Predictable Regular Continued Cotangent Expansions*

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Expansions of the form \( x = \cot(\arccot n_0 - \arccot n_1 + \arccot n_2 - \ldots) \) are discussed. It is shown that if \( x \) is of the form \( \frac{1}{2}(c + \sqrt{c^2 + 4}) \), then the \( n \)'s are predictable by a simple recurrence.

Continued fractions derived from the expansion of \( x \) are also given.

Key words: Continued cotangent; continued fraction; quadratic irrational.

1. Introduction

In "A Cotangent Analogue of Continued Fractions" [1], D. H. Lehmer discussed expansions of the form

\[ x = \cot(\arccot n_0 - \arccot n_1 + \arccot n_2 - \ldots). \]

This expansion is called a continued cotangent. The expansion is called a regular continued cotangent if

(a) \( n_s \) is a positive integer satisfying \( n_s \geq n_{s-1}^2 + n_{s-1} + 1 \) (\( s = 1, 2, \ldots \)).

(b) If the expansion (1) is finite and \( n_k \) is the last \( n \), then \( n_k > n + n_{k-1}^2 + n_{k-1} + 1 \).

Given any positive real number \( x \), its regular continued cotangent expansion is generated by the following algorithm:

\[ x_0 = x, \quad n_0 = \lfloor x_0 \rfloor \]
\[ x_{s+1} = \frac{x_n s + 1}{x_s - n_s}, \quad n_{s+1} = \lfloor x_{s+1} \rfloor \quad (s = 1, 2, \ldots). \]

As usual, the brackets denote the greatest integer function.

Lehmer called the \( x_s \)'s complete cotangents and the \( n_s \)'s incomplete or partial cotangents.

He did not find any combination of familiar constants whose regular continued cotangent expansion was in any way predictable.

Here we present an infinite sequence of quadratic irrationals with the property that each member of the sequence has a predictable regular continued cotangent expansion.

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1 Figures in brackets indicate the literature references at the end of this paper.
2. Irrationals With Predictable Expansions

We will show that, for a positive integer \( c \),

\[
\frac{c + \sqrt{c^2 + 4}}{2}
\]

has a predictable regular continued cotangent expansion. More precisely, we state the following:

**THEOREM:** Let \( \alpha, \beta \) be the roots of the quadratic \( x^2 - cx - 1 = 0 \), with \( c \) a positive integer, and \( \alpha > \beta \). Let \( \alpha \) be expanded into its regular continued cotangent expansion, with \( n_0, n_1, n_2, \ldots \) being the sequence of partial cotangent, and \( x_0, x_1, x_2, \ldots \) being the sequence of complete cotangents. Then

\[
\begin{align*}
(I) & \quad x_k = \alpha^{3^k} \\
(II) & \quad n_k = \alpha^{3^k} + \beta^{3^k}.
\end{align*}
\]

**PROOF:** We start out by stating some facts about \( \alpha \) and \( \beta \):

\[
\frac{c + \sqrt{c^2 + 4}}{2}, \quad \frac{c - \sqrt{c^2 + 4}}{2}.
\]

This follows from the definition of \( \alpha \) and \( \beta \).

\[
\alpha \beta = -1, \quad \alpha + \beta = c, \quad \alpha - \beta = \sqrt{c^2 + 4}, \quad [\alpha] = c.
\]

We now define the sequence \( V_k \):

\[
V_k = \alpha^k + \beta^k \quad (k \geq 0)
\]

It is easily shown by induction that

\[
V_k = cV_{k-1} + V_{k-2} \quad (k \geq 2).
\]

We are now ready to prove the theorem. The proof of part (I) proceeds by induction.

A.Verification for \( k = 0 \):

\[
\begin{align*}
x_0 &= \alpha^0 = \alpha^1 = \alpha \\
n_0 &= \alpha^0 + \beta^0 = \alpha^1 + \beta^1 = c \\
\therefore n_0 &= [\alpha] = [\alpha] = c.
\end{align*}
\]

B. Assume the theorem is true for \( k = s \). Then we want to show that the theorem is true for \( k = s + 1 \).

From (3), we have

\[
x_{s+1} = \frac{x_s n_s + 1}{x_s - n_s}
\]

\[
= \frac{(\alpha^{3^s} (\alpha^{3^s} + \beta^{3^s}) + 1}{\alpha^{3^s} - (\alpha^{3^s} + \beta^{3^s})}
\]

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\[
\begin{align*}
&= \frac{\alpha^{2\beta^3} + \alpha^{3\beta^3} + 1}{-\beta^3} \\
&= \frac{\alpha^{2\beta^3}}{-\beta^3} \quad \text{(since } \alpha \beta = -1) \\
x_{s+1} = \alpha^{3^{s+1}} \quad \text{(since } \beta = -1/\alpha) \\
\end{align*}
\]

The proof of part (I) of the theorem is now complete by induction. We complete the proof of the theorem by showing that

\[
n_{s+1} = [x_{s+1}] = [\alpha^{3^{s+1}}] = \alpha^{3^{s+1}} + \beta^{3^{s+1}}
\]

Since \( \beta = -1/\alpha \), and \( \alpha > 1 \), we have

\[
-1 < \beta^r < 0 \quad (r \text{ odd}, \geq 1).
\]

If we add \( \alpha^r \) to this inequality, we get

\[
\alpha^r - 1 < \alpha^r + \beta^r < \alpha^r \quad (r \text{ odd}, \geq 1)
\]

Now it is obvious that \( \alpha^r \) is a quadratic irrational, so neither \( \alpha^r - 1 \) nor \( \alpha^r \) are integers. Therefore, there must be exactly one integer between \( \alpha^r - 1 \) and \( \alpha^r \). Since \( V_0 = \alpha^0 + \beta^0 = 2 \) and \( V_1 = \alpha^1 + \beta^1 = c \), we have, from (7), that \( V_r = \alpha^r + \beta^r \) is an integer for \( r \geq 0 \). Therefore, \( V_r \) is the integer between \( \alpha^r - 1 \) and \( \alpha^r \), and we have

\[
[\alpha^r] = \alpha^r + \beta^r \quad (r \text{ odd}, \geq 1).
\]

We may now take \( r = 3^{s+1} \) to get

\[
n_{s+1} = [x_{s+1}] = [\alpha^{3^{s+1}}] = \alpha^{3^{s+1}} + \beta^{3^{s+1}} \quad (s \geq 0).
\]

The proof of both parts of the theorem is now complete.

In the following tables, we give the values of \( \alpha \) and \( \beta \) for the first few values of \( c \), and the values of \( n_k \) for the first few values of \( c \) and \( k \).

| Table 1. Values of \( \alpha \) and \( \beta \) |
|-----------------|-----------------|-----------------|
| \( c \) | \( \alpha \) | \( \beta \) |
| 1 | \( \frac{1}{2}(1 + \sqrt{5}) \) | \( \frac{1}{2}(1 - \sqrt{5}) \) |
| 2 | \( 1 + \sqrt{2} \) | \( 1 - \sqrt{2} \) |
| 3 | \( \frac{1}{2}(3 + \sqrt{13}) \) | \( \frac{1}{2}(3 - \sqrt{13}) \) |
| 4 | \( 2 + \sqrt{5} \) | \( 2 - \sqrt{5} \) |
| 5 | \( \frac{1}{2}(5 + \sqrt{29}) \) | \( \frac{1}{2}(5 - \sqrt{29}) \) |

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Table 2. \textit{Values of }$n_k$

<table>
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<th>$k$</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
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<td>2</td>
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<td>46764</td>
<td>439204</td>
<td>2744420</td>
</tr>
</tbody>
</table>

3. \textbf{Some Observations on }$n_k\text{ and }x_k$

First, we note that, as a special case of the theorem for $c = 1$, we have $\alpha = \phi$, the golden ratio, and $V_k = L_k$, the $k$th Lucas number \cite{2}. In fact, we have

$$x_k = \phi^k$$

and $n_k = L_3^k$.

Second, from part (II) of the theorem, it is not difficult to show that

\begin{equation}
(1)
\end{equation}

We also point out that, empirically, the regular continued cotangent expansion of an “average” irrational number satisfies

$$n_{k+1} = n_k^2 + 3n_k \quad (k \geq 0).$$

Third, we observe that the regular continued cotangent for $1/\alpha = -\beta = \frac{1}{2}(-c + \sqrt{c^2 + 4})$ also is predictable. For if the regular continued cotangent expansion of $x$ ($x > 1$) is $n_0, n_1, n_2, \ldots$ then the expansion for $1/x$ is $0, n_0, n_1, n_2, \ldots$. From this it easily follows that the expansion for $1/\alpha$ is predictable as follows:

$$x_0 = 1/\alpha, \quad n_0 = 0$$

$$x_k = \alpha^{k+1}, \quad n_k = \alpha^{k+1} + \beta^{k+1} \quad (k \geq 1).$$

We now introduce the sequence $U_k$, defined as follows:

\begin{equation}
U_k = \frac{\alpha^k - \beta^k}{\alpha - \beta}.
\end{equation}

The sequence $U_k$ satisfies the same recurrence as $V_k$, that is,

\begin{equation}
U_k = cU_{k-1} + U_{k-2} \quad (k \geq 2).
\end{equation}
Now $x_k = \alpha^k$ is a quadratic irrational and can obviously be put in the form

\begin{equation}
 x_k = v_k + u_k \sqrt{c^2 + 4}
\end{equation}

where $v_k$ and $u_k$ are rational numbers. From the definitions of $V_k$ and $U_k$, it is easily verified that

\begin{align}
 v_k &= \frac{1}{2} V_{3k} \tag{15} \\
 u_k &= \frac{1}{2} U_{3k}. \tag{16}
\end{align}

Letting $m_k = U_{3k}$, we have the following recurrence formula which can be verified by substitution:

\begin{align}
 m_{k+1} &= (c^2 + 4)m_k^3 - 3m_k \tag{17} \\
 v_{k+1} &= 4v_k^3 + 3v_k \tag{18} \\
 u_{k+1} &= 4(c^2 + 4)u_k^3 - 3u_k. \tag{19}
\end{align}

Lehmer [3] observed that

\begin{equation}
 2u_k = (n_0^2 + 1)(n_1^2 + 1)(n_2^2 + 1) \ldots (n_{k-1}^2 + 1) \tag{20}
\end{equation}

We also have

\begin{equation}
 (v_{k+1})/v_k = (u_{k+1})/u_k + 2. \tag{21}
\end{equation}

Many similar identities can be obtained.

4. Unusual Continued Fractions

We observe that the regular continued fractions for $\alpha$ and $1/\alpha$ are as follows:

\[ \alpha = c + \frac{1}{c + c + \frac{1}{c + c + \ldots}} \]

\[ 1/\alpha = \frac{1}{c + c + \frac{1}{c + c + \ldots}} \]

Lehmer showed that if the $n_k$'s (the partial cotangents) for a real number $x$ are known, then $x$ can be expanded into the following irregular continued fraction:

\[ x = n_0 + \frac{n_0^2 + 1}{n_1 - n_0 + \frac{n_1^2 + 1}{n_2 - n_1 + \frac{n_2^2 + 1}{n_3 - n_2 + \ldots}}} \]

It can be shown by induction that the $k$th convergent to this continued fraction, $p_k/q_k$ satisfies

\begin{equation}
 p_k/q_k = U_e/U_{e-1} \quad \text{where } e = (3^k + 1)/2 \tag{22}
\end{equation}
The first few such continued fractions are as follows:

\[
\frac{1}{2} (1 + \sqrt{5}) = 1 + \frac{2}{3 + \frac{17}{72 + \frac{5777}{439128} + \cdots}}
\]

\[
1 + \sqrt{2} = 2 + \frac{5}{12 + \frac{197}{2772 + \frac{7761797}{21624369228} + \cdots}}
\]

\[
\frac{1}{2} (3 + \sqrt{13}) = 3 + \frac{10}{33 + \frac{1297}{46728 + \frac{2186871697}{102266868085272} + \cdots}}
\]

These continued fractions converge extremely rapidly.

5. References


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