Properties of Neighboring Sequences in Stratifiable Spaces*

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In $T_0$-spaces metrizability can be characterized in terms of mutual convergence of “neighboring sequences.” In this paper Nagata spaces are characterized in terms of a convergence property of neighboring sequences and more generally it is shown that in all stratifiable spaces, neighboring sequences satisfy a similar convergence property.

Key words: Coconvergent; contraconvergent; Nagata spaces; open neighborhood assignments; stratifiable spaces; $U$-linked sequences

An open neighborhood assignment (ONA) is defined in [4] as a function

$$U : X \times Z \rightarrow \{N(x) : x \in X\}$$

such that $x \in U_n(x)$ (where $X$ is a topological space, $Z$ is the set of natural numbers and $N(x)$ is the collection of open neighborhoods of $x$). If $U$ is an ONA then the sequence $\{x_n\}$ is $U$-linked to $\{y_n\}$ if $x_n \in U_n(y_n)$ for all $n$. Using the notation $Cp\{x_n\}$ for the set of cluster points of $\{x_n\}$ a space will be called coconvergent (contraconvergent) if $Cp\{x_n\} \subset Cp\{y_n\}$ ($Cp\{y_n\} \subset Cp\{x_n\}$) whenever $\{y_n\}$ is $U$-linked to $\{x_n\}$. If on $X$ there is an ONA $U$ satisfying some property $P$ I shall say “$X$ is $P$” or “$U$ is $P$.” Also, $U$ will be said to be nested if $U_{n+1}(x) \subset U_n(x)$ for all $x$ and $n$.

It was proved in [4]:

**Theorem 1:** $X$ is metrizable iff it is a coconvergent, contraconvergent $T_0$-space.

Also, examples of a $T_2$, coconvergent space and a $T_2$, contraconvergent space, neither of which are metrizable were given in [4]. Coconvergence implies first countability whereas nonfirst countable contraconvergent spaces exist.

R. W. Heath in [3] proved:

**Theorem 2:** A $T_1$-space $X$ is a Nagata space (first countable and stratifiable) iff there is an ONA $U$ on $X$ such that (a) $U$ is first countable and (b) for every $x \in X$ and open set $R$ containing $x$ there is an $n \in Z$ such that $U_n(x) \cap U_n(y)$ implies $y \in R$.

(Note: Condition (a) is implied by (b): If $x \in X$ and $R \in N(x)$ such that for all $n$ there is a $y_n \in U_n(x) \cap R$, then $U_n(x) \cap U_n(y_n) \neq 0$.

It follows from (b) there is a $y_n \in R$ for some $k$ contradicting the way the $y_n$ were chosen.)

**Proposition 3:** A $T_1$-space $X$ is a Nagata space iff it is first countable and contraconvergent.

**Proof:** Let $U$ be an ONA on $X$ satisfying the conditions of Theorem 2. Without loss of generality we may take $U$ to be nested. Let $\{y_n\}$ be $U$-linked to $\{x_n\}$ with $y \in Cp\{y_n\}$. Given an $R \in N(y)$ and $N \in Z$ there is an $n_1 > N$ such that $U_{n_1}(y) \subset R$. Also, since $U$ is nested there is an $n_2 > n_1$ such that if $x$ satisfies $U_{n_2}(x) \cap U_{n_2}(y) \neq 0$ then $x \in R$. Finally, there is a $k > n_2$ such

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*Figures in brackets indicate the literature references at the end of this paper.*
that \( y_k \in U_{n_k}(y) \subset U_n(y) \). Since \( \{ y_n \} \) is \( U \)-linked to \( \{ x_n \} \), \( y_k \in U_k(x_k) \cap U_{n_k}(y) \). It now follows from \( k > N \) that \( y_k \in \mathcal{C}p \{ y_n \} \) and that \( U \) is contraconvergent.

Conversely, without loss of generality it may be assumed there is an \( \mathcal{O} \) on \( X \) which is contraconvergent and first countable. If \( U \) does not satisfy the condition of Theorem 2, then for some \( x \) and \( R \in \mathcal{N}(x) \) there are sequences \( \{ x_n \} \) and \( \{ y_n \} \) such that \( x_n \in U_n(x) \cap U_n(y_n) \) and \( y_n \in R \) for all \( n \). It follows from the first countability and contraconvergence of \( U \) that \( x \in \mathcal{C}p \{ y_n \} \) which is a contradiction.

**Corollary 4:** A \( T_0 \)-space is metrizable iff it is coconvergent and stratifiable.

Coconvergent spaces are those included in the class of spaces in which compact sets have countable local bases (\( D_0 \)-spaces) [5]. The set of all ordinals less than or equal to the first uncountable ordinal, with the order topology is an example of a \( D_0 \)-space which is not coconvergent. It was shown in [5] that spaces in which the stratifications of open sets satisfy a certain monotone condition, "coconvergent" can be replaced by "\( D_0 \)" in Corollary 4.

If \( X \) is stratifiable, I will use: \( R_n, (X - F)_n \) and \( \{ U_k(x) \}_n \) to denote the \( n \)th layers of stratifications of the open sets \( R, X - F \) and \( U_k(x) \) respectively and more generally, \( A \) for the closure of \( A \). Without loss of generality I shall assume \( R_n \subset R_{n+1} \) for any open \( R \). It will be shown that if \( X \) is stratifiable that the \( \mathcal{O} \) defined by \( U_n(x) = X - (X - x_n)^{-} \) satisfies a condition similar to that defining contraconvergent spaces. The \( \mathcal{O} \) will be referred to as the \( \mathcal{O} \) associated with the given stratification.

**Definition 5:** An \( \mathcal{O} \) on \( X \) satisfies property \( A \) if whenever \( \{ y_n \} \) is \( U \)-linked to \( \{ x_n \} \) and \( y \in \mathcal{C}p \{ y_n \} \) then for any \( N \in \mathbb{Z} \) there is a \( k > N \) such that \( x_k \in U_n(y) \).

**Proposition 6:** A \( T_1 \)-space is a Nagata space iff there is a first countable \( \mathcal{O} \), \( U \) on \( X \) satisfying property \( A \).

**Proof:** If \( X \) is a Nagata space, by Proposition 3 there is an \( \mathcal{O} \), \( U \) which is first countable and contraconvergent. If \( \{ y_n \} \) is \( U \)-linked to \( \{ x_n \} \) and \( y \in \mathcal{C}p \{ y_n \} \) then \( y \in \mathcal{C}p \{ x_n \} \) and it follows that \( U \) satisfies condition \( A \).

Let \( U \) be a first countable \( \mathcal{O} \) satisfying condition \( A \). If \( \{ y_n \} \) is \( U \)-linked to \( \{ x_n \} \) with \( y \in \mathcal{C}p \{ y_n \} \) and if \( N_1 \in \mathbb{Z} \) and \( R \in \mathcal{N}(y) \) then there is an \( N_2 > N_1 \) and by property \( A \) a \( k > N_2 \) such that \( x_k \in U_{N_2}(y) \subset R \) proving \( y \in \mathcal{C}p \{ x_n \} \) and that \( U \) is contraconvergent.

**Proposition 7:** Let \( X \) be a stratifiable space. Then the \( \mathcal{O} \) \( U \) associated with a given stratification satisfies property \( A \).

**Proof:** It follows from \( R_n \subset R_{n+1} \) for any open \( R \) that \( U \) is nested. If \( U \) does not satisfy property \( A \), then there exist a \( \{ y_n \} \) \( U \)-linked to \( \{ x_n \} \), a \( y \in \mathcal{C}p \{ y_n \} \) and an \( N \in \mathbb{Z} \) such that for all \( k > N \), \( x_k \in U_n(y) \). Hence \( F = \{ x_k : k > N \}^{-} \subset X - U_n(y) \).

**References**


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