On Characters of Subgroups*

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Let $H$ be a subgroup of $G$. Let $\chi$ be an irreducible character of $H$. Let $\chi^G$ be the character of $G$ induced by $\chi$. The irreducibility of $\chi^G$ is discussed. In particular, if $H$ is normal in $G$, then $\chi^G$ is irreducible if and only if $\chi$ cannot be extended to any subgroup of $G$ which properly contains $H$.

These results have application to the determination of irreducibility of a class of representations of the full linear groups.

Key words: Frobenius Reciprocity Theorem.

I. First Introduction

Let $H$ be a subgroup of $G$. Let $\chi$ be an irreducible complex character of $H$. In the course of the author’s study of a class of representations of the full linear group, the following criterion arose:

Condition 1. There is an irreducible character $\lambda$ of $G$ such that $(\lambda, \chi)_H = 1$, and $(\xi, \chi)_H = 0$, for every irreducible character $\xi$ of $G$ different from $\lambda$. (Here, of course,

$$(\xi, \chi)_H = \frac{1}{o(H)} \sum_{h \in H} \xi(h) \chi(h^{-1}).$$

It turns out that when $G = S_m$, the symmetric group, Condition 1 is equivalent to the irreducibility of a certain representation of the full linear group [3]. The main purpose of this (essentially expository) note is to investigate character theoretic statements which are related to Condition 1.

II. Second Introduction

Let $H$ be a subgroup of $G$ and let $\chi$ be an irreducible character of $H$. Can we obtain from this situation any information about the irreducible characters of $G$? It would be most pleasant, for example, if $\chi$ could be extended to a character of $G$. But, this is not always possible.

One general method to obtain a character of $G$ from $\chi$ goes as follows: Define $\chi^*$ on $G$ by $\chi^*(h) = \chi(h)$ for $h \in H$, and $\chi^*(g) = 0$ for $g \in G \setminus H$. then

$$\chi^G(g) = \frac{1}{o(H)} \sum_{f \in G} \chi^*(f^{-1}g) \chi(f), \quad g \in G,$$

turns out to be a character of $G$ whose degree is $\chi(id) [G : H]$. It is called the character of $G$ induced by $\chi$ [1, 2].

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Figures in brackets indicate the literature references at the end of this paper.
Now, of course, we would like to know something about $\chi^G$. For example, is it an irreducible character? In general, the answer is no. We are indebted to Frobenius for the following very useful result:

(Frobenius Reciprocity) Theorem: Let $H$ be a subgroup $G$. Let $\chi$ and $\lambda$ be characters of $H$ and $G$ respectively. Then

$$(\chi, \lambda)_H = (\chi^G, \lambda)_G.$$ 

As we shall see, the irreducibility of $\chi^G$ is related to the extendability of $\chi$.

**III. Results**

Suppose $g \in G$. We let $\chi^g$ denote the character of $gHg^{-1}$ defined by

$$\chi^g(gHg^{-1}) = \chi(h), \quad h \in H.$$ 

**Theorem 1:** Let $H$ be a subgroup of $G$. Let $\chi$ be an irreducible character of $H$. The following are equivalent

a. Condition 1.

b. $\chi^G$ is irreducible (in fact $\chi^G = \lambda$).

c. For all $g \in G\setminus H$, $\chi^g$ and $\chi$ are different characters of $H \cap gHg^{-1}$.

**Theorem 2:** If $\chi^G$ is irreducible, then $\chi$ cannot be extended to any subgroup of $G$ which properly contains $H$.

Unfortunately, the converse of Theorem 2 is not true in general. For example, let $G = S_4$. Let $H$ be the subgroup generated by $\{(14)(23), (1234)\}$. (Then $H$ is the dihedral group $D_4$ of order 8.) Let $\chi$ be the irreducible character of $H$ of degree 2. The only subgroup of $G$ which properly contains $H$ is $G$ itself, and $\chi$ does not extend to $G$. The character $\chi^G$, of degree 6, is the sum of the two inequivalent characters of $G$ of degree 3.

When $H$ is normal in $G$, however, the converse does hold.

**Theorem 3:** Let $H$ be a normal subgroup of $G$. Let $\chi$ be an irreducible character of $H$. If $\chi$ cannot be extended to any subgroup of $G$ which properly contains $H$, then $\chi^G$ is irreducible.

In this connection, we point out a recent result of Roth [4, Theorem 3.1].

(Roth's) Theorem. Let $\xi$ be a character of $G$ of degree 1. Let $H = \ker \xi = \{g \in G : \xi(g) = 1\}$. Suppose there exists an irreducible character $\lambda$ of $G$ such that $\lambda \xi = \lambda$. Then there exists an irreducible character $\chi$ on $H$ such that $\chi^G = \lambda$.

**IV. Proofs**

We begin with Theorem 1. The equivalence of a and b is immediate from the Frobenius Reciprocity Theorem, i.e., $\chi^g = \lambda$ if and only if $(\lambda, \chi)_H = 1$, and $(\xi, \chi)_H = 0$, for every irreducible character $\xi$ of $G$ different from $\lambda$. The equivalence of b and c is Theorem (45.2)' of [1].

The proof of Theorem 2 is equally straightforward. If $\chi^G$ is irreducible, then (by Theorem 1) for all $g \in G\setminus H$, $\chi^g$ and $\chi$ are different on $H \cap gHg^{-1} \subset <H, g>$, the group generated by $H$ and $g$. Thus, since characters are class functions, $\chi$ cannot be extended to $<H, g>$.

We proceed to the proof of Theorem 3.

**Lemma:** Let $H$ be a normal subgroup of $G$. Let $\chi$ be an irreducible character on $H$. Then $\chi$ can be extended to $<H, g>$ if and only if $\chi^g = \chi$. 

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**Proof:** As above, necessity is clear. Suppose, then, that \( \chi(g^{-1}hg) = \chi(h) \) for all \( h \in H \). Let \( h \to A(h) \) be an irreducible representation of \( H \) affording \( \chi \). Define

\[
B(h) = A(g^{-1}hg), \quad h \in H.
\]

Then \( h \to B(h) \) is a representation of \( H \) which affords \( \chi \). It follows that \( A \) and \( B \) are equivalent. Let \( U \) be nonsingular such that

\[
B(h) = U^{-1}A(h)U, \quad h \in H. \tag{1}
\]

Now, let \( r \) be minimal such that \( g^r \in H \). Observe

\[
A(g^{-r})A(h)A(g^r) = A(g^{-r}hg^r) = A(g^{-1}g^{-r+1}hg^r) = B(g^{-r+1}hg^r) = U^{-1}A(g^{-r+1}hg^r)U = \ldots = U^{-r}A(h)Ur.
\]

Thus, \( A(g^r)U^{-r} \) commutes with \( A(h) \) for all \( h \in H \). It follows from Schur’s Lemma that \( A(g^r)U^{-r} \) is a scalar matrix \( S \). We now replace \( U \) in (1) with \( U \) times any scalar \( r \)-th root of \( S^{-1} \), i.e., we may assume that \( Ur = A(g^r) \).

Next, we define \( R \) on \( <H, g> \) by

\[
R(hg^k) = A(h)U^k
\]

for all \( h \in H \) and \( k = 0, 1, \ldots, r-1 \). We claim \( R \) is a representation of \( <H, g> \). Observe

\[
R(h_1g^s)R(h_2g^t) = A(h_1)U^sA(h_2)U^t \tag{2}
\]

and

\[
R(h_1g^s)R(h_2g^t) = R(h_1h_2g^{s+t}) = A(h_1)A(h_2)U^{s+t}, \tag{3}
\]

where \( h_2' = g^r h_2 g^{-r} \). To obtain equality between (2) and (3), it remains to show that

\[
U^sA(h_2) = A(h_2')U^s.
\]

But, this follows as above. This establishes our claim that \( R \) is a representation of \( <H, g> \). Since the restriction of \( R \) to \( H \) is \( A \), the character afforded by \( R \) extends \( \chi \). The proof of the lemma is complete.

Now, to complete the proof of Theorem 3, we appeal to the implication \( c \to b \) of Theorem 1.\(^2\)

Since \( gHg^{-1} = H \), this implication establishes that \( \chi^G \) is irreducible if \( \chi \neq \chi^g \) for all \( g \in G \setminus H \), i.e., by the lemma, if \( \chi \) cannot be extended to \( <H, h> \) for all \( g \in G \setminus H \).

**Corollary:** Let \( H \) be a normal subgroup of \( G \). Suppose \( [G: H] \) is prime. Let \( \lambda \) be an irreducible character of \( G \). Then either the restriction of \( \lambda \) to \( H \) is irreducible or \( \lambda = \chi^G \) for some irreducible character \( \chi \) of \( H \).

\(^2\) For a slightly different proof, one could appeal at this point to [2, (9.11)].
5. References


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