A Minimax-Measure Intersection Problem

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(March 16, 1976)

The problem solved is that of selecting $n$ subsets of the unit interval, each of measure $\alpha$, so as to minimize the maximum of the measures of their $p$-fold intersections. This is achieved by minimizing the sum of the measures of these $p$-fold intersections.

Key words: Combinatorial analysis; combinatorial probability; measure theory; minimax.

1. Introduction

Some years ago, NBS colleague S. Haber communicated the following problem: To select $n$ subsets of the unit interval, each of measure $\frac{1}{2}$, so as to minimize the maximum of the measures of the pairwise intersections of these subsets. The problem is suggested by a paper [1] of Gillis which, settling "an unpublished conjecture of Erdos," proves that for denumerably infinite collections of sets of measure $\alpha$, the value corresponding to the maximum pairwise-intersection measure has infimum $\alpha^2$. (Collections with higher transfinite cardinality are treated by Gillis in [2].) Here we provide an explicit solution for collections of finite cardinalities $n$. Further, and also corresponding to [1], we consider as well the case of $p$-fold intersections with $2 \leq p \leq n$, and provide the corresponding explicit solution. (As noted in [2], the argument of [1] easily extends to show that $\alpha^n$ is the limiting value for a denumerably infinite collection.)

As preliminary, we introduce a second minimization and point out its relationship to our minimax problem, to wit: Select $n$ subsets $A_1, A_2, \ldots, A_n$ of the unit interval, each of measure $\alpha$, so that the sum of the measures of their $p$-fold intersections is minimum. If now $X = \{S_1, \ldots, S_n\}$, a solution to this minimum problem, can be chosen so that all its $p$-fold intersections have the same measure $s$, and if $M$ is the maximum of the measures of the $p$-fold intersections of an arbitrary collection $A_1, A_2, \ldots, A_n$ with all $\mu(A_j) = \alpha$, then

$$\binom{n}{p} M \geq \sum_{i_1 < i_2 < \ldots < i_p} \mu \left( A_{i_1} \cap A_{i_2} \cap \ldots \cap A_{i_p} \right)$$

$$\geq \sum_{i_1 < i_2 < \ldots < i_p} \mu \left( S_{i_1} \cap S_{i_2} \cap \ldots \cap S_{i_p} \right)$$

$$= \binom{n}{p} s.$$

Thus $s \geq M$, demonstrating that $X$ solves the minimax problem. This observation suggested the analysis which follows.

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AMS Subject Classification: Primary 05A05; Secondary 05B99, 10E30, 28A75.

1 Figures in brackets indicate the literature references at the end of this paper.
2. Analysis

We will use the following notation: \( N = \{1, 2, \ldots, n\} \); \( p \) is a fixed positive integer with \( 2 \leq p \leq n \). The underlying space is the unit interval \( I \) with Lebesgue measure \( \mu \) (but the analysis actually carries over to any "atomless" probability space). Set complements of \( A \subseteq I \) and \( R \subseteq N \) are denoted \( A^c \) and \( R^c \) respectively. Let, for \( 0 \leq r \leq n \),

\[
K_r = \{R \subseteq N: |R| = r\}.
\]

Given the real number \( \alpha \) with \( 0 < \alpha \leq 1 \), let

\[
F(\alpha) = \{A \subseteq I: \mu(A) = \alpha\}
\]

and let \( F^n(\alpha) \) denote the \( n \)-fold Cartesian power of \( F(\alpha) \), consisting of all \( n \)-tuples

\[
X = \{A_1, A_2, \ldots, A_n\}
\]

with each \( A_i \in F(\alpha) \). For each such \( X \), and each \( R \subseteq N \), set

\[
X_R = \{x \in I: x \in A_i \text{ iff } i \in R\};
\]

an easily-proved property of these sets, to be used repeatedly below, is that

\[
X_R \cap (\bigcap_{i \in P} A_i) = \begin{cases} X_R & \text{if } P \subseteq R, \\ \emptyset & \text{otherwise.} \end{cases} \tag{1}
\]

Note that the disjoint union

\[
X_r = \bigcup_{R \in K_r} X_R
\]

consists of those points \( x \in I \) which are members of exactly \( r \) sets \( A_i \in X \). Finally, for measurable \( B \)
\( \subseteq I \), it is convenient to define

\[
S(X,B) = \sum_{p \in \mathbb{K}_p} \mu \left[ \left( \bigcap_{i \in p} A_i \right) \cap B \right].
\]

The “objective function” for the minimax problem is

\[
M(X) = \max_{p \in \mathbb{K}_p} \mu \left[ \bigcap_{i \in p} A_i \right],
\]

while that for the related minimization problem introduced in section 1 is

\[
S(X) = S(X,I) = \sum_{p \in \mathbb{K}_p} \mu \left[ \bigcap_{i \in p} A_i \right].
\]

An alternative formula for \( S(X) \) will first be developed (Lemma 1), and then a necessary condition (Lemma 2) for some \( X \in F^n(\alpha) \) to minimize \( S \) will be presented.

**Lemma 1:** For each \( X \in F^n(\alpha) \),

\[
S(X) = \sum_{r=p}^{n} \binom{r}{p} \mu \left( X_r \right). \tag{2}
\]

**Proof:** Since \( \{ X_r : r = 0, 1, \ldots, n \} \) is a partition of \( I \),

\[
S(X) = \sum_{r=0}^{n} S(X,X_r)
= \sum_{r=0}^{n} \sum_{|R|=r} S(X,X_R).
\]

Applying (1) to each summand, we obtain

\[
S(X) = \sum_{r=p}^{n} \sum_{|R|=r} \binom{r}{p} \mu \left( X_R \right),
\]

yielding (2).
LEMMA 2. If $X$ minimizes $S$ over $F^n(\alpha)$, and $\mu(X_i) > 0$ for some $r \geq p$, then $\mu(X_t) = 0$ for all $t < r - 1$.

PROOF: Suppose, to the contrary, that there exist $r \geq p$ and $t < r - 1$ such that $\mu(X_r) > 0$ and $\mu(X_t) > 0$. We will prove the existence of an $X' \in F^n(\alpha)$ for which $S(X') < S(X)$, thus contradicting the hypothesis about $X$.

Since $\mu(X_r) > 0$ and $\mu(X_t) > 0$, $K_r$ and $K_t$ must contain respective members $R$ and $T$ with $\mu(X_R) > 0$ and $\mu(X_T) > 0$. Choose subsets $Y$ and $Z$ of $I$ with

$$Y \subseteq X_R, \quad Z \subseteq X_T, \quad \mu(Y) = \mu(Z) > 0.$$ 

Also choose a member $i$ of the nonempty set $R - T$; then

$$Y \subseteq A_i, \quad Z \subseteq A'_i.$$ 

Now define $X' = \{A_1, A_2, \ldots, A'_i, \ldots, A_n\}$, where

$$A'_i = (A_i - Y) \cup Z;$$

Since $\mu(A'_i) = \mu(A_i - Y) + \mu(Z) = \mu(A_i)$, we have $X' \in F^n(\alpha)$.

To prove that $S(X') < S(X)$, observe that $I$ is partitioned into $Y$, $Z$, and $I - Y - Z$.

Thus

$$S(X) = S(X,Y) + S(X,Z) + S(X,I - Y - Z),$$


Since $X$ and $X'$ differ only on $Y \cup Z$, it follows that

$$S(X) - S(X') = [S(X,Y) - S(X',Y)] - [S(X',Z) - S(X,Z)].$$

Since $Y \subseteq X_r$ and $Y \subseteq X'_{r-1}$, application of (1) to the summands of $S(X,Y)$ and $S(X',Y)$ yields

$$S(X,Y) - S(X',Y) = \binom{r}{p}\mu(Y) - \binom{r}{p-1}\mu(Y) = \binom{r-1}{p-1}\mu(Y).$$
Similarly, it follows from $Z \subseteq X_t$ and $Z \subseteq X_{t+1}$ that

$$S(X', Z) - S(X, Z) = \left( \frac{t + 1}{p} \right) \mu(Z) - \left( \frac{t}{p} \right) \mu(Z) = \left( \frac{t}{p - 1} \right) \mu(Z).$$

Since $r - 1 > t$ and $\mu(Y) = \mu(Z) > 0$,

$$S(X) - S(X') = \left( \frac{r - 1}{p - 1} \right) \mu(Y) - \left( \frac{t}{p - 1} \right) \mu(Z) > 0,$$

completing the proof.

We will subsequently show that if $\mu(X_r) > 0$ for some $r > p$ then $\mu(X_t) = 0$ for $t < r - 1$ is a sufficient condition for $X$ to minimize $S$ over $F^n(\alpha)$.

**Lemma 3:** For all $X \in F^n(\alpha)$,

$$n\alpha = \sum_{r=0}^{n} \mu(X_r). \quad (3)$$

**Proof:** Let $c_i$ denote the characteristic function of $A_i$. Then

$$n\alpha = \sum_{r=1}^{n} \int c_i(x) \, d\mu(x) = \int \left[ \sum_{i=1}^{n} c_i(x) \right] \, d\mu(x) = \sum_{r=0}^{n} \int_{X_r} \left[ \sum_{i=1}^{n} c_i(x) \right] \, d\mu(x) = \sum_{r=0}^{n} r \mu(X_r).$$

It is now possible to prove:

**Lemma 4:** If $n\alpha \leq p - 1$, then $S_{\text{min}} = \min \{ S(Y) : Y \in F^n(\alpha) \} = 0$.

**Proof:** It suffices to exhibit an $X \in F^n(\alpha)$ for which

$$\mu(X_r) = 0 \quad \text{for } r \geq p. \quad (4)$$

To this end, let

$$A_i = [(i - 1)\alpha, i\alpha) \pmod{1} \quad \text{for } 1 \leq i \leq n.$$ 

Each point of $[0,1)$ corresponds (mod 1) to exactly $p - 1$ points of the interval $[0, p - 1)$, and thus to at most $p - 1$ points of the subinterval $[0, n\alpha)$; thus $X_r \cap [0, 1) = \phi$ for $r \geq p$, verifying (4).

**Lemma 5:** If for given $X \in F^n(\alpha)$, the largest $r$ such that $\mu(X_r) > 0$ satisfies $r \geq p$ and further for $t < r - 1$, $\mu(x_t) = 0$, then

$$S(X) = S_{\text{min}}.$$ 

**Proof:** It suffices to show that $S(X)$ has the same value for all $X \in F^n(\alpha)$ satisfying the conditions of the lemma. Consider such an $X$, and the greatest $r$ for which $\mu(X_r) > 0$. Since

$$\sum_{i=0}^{n} \mu(X_i) = \mu(I) = 1, \quad (5)$$

such an $r$ must exist. By the above condition $\mu(X_r) = 0$ for $t \neq r, r - 1$, and so by (5),
\[
\mu(X_{r-1}) = 1 - \mu(X_r).
\]

Let \( n\alpha = m + \beta \) with \( m \) integral and \( 0 \leq \beta < 1 \). It follows from (3) that

\[
m + \beta = r\mu(X_r) + (r - 1)\mu(X_{r-1}) = (r - 1) + \mu(X_r),
\]

and from (2) that

\[
S(X) = \binom{r}{p}\mu(X_r) + \binom{r-1}{p}\mu(X_{r-1}) = \binom{r-1}{p} + \binom{r-1}{p-1}\mu(X_r).
\]

If \( \beta = 0 \), then since \( m \) is integral and \( 0 < \mu(X_r) \leq 1 \), it follows from (6) that \( \mu(X_r) = 1 \) and \( r = m \), and then it follows from (7) that

\[
S(X) = \binom{m-1}{p} + \binom{m-1}{p-1} = \binom{m}{p}.
\]

If \( \beta > 0 \), then it follows from (6) that \( \mu(X_r) = \beta \) and \( m = r - 1 \), and then it follows from (7) that

\[
S(X) = \binom{m}{p} + \binom{m}{p-1}\beta.
\]

Thus \( S(X) \) is uniquely determined by the pair \((m, \beta)\), i.e., by \( n\alpha \). Note that (8) and (9) are consistent with Lemma 4, since both yield \( S(X) = 0 \) if \( n\alpha \leq p - 1 \).

We are now able to provide the solutions, both to the problem of minimizing \( S(X) \) over \( F^n(\alpha) \) and to the original problem of minimizing

\[
M(X) = \max_{p\in\mathbb{R}_p} \mu \left( \bigcap_{i\in\mathbb{R}} A_i \right)
\]

over \( F^n(\alpha) \). Let \( M_{\min} \) denote the value of this latter minimum. Then the solution takes the following form.

**Theorem.** Let \( n\alpha = m + \beta \) with \( m \) integral and \( 0 \leq \beta < 1 \). Then

\[
S_{\min} = M_{\min} = 0 \quad \text{if} \ n\alpha \leq p - 1,
\]

\[
S_{\min} = \binom{m}{p} + \binom{m}{p-1}\beta, \quad M_{\min} = S_{\min}/\binom{n}{p} \quad \text{if} \ n\alpha > p - 1.
\]

Thus, in particular, for the problem as originally posed where \( p = 2 \) and \( \alpha = 1/2 \).

\[
S_{\min} = \begin{cases} 
k(k - 1)/2 & \text{if } n = 2k \\
(k - 1)^2/2 & \text{if } n = 2k - 1,
\end{cases}
\]

and \( M_{\min} = (k - 1)/2(2k - 1) \).

**Proof:** First suppose \( n\alpha \leq p - 1 \). Then \( S_{\min} = 0 \) follows from Lemma 4, whose proof constructed an \( X \in F^n(\alpha) \) for which \( \mu \left( \bigcup_{r=p}^n X_r \right) = 0 \). Since every \( p \)-fold intersection of the members of \( X \) lies in this union, it follows that \( M(X) = 0 \), implying \( M_{\min} = 0 \).
Now suppose $n\alpha > p - 1$. The formula for $S_{\min}$ follows from (8) and (9). We will prove the result for $M_{\min}$ by constructing an $X \in F^n(\alpha)$ which satisfies the condition of Lemma 2, and which furthermore (see the end of sec. 1) has equal measures for each of its $p$-fold intersections. For this purpose, partition the interval $[0, \beta)$ into $\binom{n}{m+1}$ equal subintervals and the interval $[\beta, 1]$ into $\binom{n}{m}$ equal subintervals. Label the second family of subintervals as $\{X_M: M \in K_m\}$ and the first family as $\{X_Q: Q \in K_{m+1}\}$. Define

$$A_i = \left[ \bigcup \{X_M: i \in M\} \right] \cup \left[ \bigcup \{X_Q: i \in Q\} \right].$$

Then each $A_i$ consists of $\binom{n-1}{m-1}$ intervals $X_M$ and $\binom{n-1}{m}$ intervals $X_Q$, all disjoint, so all $A_i$ have equal measure. If $c_i$ denotes the characteristic function of $A_i$, then

$$\sum_{i=1}^{\binom{n}{m+1}} \mu(A_i) = \int_{\beta}^{1} \left( \sum_{i=1}^{\binom{n}{m+1}} c_i \right) d\mu = \int_{0}^{\beta} \left( \sum_{i=1}^{\binom{n}{m+1}} c_i \right) d\mu + \int_{\beta}^{1} \left( \sum_{i=1}^{\binom{n}{m}} c_i \right) d\mu$$

$$= (m + 1)\beta + m(1 - \beta) = m + \beta = n\alpha.$$

Thus each $\mu(A_i) = \alpha$, i.e., $X \in F^n(\alpha)$. For $r \geq p$, $\mu(X_r) > 0$ holds only for $r = m$ and $r = m + 1$, so the condition of Lemma 2 is satisfied. The symmetry of the construction assures that all $p$-fold intersections of the members of $X$ have equal measure; explicitly, for $P \in K_p$, we have

$$\bigcap_{i \in P} A_i = \left[ \bigcup \{X_M: P \subset M\} \right] \cap \left[ \bigcup \{X_Q: P \subset Q\} \right],$$

implying

$$\mu\left[ \bigcap_{i \in P} A_i \right] = \binom{n-p}{m-p} (1 - \beta) \binom{n}{m} + \binom{n-p}{m+1-p} \beta \binom{n}{m+1},$$

independently of $P$.

3. References


(Paper 80B2–438)