Rational Equivalence of Unimodular Circulants*

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(December 5, 1973)

We answer a question of M. Newman by providing all unimodular positive circulants are rationally equivalent to the identity.

Key words: Circulant, totally positive unit.

Let $P$ be the $n \times n$ matrix satisfying $P_{12} = \ldots = P_{n-1,n} = P_{n,1} = 1$ and all other entries 0. The elements of the group ring $\mathbb{Z}[P]$ are called integral circulants. Let $G$ be the group in $\mathbb{Z}[P]$ consisting of the positive definite symmetric unimodular elements. M. Newman [2, p. 198] asks which members of $G$ are rationally equivalent to $I_n$. The answer to this question is in fact an easy consequence of the Hasse norm theorem [1, p. 186].

**Theorem:** All members of $G$ are rationally equivalent to $I_n$.

**Proof:** Let $A \in G$. As noted in [2, p. 198], we must show that any eigenvalue $\lambda$ of $A$ is of the form $a \bar{\alpha}$ for some $\alpha$ in $Q_n$, the $n$th cyclotomic field. Let $K$ be the real subfield of index 2 in $Q_n$. We must show $\lambda$ is a norm from $Q_n$ to $K$. If $n = 2p^m$, $p$ a prime, then $p$ is fully ramified in $Q_n$. In any other case, $Q_n$ is unramified over $K$ at all finite primes. So at most one finite prime ramifies from $K$ to $Q_n$.

Now $\lambda$ is totally positive, so $\lambda$ is a norm at all Archimedean localizations. And $\lambda$ is a unit; thus $\lambda$ is a norm at all finite localizations, except, possibly, one. By the product formula, $\lambda$ is a norm everywhere, and hence, by the Hasse norm theorem, $\lambda$ is a global norm.

**References**


(Paper 78B2–398)