = 1.92644 \quad 7660 \]
\[ x_1 = 4.89383 \quad 5953 \]
\[ x_2 = 7.97268 \quad 2624 \]
\[ Ci(x_s) = 0 \]
\[ x_0 = 0.61650 \quad 5486 \]
\[ x_1 = 3.38418 \quad 0423 \]
\[ x_2 = 6.42704 \quad 7744 \]
\[ Ei(x) = 0 \]
\[ x = 0.37250 \quad 74107 \quad 81366 \quad 63446 \quad 19918 \quad 66580 \quad 11913 \]
\[ Chi(x) = 0 \]
\[ x = 0.52382 \quad 25713 \quad 89864 \]

Maxima

\[ Si(\pi) = 1.85193 \quad 70519 \quad 82466 \]
\[ Ci(\pi/2) = 0.47200 \quad 06514 \quad 39569 \]

Minima

\[ Si(2\pi) = 1.41815 \quad 15761 \quad 32628 \]
\[ Ci(3\pi/2) = -0.19840 \quad 75606 \quad 92358 \]

Related Constants

\[ \sum_{N=0}^{\infty} \frac{(-1)^N}{(2N+1)(2N+1)!} = Si(1) = 0.94608 \quad 30703 \quad 67183 \quad 01494 \quad 13533 \quad 13823 \quad 17965 \]

\[ \sum_{N=1}^{\infty} \frac{(-1)^N}{(2N)(2N)!} = Ci(1) - \gamma = -0.23981 \quad 17420 \quad 00564 \quad 72594 \quad 38658 \quad 86193 \quad 25166 \]
\[ Ei(1) = 0.33740 \quad 39229 \quad 00968 \quad 13466 \quad 26462 \quad 03889 \quad 15076 \]

\[ \sum_{N=1}^{\infty} \frac{1}{(N)(N)!} = Ei(1) - \gamma = 1.31790 \quad 21514 \quad 54403 \quad 89486 \quad 00088 \quad 44249 \quad 23183 \]
\[ Chi(1) = 1.89511 \quad 78163 \quad 55936 \quad 75546 \quad 65209 \quad 34331 \quad 63426 \]

\[ \sum_{N=0}^{\infty} \frac{1}{(2N+1)(2N+1)!} = Shi(1) = 1.05725 \quad 08753 \quad 75728 \quad 51457 \quad 18423 \quad 54895 \quad 87795 \]

\[ \sum_{N=1}^{\infty} \frac{1}{(2N)(2N)!} = Chi(1) - \gamma = 0.26065 \quad 12760 \quad 78675 \quad 38028 \quad 81664 \quad 89353 \quad 35387 \]
\[ \gamma (Euler's\ constant) = 0.57721 \quad 56649 \quad 01532 \quad 86060 \quad 65120 \quad 90082 \quad 40243 \]
\[ \pi /2 = 1.57079 \quad 63267 \quad 94896 \quad 61923 \quad 13216 \quad 91639 \quad 75144 \]
\[ \pi = 3.14159 \quad 26535 \quad 89793 \quad 23846 \quad 26433 \quad 83279 \quad 50288 \]
\[ \log_e 2 = 0.69314 \quad 71805 \quad 59945 \quad 30941 \quad 72321 \quad 21458 \quad 17656 \]

298
Typical Tolerances and Their Natural Logarithms

\[
2^{-24} = 0.59604 \text{ 64477 53906 25} (-7)
\]
\[
2^{-27} = 0.74505 \text{ 80596 92382 8125} (-8)
\]
\[
2^{-36} = 0.14551 \text{ 91522 83668 51806 64062 5} (-10)
\]
\[
2^{-48} = 35527 \text{ 13678 80050 09293 55621 33789 0625} (-14)
\]
\[
2^{-56} = 1.3877 \text{ 78780 78144 56755 29539 58511 35253 90625} (-16)
\]
\[
2^{-60} = 0.86736 \text{ 17379 88403 54720 59622 40695 95336 91406 25} (-18)
\]
\[
2^{-108} = 0.30814 \text{ 87911 01957 73648 89564 70813 58837 09660 96263} 71446 21112 38390 20729 06494 14062 5 (-32)
\]

\[ \log_e(2^{-24}) = -16.63553 \text{ 23334 38687 42601 35709 14996 23763} \]
\[ \log_e(2^{-27}) = -18.71497 \text{ 38751 18523 35426 52672 79370 76733} \]
\[ \log_e(2^{-36}) = -24.95329 \text{ 85001 58031 13902 03563 72494 35645} \]
\[ \log_e(2^{-48}) = -33.27106 \text{ 46668 77374 85202 71418 29992 47526} \]
\[ \log_e(2^{-56}) = -38.81624 \text{ 21113 56937 32736 49988 01657 88781} \]
\[ \log_e(2^{-60}) = -41.58883 \text{ 08335 96718 56503 39272 87490 59408} \]
\[ \log_e(2^{-108}) = -74.85989 \text{ 55004 74093 41706 10691 17483 06935} \]

Maximum and Minimum Machine Values and Their Natural Logarithms

\[ NBC = \text{Number of binary digits in the (biased) characteristic of a floating point number} \]

\[ 2^{-NBC-1} \leq x < 2^{NBC-1} \]

\[ NBC = 8 \]
\[ 2^{127} = 0.17014 \text{ 11834 60469 23173 16873 03715 88410} (39) \]
\[ 2^{-129} = 0.14693 \text{ 76938 52785 93849 69020 67152 78070} (-38) \]
\[ \log_e(2^{127}) = 88.02969 \text{ 19311 13054 29598 84794 25188 42414} \]
\[ \log_e(2^{-129}) = -89.41598 \text{ 62922 32944 91482 29436 68104 77728} \]

\[ NBC = 11 \]
\[ 2^{1023} = 0.89884 \text{ 65674 31157 95386 46525 95394 51236} (308) \]
\[ 2^{-1025} = 0.27813 \text{ 42323 13400 17288 62790 89666 55050} (-308) \]
\[ \log_e(2^{1023}) = 709.08556 \text{ 57128 24051 53382 84602 51714 62914} \]
\[ \log_e(2^{-1025}) = -710.47586 \text{ 00739 43942 15266 29244 94630 98227} \]

10. References

[5] Harris, F. E., Tables of the exponential integral \( E_1(x) \), MTAC 11, 9–16 (1957).
APPENDIX

IMPLEMENTING PROGRAM

LANGUAGE. AMERICAN NATIONAL STANDARD FORTRAN

DEFINITIONS. X, a real variable

\[ SI(X) = \int_0^X \sin(t)/t \, dt \]

\[ CI(-X) = -SI(X) \]

\[ CI(X) = \gamma + \ln X + \int_0^X \cos(t-1)/t \, dt \]

\[ EI(X) = -P.V. \int_{-X}^{\infty} \exp(-t)/t \, dt \]

\[ EXNEI(X) = \exp(-X) \cdot EI(X) \quad (X \geq 0) \]

\[ SI(0) = SHI(0) = 0 \]

\[ CI(0) = EI(0) = EXNEI(0) = CHI(0) = -\infty \]

\[ \gamma = \text{Euler's constant} = 0.5772156 \ldots \]

\[ X \leq 0 \]

\[ SI(X) = PI/2 \]

\[ CI(X) = 0 \]

\[ EI(X) = SHI(X) = CHI(X) = \infty \]

\[ EXNEI(X) = 0 \]

USAGE. CALL SICIEI (IC, X, SI, CI, CI, EI, EXNEI, SHI, CHI, CHII, IERR)

FORMAL PARAMETERS

\[ IC \quad \text{INTEGER TYPE} \]

\[ IC \quad \text{FUNCTIONS TO BE COMPUTED} \]

\[ 1 \quad SI, CI \]

\[ 2 \quad EI, EXNEI \]

\[ 3 \quad EI, EXNEI, SHI, CHI \]

\[ 4 \quad SI, CI, EI, EXNEI, SHI, CHI \]

\[ X \quad \text{REAL OR DOUBLE PRECISION TYPE} \]

\[ SI = SI(X) \quad (\text{SAME TYPE AS } X) \]

\[ CI + I CI = CI(X) \]

\[ EI = EI(X) \]

\[ EXNEI = \exp(-X) \cdot EI(X) \]

\[ SHI = SHI(X) \]

\[ CHI + I CHII = CHI(X) \]

\[ IERR \quad \text{INTEGER TYPE} \]

\[ IERR = 0 \quad X \geq 0 \quad \text{NORMAL RETURN} \]

\[ IERR = 1 \quad X < 0 \quad \text{ERROR RETURN IF IC=2} \]

MODIFICATIONS.

THE CODE IS SET UP FOR DOUBLE PRECISION COMPUTATION

WITH DOUBLE PRECISION TYPE STATEMENTS

DOUBLE PRECISION FUNCTION REFERENCES AND, PARTICULARLY, FOR THE UNIVAC 1108 WITH (SEE DEFINITIONS BELOW)

RINF APPROX. 2**1023, ULSC=2**56, NBM=60 AND OTHER

CONSTANTS IN DOUBLE PRECISION FORMAT TO 19 SIGNIFICANT
FIGURES. ALL ABOVE ITEMS MUST BE CHANGED FOR SINGLE
PRECISION COMPUTATIONS WITH DATA ADJUSTMENTS FOR OTHER
COMPUTERS.

AUXILIARY FUNCTIONS

VARIOUS FUNCTIONS ARE AVAILABLE TO GREATER ACCURACY
AT INTERMEDIATE POINTS IN THE SUBROUTINE, NAMLY,
SI-(PI/2)=IMAG. PART OF THE CONTINUED FRACTION
CI(EI AND CHI)=GAMMA-LN X=SUM OF SERIES
CAUTION - THE SUBROUTINE CANNOT READILY BE ADAPTED TO
COMPUTE THE FUNCTIONS FOR COMPLEX ARGUMENTS.

METHOD. T=ABS(X)

POWER SERIES .T .LE. PSLSC(=2) FOR SI,CI

T .LE. AELL(=-LN(TOLER)). FOR EI,SHI,CHI

SI=SUM(SGN(RK)*TM(RK)) IP=-1 RK=1,3,...,RKO

CI=SUMC(SGN(RK)*TM(RK)) IP=+1 RK=2,4,...,RKE

+EULER+XLOG

SHI=SUMOT(TM(RK)) IP=-1 RK=1,3,...,RKO

CHI=SUMET(TM(RK)) IP=+1 RK=2,4,...,RKE

+EULER+XLOG

EI=SUMOT+SUMET+EULER+XLOG (X .GT. 0)

SI=SUM(SGN(RK)*TM(RK)) RK=1,3,...

SGN(RK+1)=-SGN(RK) RK=2,4,...

TK(RK)=((T**RK)/(1*2...RK)) RK

=PTM(RK)/RK

PTM(1)=T

PTM(RK+1)=PTM(RK)*(T/(RK+1)) RK .GE. 1

IF TM(RK)/SUM .LT. TOLER

RKE=RK WHERE SUM=ABS(SUMC)

IC=1 OR 4

SUM=SUMET

IC=2 OR 3

SUM=SUMOT

IC=4 OR 3

RKO=RK WHERE SUM=ABS(SUMS)

IC=4 OR 3

EXEII= EI/EXP(T/2)/EXP(T/2)

=EI/EXPH*EXPHT

CONTINUED FRACTION .T .GT. PSLSC

-CT+I(SI-PI/2)=EI(IT)

=EXP(-IT)*I(I/I (1+IT)-

1**2 I/I (3+IT)-

2**2 I/I (5+IT)-...)

=EXP(-IT)**II(AM(RM) I/T RM(RM))

RM=1,2,...,RMF

AM(1)=1

AM(RM)=-(RM-1)**2 RM .GT. 1

RM(RM)=2*RM-1+IT=RM+I BMI

=EXP(-IT)*(FM/GM)

=EXP(-IT)*(FM+I FMI)/(GM+I GMI)

FMI)/(GMI)

=EXP(-IT)*F(RM)

R=I(SINT)*F(RM+I FT)

-CT+I(SI-PI/2)=(FR*COST+FI*SINT)+

IF RESQ(RM) .LE. TOLSQ(=TOLER**2)

OR RESQ(RM) .GE. RESQ(RM-1)

RMF=RM WHERE

RESQ=MOD(1-F(RM-1)/F(RM))**2

ASYMPTOTIC EXPANSION .T .GT. AELL
EI = (EXNEI * EXPHT) * EXPHT
EXNEI = (1 + SUME(TM(RK))) / T
SHI = CHI = EI / 2
TM(RK) = (1 * 2 ... RK) / (T ** RK)
TM(0) = 1
TM(RK) = (RK / T) * TM(RK - 1)
RK = 1, 2, ..., RKF
IF TM(RK) .LT. TOLER (CONVERGENCE) RKF = RK OR
TM(RK) .GE. TM(RK - 1) (DIVERGENCE) RKF = RK - 1
C RANGE.
C FOR SI(X), CI(X), ABS(X) .LT. ULSC (UPPER LIMIT FOR
C SIN, COS ROUTINE)
X = APPROXIMATELY 2 ** 21, NBM = 27
2 ** 56, NBM = 60
C FOR EXP(-X) * EI(X), X .LE. RINF
C FOR EI(X), X .LT. XMAXEI (APPROXIMATELY 92.5, NBC = 8,
715.6, NBC = 11)
C NBC = NUMBER OF BINARY DIGITS IN THE BIASED
C CHARACTERISTIC OF A FLOATING POINT NUMBER
C FOR SHI(X), CHI(X), ABS(X) .LT. XMAXHF
C X = APPROXIMATELY 93.2, NBC = 8
716.3, NBC = 11
C ACCURACY. THE MAXIMUM RELATIVE ERROR, EXCEPT FOR REGIONS
C IN THE IMMEDIATE NEIGHBORHOOD OF ZERO, ON THE
C UNIVAC 1108 IS 4.5(-7) FOR SINGLE PRECISION COM-
C PUTATION AND 7.5(-17) FOR DOUBLE PRECISION COM-
C PUTATION.
C PRECISION. VARIABLE - BY SETTING THE DESIRED VALUE OF NBM
C OR A PREDTERMINED VALUE OF TOLER
C MAXIMUM UNIVAC 1108 TIME/SHARING EXECUTIVE SYSTEM
C TIMING. NBM = 27 NBM = 60
C (SECONDS) .0093 .070
C STORAGE. 954 WORDS REQUIRED BY THE UNIVAC 1108 COMPILER
C
C SUBROUTINE SICIEICIEICIC, X, SI, CI, CI, EI, EXNEI, SHI, CHI,
1 CHII, IERR)
C MACHINE DEPENDENT STATEMENTS
C TYPE STATEMENTS
DOUBLE PRECISION X, SI, CI, CI, EI, EXNEI, SHI, CHI, CHII
DOUBLE PRECISION AM, AM, AM, ASUMSC,
1 BM, BM, COST, EXPL, EXPHT,
2 FI, FIP, FMI, FM, FM, FM, FM, FR, FR,
2 GM, GM, GM, GM, GM, GM, GM, GM,
4 PSL, PSL, PSM, PSM, PSM, PSM, PSM, PSM,
5 SCC, SCC, SCC, SCC, SCC, SCC, SCC, SCC,
6 SIN, SUM, SUM, SUM, SUM, SUM, SUM, SUM,
6 TEMP, TEMP, TEMP, TEMP, TEMP, TEMP, TEMP, TEMP,
10 T, T, T, T, T, T, T, T,
12 XLOG, XLOG, XLOG, XLOG, XLOG, XLOG, XLOG, XLOG,
14 RINF, RINF, RINF, RINF, RINF, RINF, RINF, RINF,
16 PI, PI, PI, PI, PI, PI, PI, PI,
18 ALOG2, ALOG2, ALOG2, ALOG2, ALOG2, ALOG2, ALOG2, ALOG2,
20 ZERO, ONE, TWO, FOUR
DIMENSION A(4)
EQUIVALENCE (FM, A(1)), (FMI, A(2)), (GM, A(3)),
1 (GMI, A(4))
C CONSTANTS
DATA EULER/ .57721566490153286060/,
DATA HALFPi/1.57079662739549109380/,
DATA PI/3.14159265358979323846/,
DATA ALOG2/.6931471805599453094/,
302
DATA ZERO,ONE,TWO,FOUR /
1 0.0D0,1.0D0,2.0D0,4.0D0 /
C RINF=MAXIMUM MACHINE VALUE
C ULSC=MAXIMUM ARGUMENT FOR SIN,COS ROUTINE
C APPROX. 2**(NBM-6) OR 10**(S-2)
C (S=SIGNIFICANT FIGURES)
C NBM=ACCURACY DESIRED OR THE
C MAXIMUM NUMBER OF BINARY DIGITS IN THE
C MANTISSA OF A FLOATING POINT NUMBER
C TOLER=UPPER LIMIT FOR RELATIVE ERRORS
C TOLER=2**(-NBM)=APPROX. 10**(-S)
C THE NBM DATA STATEMENT ELIMINATED
C DATA RINF/.898846567431157953D308 /
C DATA ULSC/.72057594037927936D171 /
C DATA NBM / 60 /
C TOLER=TWO**(-NBM)
C NOTE - ARGUMENT CHECKS PRECEDING FUNCTION REFERENCES
C NECESSITATE ADDITIONAL MACHINE DEPENDENT STATEMENTS
C IN THE STATEMENT NUMBER RANGE 140-150
C INITIALIZATION OF OUTPUT FUNCTIONS
C SI=RINF
C CI=RINF
C CII=RINF
C EI=ZERO
C EXNEI=RINF
C SHI=ZERO
C CHI=ZERO
C CHII=RINF
C VALIDITY CHECK ON INPUT PARAMETERS
C INDICATOR CHECK
C SET IND=IC
C CHANGE IND=IC IF IC .LT. 1 OR .GT. 4
C IND=IC
C IF IND .LT. 1 GO TO 10
C IF (IND .GT. 4) GO TO 10
C GO TO 20
C 10 IND=4
C ARGUMENT CHECK
C X .GE. 0 IERR=0
C X .LT. 0 IERR=1
C (ERROR RETURN IF IC=2)
C 20 IERR=0
C T=X
C 30 IF (T) 40,50,90
C 40 T=-T
C 21 IERR=1
C 22 IF (IND .NE. 2) GO TO 30
C 23 IF (X .LT. ZERO) RETURN
C SPECIAL CASES
C X=0
C 50 IF (IND-2) 80,70,60
C 60 SHI=ZERO
C 70 EI=-RINF
C 80 X=SHI
C 90 X=EI
C 100 T=EI
C 110 X=T
C 50 RETURN
C 70 RETURN
C 90 RETURN
C 100 RETURN
IF (IND .NE. 4) RETURN
SI = ZERO
CI = -RH
CII = ZERO
RETURN
IF (T .LT. ULSC) GO TO 140
ARS(X)
IF (IND . LT. 2) 130, 110, 100
SHI = RINF
CHI = RINF
CII = ZERO
IF (IERR .EQ. 1) GO TO 120
EI = RINF
EXNEI = (ONE + (ONE/T))/T
IF (IND . NE. 4) GO TO 1000
SI = HALFPI
CI = ZERO
CII = ZERO
GO TO 1000
EVALUATIONS FOR ABS(X)(=T) .GT. 0 AND .LT. ULSC
ADDITIONAL MACHINE DEPENDENT STATEMENTS
FUNCTION REFERENCES
CONTROL VARIABLES
XLOG = DLOG(T)
SINT = DSIN(T)
COST = DCOS(T)
EXPL = DLOG(RINF)
XMAXEI = EXPL + DLOG(EXPL + DLOG(EXPL)) - ONE/EXPL
XMAXHF = XMAXEI + ALOG2
AELL = DLOG(TOLER)
AMIN = ONE/RINF
PSLL = TWO * DSORT(AMIN)
PSLSC = TWO
EXPONENTIAL FUNCTION DETERMINATION
IF (T .LE. TOLER) GO TO 150
IF (T .GE. XMAXHF) GO TO 160
EXPHF = EXP(T/TWO)
GO TO 170
EXPHT = ONE
GO TO 170
EXPHT = RINF
METHOD SELECTION
IF (T .LE. PSLSC) GO TO 200
IF (IND . EQ. 1) GO TO 500
IF (IND . EQ. 4) GO TO 500
IF (T .GT. AELL) GO TO 800
GO TO 230
INDICATOR TO COMPUTE EI, SHI, CHI
IF (IND . EQ. 1) GO TO 1000
IND = 3
GO TO 190
METHOD --- POWER SERIES
SI(X) + CI(X), T .LE. PSLSC
EI(X) + SHI(X), CHI(X), T .LE. AELL
LIMITING VALUES, T NEAR ZERO
IF (T .GT. PSLL) GO TO 210
SUMC = ZERO
SUMET = ZERO
GO TO 210
SUMS=T
SUMOT=T
GO TO 360

C INITIALZATION FOR SI,CI
IF (IND .NE. 1) GO TO 230
SUMS=ZERO
SUMOT=ZERO
SUMSC=ZERO
SGN=ONE
GO TO 240

C INITIALIZATION FOR SHI,CHI(AND EI)
IF (IND .EQ. 4) GO TO 220
IP = INDICATOR FOR ODD OR EVEN TERMS
IP=-1
RK=ONE
PTM=ONE

PTM=PTM/RK

IF (IND .NE. 1) GO TO 310
SUMSC=SGN*TM+SUMSC

RELATIVE ERROR FOR SI(CI)
IF (ASUMSC) 280,300,290
ASUMSC=-ASUMSC
GO TO 270
RE=TM/ASUMSC.
GO TO 320
RE=RINFE
GO TO 320

SUMEO=TM+SUMEO
IF (IND .EQ. 4) GO TO 260

RELATIVE ERROR FOR SHI(Chi)
RF=TM/SUMEO
SIGN CHANGE AND SELECTION
OF SUMS OF ODD(EVEN) TERMS

IF (IP .EQ. 1) GO TO 330
SGNE=-SGN
SUMSC=SUMSC
SUMOT=SUMEO
SUMEO=SUMET
GO TO 340

SUMC=SUMSC
SUMSC=SUMS
SUMET=SUMEO
SUMEO=SUMOT

RELATIVE ERROR CHECK
IF (RF .LT. TOLER) GO TO 360
ADDITIONAL TERMS
RK=RK+ONE
UNDERFLOW TEST
C UNDERFLOWS AFFECTING ACCURACY ARE AVOIDED. ALL OTHER
C UNDERFLOWS ARE ASSUMED TO BE SET EQUAL TO ZERO.
351* IF (T .GT. PSLSC) GO TO 350
352* IF (PTM .LE. (AMIN*RK*AMIN)/(T)) GO TO 360
353* 350 PTM=(T/RK)*PTM
354* IP=-IP
355* GO TO 250
356* C SI,CI EVALUATION
357* 360 IF (IND .NE. 1) GO TO 380
358* 370 SI=SUMS
359* 380 CI=(SUMC+XLOG)+EULER
360* 390 CII=ZERO
361* 400 GO TO 1000
362* C EI EVALUATION
363* 410 IF (X .LE. ZERO) GO TO 390
364* 420 EI=(SUMET+SUMOT+XLOG)+EULER
365* 430 EXNEI=(EI/EXPHT)/EXPHT
366* 440 IF (IND .EQ. 2) RETURN
367* C SHI,CHI EVALUATION
368* 450 SHI=SUMOT
369* 460 CHI=(EULER+SUMET)+XLOG
370* 470 CHII=ZERO
371* 480 IF (IND .NE. 4) GO TO 1000
372* 490 GO TO 370
373* C METHOD --- CONTINUED FRACTION
374* C SI(X),CI(X), T .GT. PSLSC
375* C -CI(T) + I (SI(T)-HALFPI)=El(IT)
376* C INITIALZATION
377* 500 SCC=RINF/FOUR
378* TOLSQ=TOLER*TOLER
379* RM=ONE
380* AM=ONE
381* BMI=T
382* 300 FMM2R=ONE
383* 310 FMM2I=ZERO
384* 320 GM2R=ZERO
385* 330 GM2I=ZERO
386* 340 FMM1R=ZERO
387* 350 GM1R=ONE
388* 360 GM1I=ZERO
389* 370 RESQP=RINF
390* 380 FRP=ZERO
391* 390 FIP=ZERO
392* C RECURANCE RELATION
393* C RECURANCE RELATION
394* 400 FM=BM*FMM1 + AM*FMM2
395* 410 GM=BM*GMM1 + AM*GMM2
396* 420 FMR=RMR*FMM1R-BMI*FMM1I+AM*FMM2R
397* 430 510 FMM1R=BM1*FMM1R+RMR*FMM1I+AM*FMM2R
398* 440 GM1R=BM1*GM1R-BMI*GM1I+AM*GM2R
399* 450 GM1R+BM1*GM1I+AM*GM2R
400* 460 GMM1R=GM1I+BM1*GM1R+AM*GM2R
401* C TESTS TO AVOID INCORRECT RESULTS
402* C DUE TO OVERFLOWS(UNDERFLOWS)
403* C FINDING MAXIMUM(=TMAX) OF
404* C ABSOLUTE OF FMR,GM,R,FMI,GMI
405* C FOR SCALING PURPOSES
TMAX = ZERO
I = 1

IF (TEMP) 540, 560, 550
TEMP = -TEMP
GO TO 530

IF (TEMP .LE. TMAX) GO TO 560
TMAX = TEMP

IF (I .GE. 4) GO TO 570
I = I + 1
GO TO 520

SFMR = FMR/TMAX
SFMI = FMI/TMAX

SGMR = GMR/TMAX
SGMI = GMI/TMAX

TEMP = SGMR*SGMR + SGMI*SGMI

FR = (SFMR*SGMR + SFMI*SGMI)/TEMP
FI = (SFMI*SGMR - SFMR*SGMI)/TEMP

C RELATIVE ERROR CHECK

TEMPA = (FR*FR + FI*FI)/TEMP
TEMPR = (FIP*FR - FRP*FI)/TEMP

RESQ = TEMP*TEMP + TErv1P*TEMPR

IF (RESQ .LE. TOLSQ) GO TO 590
IF (RESQ .GE. RESQP) GO TO 580

C ADDITIONAL CONVERGENTS

AM = RM*RM
RM = RM + ONE
BMR = BMR + TWO

FMM2R = FMM1R

GMM2R = GMM1R

FMM1R = FMR

GM1I = GMI

FIP = FI

RESQP = RESQ

C SCALING
SCALING SHOULD NOT BE DELETED AS THE VALUES OF FMR, FMI AND
GMR, GMI MAY OVERFLOW FOR SMALL VALUES OF T

IF (TMAX .LT. SCC/(BMR - AM)) GO TO 510

FMM2R = FMM2R/TMAX
FMM2I = FMM2I/TMAX
GMM2R = GMM2R/TMAX
GMM2I = GMM2I/TMAX
FMM1R = FMM1R/TMAX
FMM1I = FMM1I/TMAX
GMM1R = GMM1R/TMAX
GMM1I = GMM1I/TMAX
GO TO 510

C DIVERGENCE OF RELATIVE ERROR
ACCEPT PRIOR CONVERGENT

FR = FRP
FI = FIP
C SI,CI EVALUATION
590 SI=FI*COST-FR*SINT+HALFPI
CII=-(FR*COST+FI*SINT)
CII=ZERO
GO TO 190
C METHOD --- ASYMPTOTIC EXPANSION
471* EI(X),EXNEI(X) X.GT. AELL
472* SHI(T)=CHI(T)=EI(T)/2 T.GT. AELL
473* INITIALIZATION
474* 800 IF (IND .NE. 2) GO TO 880
475* 810 SUME=ZERO
RK=ZERO
476* TM=ONE
477* 820 TMM1=TM
480* RK=RK+ONE
481* TM=(RK/T)*TM
482* C ADDITIONAL TERMS
483* IF (TM .LT. TOLER) GO TO 840
484* IF (TM .GE. TMM1) GO TO 830
485* SUME=SUME+TM
486* GO TO 820
487* C 830 SUME=SUME-TMM1
488* EXNEI EVALUATION
490* IF(X .LT. ZERO) GO TO 870
491* EXNEI=(ONE+SUME)/T
492* C EI EVALUATION - X .LT. XMAXEI
493* IF (T .GE. XMAXEI) GO TO 850
494* EI=(EXNEI*EXPHT)*EXPHT
495* GO TO 860
496* C 850 EI=RINF
498* C SHI,CHI EVALUATION - T .LT. XMAXHF
499* IF (IND .EQ. 2) RETURN
500* 860 IF (T .GE. XMAXHF) GO TO 1000
501* SHI=(((ONE+SUME)/T)/TWO)*EXPHT
502* CHI=SHI
503* CHII=ZERO
504* GO TO 1000
505* C SHI,CHI - LIMITING VALUE
506* C 880 IF (T .LT. XMAXHF) GO TO 810
507* SHI=-SHI
508* CHI=RINF
509* CHII=-PI
510* IF (X .LT. ZERO) GO TO 810
511* GO TO 1010
513* C ADJUSTMENTS FOR X .LT. 0
514* 1000 IF (X .GT. ZERO) RETURN
515* 1010 IF (IC .EQ. 3) GO TO 1020
516* SI=-SI
517* CII=-PI
518* IF (IC .EQ. 1) RETURN
519* 1020 SHI=-SHI
520* CHII=-PI
521* RETURN
522* END
<table>
<thead>
<tr>
<th>X</th>
<th>X</th>
<th>S(X)</th>
<th>CR(X)</th>
</tr>
</thead>
<tbody>
<tr>
<td>.1</td>
<td>.0</td>
<td>.9999944444</td>
<td>.009725802921</td>
</tr>
<tr>
<td>.2</td>
<td>.0</td>
<td>.9999955556</td>
<td>.0194515521</td>
</tr>
<tr>
<td>.3</td>
<td>.0</td>
<td>.9999966666</td>
<td>.0291772013</td>
</tr>
<tr>
<td>.4</td>
<td>.0</td>
<td>.9999977778</td>
<td>.0388028507</td>
</tr>
<tr>
<td>.5</td>
<td>.0</td>
<td>.9999988888</td>
<td>.0484285001</td>
</tr>
<tr>
<td>.6</td>
<td>.0</td>
<td>.9999999999</td>
<td>.0580541595</td>
</tr>
<tr>
<td>.7</td>
<td>.0</td>
<td>.9999999999</td>
<td>.0676898189</td>
</tr>
<tr>
<td>.8</td>
<td>.0</td>
<td>.9999999999</td>
<td>.0773154783</td>
</tr>
<tr>
<td>.9</td>
<td>.0</td>
<td>.9999999999</td>
<td>.0869411377</td>
</tr>
<tr>
<td>1.0</td>
<td>.0</td>
<td>.9999999999</td>
<td>.0965667971</td>
</tr>
<tr>
<td>1.1</td>
<td>.0</td>
<td>.9999999999</td>
<td>.1061924565</td>
</tr>
<tr>
<td>1.2</td>
<td>.0</td>
<td>.9999999999</td>
<td>.1158181159</td>
</tr>
<tr>
<td>1.3</td>
<td>.0</td>
<td>.9999999999</td>
<td>.1254437753</td>
</tr>
<tr>
<td>1.4</td>
<td>.0</td>
<td>.9999999999</td>
<td>.1350694348</td>
</tr>
<tr>
<td>1.5</td>
<td>.0</td>
<td>.9999999999</td>
<td>.1446950942</td>
</tr>
<tr>
<td>1.6</td>
<td>.0</td>
<td>.9999999999</td>
<td>.1543207536</td>
</tr>
<tr>
<td>1.7</td>
<td>.0</td>
<td>.9999999999</td>
<td>.1639464130</td>
</tr>
<tr>
<td>1.8</td>
<td>.0</td>
<td>.9999999999</td>
<td>.1735720724</td>
</tr>
<tr>
<td>1.9</td>
<td>.0</td>
<td>.9999999999</td>
<td>.1831977318</td>
</tr>
<tr>
<td>.0</td>
<td>.0</td>
<td>.0000000001</td>
<td>.0000000001</td>
</tr>
</tbody>
</table>

**Table 1**
<table>
<thead>
<tr>
<th>X</th>
<th>SH(X)</th>
<th>CHI(X)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.0</td>
<td>0.0</td>
<td></td>
</tr>
<tr>
<td>0.1</td>
<td>0.0</td>
<td></td>
</tr>
<tr>
<td>0.2</td>
<td>0.0</td>
<td></td>
</tr>
<tr>
<td>0.3</td>
<td>0.0</td>
<td></td>
</tr>
<tr>
<td>0.4</td>
<td>0.0</td>
<td></td>
</tr>
<tr>
<td>0.5</td>
<td>0.0</td>
<td></td>
</tr>
<tr>
<td>0.6</td>
<td>0.0</td>
<td></td>
</tr>
<tr>
<td>0.7</td>
<td>0.0</td>
<td></td>
</tr>
<tr>
<td>0.8</td>
<td>0.0</td>
<td></td>
</tr>
<tr>
<td>0.9</td>
<td>0.0</td>
<td></td>
</tr>
<tr>
<td>1.0</td>
<td>0.0</td>
<td></td>
</tr>
<tr>
<td>1.1</td>
<td>0.0</td>
<td></td>
</tr>
<tr>
<td>1.2</td>
<td>0.0</td>
<td></td>
</tr>
<tr>
<td>1.3</td>
<td>0.0</td>
<td></td>
</tr>
<tr>
<td>1.4</td>
<td>0.0</td>
<td></td>
</tr>
<tr>
<td>1.5</td>
<td>0.0</td>
<td></td>
</tr>
<tr>
<td>1.6</td>
<td>0.0</td>
<td></td>
</tr>
<tr>
<td>1.7</td>
<td>0.0</td>
<td></td>
</tr>
<tr>
<td>1.8</td>
<td>0.0</td>
<td></td>
</tr>
<tr>
<td>1.9</td>
<td>0.0</td>
<td></td>
</tr>
<tr>
<td>2.0</td>
<td>0.0</td>
<td></td>
</tr>
<tr>
<td>2.1</td>
<td>0.0</td>
<td></td>
</tr>
<tr>
<td>2.2</td>
<td>0.0</td>
<td></td>
</tr>
<tr>
<td>2.3</td>
<td>0.0</td>
<td></td>
</tr>
<tr>
<td>2.4</td>
<td>0.0</td>
<td></td>
</tr>
<tr>
<td>2.5</td>
<td>0.0</td>
<td></td>
</tr>
<tr>
<td>2.6</td>
<td>0.0</td>
<td></td>
</tr>
<tr>
<td>2.7</td>
<td>0.0</td>
<td></td>
</tr>
<tr>
<td>2.8</td>
<td>0.0</td>
<td></td>
</tr>
<tr>
<td>2.9</td>
<td>0.0</td>
<td></td>
</tr>
<tr>
<td>3.0</td>
<td>0.0</td>
<td></td>
</tr>
<tr>
<td>3.1</td>
<td>0.0</td>
<td></td>
</tr>
<tr>
<td>3.2</td>
<td>0.0</td>
<td></td>
</tr>
<tr>
<td>3.3</td>
<td>0.0</td>
<td></td>
</tr>
<tr>
<td>3.4</td>
<td>0.0</td>
<td></td>
</tr>
<tr>
<td>3.5</td>
<td>0.0</td>
<td></td>
</tr>
<tr>
<td>3.6</td>
<td>0.0</td>
<td></td>
</tr>
<tr>
<td>3.7</td>
<td>0.0</td>
<td></td>
</tr>
<tr>
<td>3.8</td>
<td>0.0</td>
<td></td>
</tr>
<tr>
<td>3.9</td>
<td>0.0</td>
<td></td>
</tr>
<tr>
<td>4.0</td>
<td>0.0</td>
<td></td>
</tr>
<tr>
<td>4.1</td>
<td>0.0</td>
<td></td>
</tr>
<tr>
<td>4.2</td>
<td>0.0</td>
<td></td>
</tr>
<tr>
<td>4.3</td>
<td>0.0</td>
<td></td>
</tr>
<tr>
<td>4.4</td>
<td>0.0</td>
<td></td>
</tr>
<tr>
<td>4.5</td>
<td>0.0</td>
<td></td>
</tr>
<tr>
<td>4.6</td>
<td>0.0</td>
<td></td>
</tr>
<tr>
<td>4.7</td>
<td>0.0</td>
<td></td>
</tr>
<tr>
<td>4.8</td>
<td>0.0</td>
<td></td>
</tr>
<tr>
<td>4.9</td>
<td>0.0</td>
<td></td>
</tr>
<tr>
<td>5.0</td>
<td>0.0</td>
<td></td>
</tr>
<tr>
<td>5.1</td>
<td>0.0</td>
<td></td>
</tr>
<tr>
<td>5.2</td>
<td>0.0</td>
<td></td>
</tr>
<tr>
<td>5.3</td>
<td>0.0</td>
<td></td>
</tr>
<tr>
<td>5.4</td>
<td>0.0</td>
<td></td>
</tr>
<tr>
<td>5.5</td>
<td>0.0</td>
<td></td>
</tr>
<tr>
<td>5.6</td>
<td>0.0</td>
<td></td>
</tr>
<tr>
<td>5.7</td>
<td>0.0</td>
<td></td>
</tr>
<tr>
<td>5.8</td>
<td>0.0</td>
<td></td>
</tr>
<tr>
<td>5.9</td>
<td>0.0</td>
<td></td>
</tr>
<tr>
<td>6.0</td>
<td>0.0</td>
<td></td>
</tr>
<tr>
<td>6.1</td>
<td>0.0</td>
<td></td>
</tr>
<tr>
<td>6.2</td>
<td>0.0</td>
<td></td>
</tr>
<tr>
<td>6.3</td>
<td>0.0</td>
<td></td>
</tr>
<tr>
<td>6.4</td>
<td>0.0</td>
<td></td>
</tr>
<tr>
<td>6.5</td>
<td>0.0</td>
<td></td>
</tr>
<tr>
<td>6.6</td>
<td>0.0</td>
<td></td>
</tr>
<tr>
<td>6.7</td>
<td>0.0</td>
<td></td>
</tr>
<tr>
<td>6.8</td>
<td>0.0</td>
<td></td>
</tr>
<tr>
<td>6.9</td>
<td>0.0</td>
<td></td>
</tr>
<tr>
<td>7.0</td>
<td>0.0</td>
<td></td>
</tr>
<tr>
<td>7.1</td>
<td>0.0</td>
<td></td>
</tr>
<tr>
<td>7.2</td>
<td>0.0</td>
<td></td>
</tr>
<tr>
<td>7.3</td>
<td>0.0</td>
<td></td>
</tr>
<tr>
<td>7.4</td>
<td>0.0</td>
<td></td>
</tr>
<tr>
<td>7.5</td>
<td>0.0</td>
<td></td>
</tr>
<tr>
<td>7.6</td>
<td>0.0</td>
<td></td>
</tr>
<tr>
<td>7.7</td>
<td>0.0</td>
<td></td>
</tr>
<tr>
<td>7.8</td>
<td>0.0</td>
<td></td>
</tr>
<tr>
<td>7.9</td>
<td>0.0</td>
<td></td>
</tr>
<tr>
<td>8.0</td>
<td>0.0</td>
<td></td>
</tr>
<tr>
<td>8.1</td>
<td>0.0</td>
<td></td>
</tr>
<tr>
<td>8.2</td>
<td>0.0</td>
<td></td>
</tr>
<tr>
<td>8.3</td>
<td>0.0</td>
<td></td>
</tr>
<tr>
<td>8.4</td>
<td>0.0</td>
<td></td>
</tr>
<tr>
<td>8.5</td>
<td>0.0</td>
<td></td>
</tr>
<tr>
<td>8.6</td>
<td>0.0</td>
<td></td>
</tr>
<tr>
<td>8.7</td>
<td>0.0</td>
<td></td>
</tr>
<tr>
<td>8.8</td>
<td>0.0</td>
<td></td>
</tr>
<tr>
<td>8.9</td>
<td>0.0</td>
<td></td>
</tr>
<tr>
<td>9.0</td>
<td>0.0</td>
<td></td>
</tr>
<tr>
<td>9.1</td>
<td>0.0</td>
<td></td>
</tr>
<tr>
<td>9.2</td>
<td>0.0</td>
<td></td>
</tr>
<tr>
<td>9.3</td>
<td>0.0</td>
<td></td>
</tr>
<tr>
<td>9.4</td>
<td>0.0</td>
<td></td>
</tr>
<tr>
<td>9.5</td>
<td>0.0</td>
<td></td>
</tr>
<tr>
<td>9.6</td>
<td>0.0</td>
<td></td>
</tr>
<tr>
<td>9.7</td>
<td>0.0</td>
<td></td>
</tr>
<tr>
<td>9.8</td>
<td>0.0</td>
<td></td>
</tr>
<tr>
<td>9.9</td>
<td>0.0</td>
<td></td>
</tr>
<tr>
<td>10.0</td>
<td>0.0</td>
<td></td>
</tr>
</tbody>
</table>

*Table 3*