The Factorization of a Matrix as the Commutator of Two Matrices

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Let \( P = P_+ = (\pm I_p) \), the direct sum of the \( p \times p \) identity matrix and the negative of the \( q \times q \) identity matrix. The following theorem is proved.

**Theorem:** If \( X = cZ \) where \( Z \) is a \( 4 \times 4 \) \( P \)-orthogonal, \( P \)-skew-symmetric matrix and \( |c| \leq 2 \), there exist matrices \( A \) and \( B \), both of which are \( P \)-orthogonal and \( P \)-skew-symmetric, such that \( X = AB - BA \). Methods for obtaining certain matrices which satisfy \( X = AB - BA \) are given. Methods are also given for determining pairs of anticommuting \( P \)-orthogonal, \( P \)-skew-symmetric matrices.

Key words: Anticommuting; commutator; factorization; matrix; orthogonal; skew-symmetric.

1. Introduction

Let \( P = P_+ = (\pm I_p) \), the direct sum of the \( p \times p \) identity matrix and the negative of the \( q \times q \) identity matrix. Katz and Olkin [2] define a real matrix \( A \) to be orthogonal with respect to \( P \) (\( P \)-orthogonal) if and only if

\[
APA' = P
\]

where \( A' \) is the transpose of \( A \). Furthermore, they define \( B \) to be skew-symmetric with respect to \( P \) (\( P \)-skew-symmetric) if and only if \( BP \) is skew-symmetric in the ordinary sense.

The main result of this paper is concerned with matrices which are both \( P \)-orthogonal and \( P \)-skew-symmetric of order \( n = 4 = p + q \). Smith [7] proved that such matrices exist in only two cases, \( p = 4 \), \( q = 0 \) and \( p = q = 2 \). In the first case \( P \)-orthogonal and \( P \)-skew-symmetric reduce to orthogonal and skew-symmetric in the ordinary sense.

Pearl [4] and Smith [6] proved the following theorem in the cases \( p = 4 \), \( q = 0 \) and \( p = q = 2 \) respectively.

**Theorem 1:** If the \( 4 \times 4 \) matrices \( A \) and \( B \) are both \( P \)-orthogonal and \( P \)-skew-symmetric then their commutator, \( [A, B] = AB - BA \), is a scalar multiple of a \( 4 \times 4 \) \( P \)-orthogonal, \( P \)-skew-symmetric matrix.

The purpose of this paper is to prove a converse to Theorem 1. Shoda [5] proved that if \( X \) is a square matrix with zero trace having elements in an algebraically closed field then there exist matrices \( A \) and \( B \) such that \( X = AB - BA \). Albert and Muckenhoupt [1] removed the restriction that the field be algebraically closed. However, both the method of Shoda and the method of Albert and Muckenhoupt give a singular matrix \( B \). The main result of this paper is:

**Theorem 2:** If \( X = cZ \) where \( Z \) is a \( 4 \times 4 \) \( P \)-orthogonal, \( P \)-skew-symmetric matrix and \( |c| \leq 2 \), there exist matrices \( A \) and \( B \), both of which are \( P \)-orthogonal and \( P \)-skew-symmetric, such that \( X = AB - BA \).
Methods for obtaining certain matrices which satisfy $X = AB - BA$ are given. Methods are also given for determining pairs of anticommuting $P$-orthogonal, $P$-skew-symmetric matrices.

2. Anticommuting Matrices

In examining the structure of $P$-orthogonal, $P$-skew-symmetric matrices in the case $p = 4, q = 0$, Pearl [4] shows that any such matrix has exactly one of the following forms:

(i) $\alpha_1 R_1 + \alpha_2 R_2 + \alpha_3 R_3, \quad \alpha_1^2 + \alpha_2^2 + \alpha_3^2 = 1$

(ii) $\alpha_1 S_1 + \alpha_2 S_2 + \alpha_3 S_3, \quad \alpha_1^2 + \alpha_2^2 + \alpha_3^2 = 1$

where the $\alpha_i$ are real scalers and the $R_i$ and $S_i$ are the first and second regular representations respectively of the real quaternions [3].

Similarly, in the case $p = q = 2$, Smith [6] shows that any such matrix has exactly one of the following forms:

(iii) $\alpha_1 R_1 P + \alpha_2 S_2 P + \alpha_3 S_3 P, \quad \alpha_1^2 + \alpha_2^2 - \alpha_3^2 = 1$

(iv) $\alpha_1 S_1 P + \alpha_2 R_2 P + \alpha_3 R_3 P, \quad \alpha_1^2 - \alpha_2^2 - \alpha_3^2 = -1$

where $P = I_2 + (-I_2)$.

A further examination of these papers leads to

**Theorem 3:** If $Z$ is a $4 \times 4$ $P$-orthogonal, $P$-skew-symmetric matrix there exists a $4 \times 4$ $P$-orthogonal, $P$-skew-symmetric matrix $B$ such that $ZB = -BZ$.

**Proof:** There are four cases to consider.

**Case 1, $Z = \alpha_1 R_1 + \alpha_2 R_2 + \alpha_3 R_3$.** If $\alpha_3 \neq 0$, choose arbitrary $\beta_1', \beta_2'$ and set

$$\beta_3' = -\frac{1}{\alpha_3} (\alpha_1 \beta_1' + \alpha_2 \beta_2').$$

Let $x = \beta_1'^2 + \beta_2'^2 + \beta_3'^2$ and set $\beta_i = \frac{\beta_i'}{\sqrt{x}}, i = 1, 2, 3$. If $\alpha_3 = 0$, let $\beta_1 = \beta_2 = 0$ and $\beta_3 = 1$. Clearly, in either situation

$$\alpha_1 \beta_1 + \alpha_2 \beta_2 + \alpha_3 \beta_3 = 0 \quad (2)$$

and

$$\beta_1^2 + \beta_2^2 + \beta_3^2 = 1. \quad (3)$$

Letting $B = \beta_1 R_1 + \beta_2 R_2 + \beta_3 R_3$, by (3) $B$ is $P$-orthogonal, $P$-skew-symmetric and by (2)

$$ZB = -BZ.$$

**Case 2, $Z = \alpha_1 S_1 + \alpha_2 S_2 + \alpha_3 S_3$.** Choose $\beta_i, i = 1, 2, 3$ as in Case 1 and let

$$B = \beta_1 S_1 + \beta_2 S_2 + \beta_3 S_3.$$

**Case 3, $Z = \alpha_1 R_1 P + \alpha_2 S_2 P + \alpha_3 S_3 P$.** The matrix $B = \beta_1 R_1 P + \beta_2 S_2 P + \beta_3 S_3 P$ will be $P$-orthogonal, $P$-skew-symmetric if

$$\beta_2^2 + \beta_3^2 - \beta_1^2 = 1 \quad (4)$$

and $ZB = -BZ$ if

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\[
\alpha_2 \beta_2 + \alpha_3 \beta_3 - \alpha_1 \beta_1 = 0.
\] (5)

If \(\alpha_1 + \alpha_2 \neq 0\), set
\[
\beta_1 = \frac{-\alpha_3}{\alpha_1 + \alpha_2}, \quad \beta_2 = \frac{-\alpha_3}{\alpha_1 + \alpha_2}, \quad \text{and} \quad \beta_3 = 1.
\]
Clearly, (4) and (5) are satisfied. If \(\alpha_1 = \alpha_2 = 0\), set \(\beta_1 = \beta_3 = 0, \beta_2 = 1\). Again (4) and (5) are satisfied. If \(\alpha_1 = -\alpha_2 \neq 0\), since clearly \(\alpha_3 = \pm 1\), set \(x = \frac{1}{1 + \alpha_1^2}\) and let \(\beta_1 = 0, \beta_3 = \alpha_3 \alpha_1 \sqrt{x}, \beta_2 = \sqrt{x}\). Again (4) and (5) are satisfied.

Case 4, \(Z = \alpha_1 S_1 P + \alpha_2 R_2 P + \alpha_3 R_3 P\). Let \(B = \beta_1 S_1 P + \beta_2 R_2 P + \beta_3 R_3 P\) where the \(\beta_i\) are chosen as in case 3.

### 3. Proof of Theorem 2

In order to prove Theorem 2 it is convenient to first prove the following lemmas.

**Lemma 1:**
(i) If \(B\) is \(P\)-skew-symmetric then \(B' = -PBP\).
(ii) If \(B\) is \(P\)-skew-symmetric and \(P\)-orthogonal than \(B^2 = -1\).

**Lemma 2:** If \(Z\) is a \(P\)-orthogonal, \(P\)-skew-symmetric matrix and \(|c| \leq 2\) then
\[
Y = \frac{\sqrt{4 - c^2}}{2} I + \frac{c}{2} Z \text{ is } P\text{-orthogonal}.
\]

**Proof:** By direct computation,
\[
YPY' = \left(\frac{\sqrt{4 - c^2}}{2} I + \frac{c}{2} Z\right) P \left(\frac{\sqrt{4 - c^2}}{2} I + \frac{c}{2} Z'\right)
\]
\[
= \frac{4 - c^2}{4} P + \frac{c^2}{4} ZPZ' + \frac{\sqrt{4 - c^2}}{2} (ZP + PZ').
\]

However, \(ZPZ' = P\) by (1) and by Lemma 1
\[
ZP + PZ' = ZP + P(-PZP) = ZP - ZP = 0.
\]

Thus \(YPY' = \frac{4 - c^2}{4} P + \frac{c^2}{4} P + O = P\) and by (1) \(Y\) is \(P\)-orthogonal.

**Lemma 3:** If \(Z\) is \(P\)-orthogonal, \(P\)-skew-symmetric and \(|c| \leq 2\), and if \(B\) is \(P\)-orthogonal, \(P\)-skew-symmetric such that \(ZB = -BZ\), then
\[
A = \left(\frac{\sqrt{4 - c^2}}{2} I + \frac{c}{2} Z\right) PB'P \text{ satisfies } [A,B] = cZ.
\]

**Proof:**
\[
AB = (YPB')P = (YP) (B'PB) = (YP) P = Y
\]
\[
= \frac{\sqrt{4 - c^2}}{2} I + \frac{c}{2} Z
\]
\[
BA = B (YPB') = \frac{\sqrt{4 - c^2}}{2} BPB'P + \frac{c}{2} BZPB'P
\]
\[
= \frac{\sqrt{4 - c^2}}{2} I + \frac{c}{2} BZPB'P
\]
\[
= \frac{\sqrt{4 - c^2}}{2} I - \frac{c}{2} ZBPB'P
\]
Thus \( [A,B] = AB - BA = c Z \).

**Corollary:** The matrix \( A \) defined in Lemma 3 is \( P \)-orthogonal, \( P \)-skew-symmetric.

**Proof:** By Lemmas 1 and 2 \( A \) is the product of two \( P \)-orthogonal matrices. Hence \( A \) is \( P \)-orthogonal. Also

\[
A = \left( \frac{\sqrt{4 - c^2}}{2} I + \frac{c}{2} Z \right) PB'P
\]

\[
= \frac{\sqrt{4 - c^2}}{2} PB'P + \frac{c}{2} ZPB'P.
\]

By Lemma 1, \( \frac{\sqrt{4 - c^2}}{2} PB'P = -\frac{\sqrt{4 - c^2}}{2} B \) which is \( P \)-skew-symmetric. Furthermore

\[
\left( \frac{c}{2} ZPB'P \right)' = \frac{c}{2} PBPZ'
\]

\[
= -\frac{c}{2} B'Z'
\]

by Lemma 1

\[
= \frac{c}{2} Z'B'
\]

since \( ZB = -BZ \)

\[
= -\frac{c}{2} PZPB'
\]

by Lemma 1

\[
= -\frac{c}{2} P(ZPB'P)'.
\]

Thus \( A \) is the sum of two \( P \)-skew-symmetric matrices and hence \( A \) is \( P \)-skew-symmetric.

In the \( 4 \times 4 \) case, the existence of the matrix \( B \) is given by Theorem 3. Thus Theorem 3, Lemma 3, and the Corollary complete the proof of Theorem 2.

### 4. Conclusion

Theorem 2 provides a converse to the theorems of Pearl [4] and Smith [6]. While Theorem 2 is restricted to the \( 4 \times 4 \) case, the results of section 3 refer to the general \( n \times n \) case. Smith [8] has generalized Theorem 1 to the \( n \times n \) case. Perhaps the results of section 3 can be applied to find a converse of that result.

### 5. References


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