Distance Coordinates With Respect to a Triangle of Reference

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(July 21, 1972)

With respect to a triangle of reference \( A_1 A_2 A_3 \), each point \( P \) in the plane of the triangle, has unique area coordinates: \( P = (b_1, b_2, b_3) \) with \( b_1 + b_2 + b_3 = 1 \). Distance coordinates are introduced such that \( P = \{d_1, d_2, d_3\} \), with \( d_k \) the distance from \( P \) to \( A_k \). It is shown that there is an explicit function \( f(x_1, x_2, x_3) \) such that \( f(d_1, d_2, d_3) = 0 \) is necessary and sufficient for \( P = \{d_1, d_2, d_3\} \), each \( d_k \) nonnegative. The partial derivatives \( f_k(x_1, x_2, x_3) = \partial f(x_1, x_2, x_3)/\partial x_k \) are such that \( b_k = f_k(d_1, d_2, d_3) \) for each \( k \). Other results relating the \( b_k \) and the \( d_k \) are given. The use of \( f(x_1, x_2, x_3) \) in solving geometric problems is shown.

Key words: Area coordinates; distance coordinates; Plane Geometry; radical center; triangle of reference.

We are given three noncollinear points \( A_1, A_2, A_3 \) and all other points are in the plane of the triangle of reference \( A_1 A_2 A_3 \).

Notationally, two distinct points \( X, Y \) determine an infinite line \( XY \), with the finite line segment \( XY \) having length \( |X, Y| \). If \( X, Y \) are centers of circles with radii \( x, y \) respectively, the radical axis of those circles is a line perpendicular to \( XY \), at a point which is \((x_1^2 + x_2^2 - y_2^2)/2 |X, Y| \) from \( X \) in the direction of \( Y \). Given a third point \( Z \), not on \( XY \), as the center of a circle, the three radical axes meet in a common point called their radical center. The area of the triangle \( XYZ \) is denoted by \( |X, Y, Z| \). The function

\[
F(x_1, x_2, x_3) = 2(x_1x_2 + x_1x_3 + x_2x_3) - x_1^2 - x_2^2 - x_3^2
\]

has the well-known property

\[
16 |X, Y, Z|^2 = F( |X, Y|^2, |X, Z|^2, |Y, Z|^2).
\]

Let \( \Delta = |A_1, A_2, A_3| \) denote the area of the triangle of reference. With respect to this triangle every point \( P \) has unique area coordinates\(^1\) \( b_1, b_2, b_3 \), which are real numbers restricted by

\[
b_1 + b_2 + b_3 = 1.
\]

We write

\[
P = (b_1, b_2, b_3),
\]

with \( A_1 = (1, 0, 0), A_2 = (0, 1, 0), A_3 = (0, 0, 1) \).

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\(^1\) Also called "normalized barycentric" or "areal" coordinates \( [1] \)

The area coordinate \( b_1 \) is defined as the ratio \( \pm |P, A_2, A_3|/\Delta \) with \( b_1 > 0 \) if \( \overline{A_1P} \) does not intersect \( A_2A_3 \), \( b_1 \leq 0 \) if it does. Similarly for the other \( b_k \), so that the diagram of signs is

\[
\begin{array}{c c c}
(+,+,+)
\end{array}
\]

Conversely, any three real numbers \( b_1, b_2, b_3 \), restricted by (3), define a unique point \( P = (b_1, b_2, b_3) \) with respect to \( A_1A_2A_3 \).

If a point \( P \) is at a distance \( d_1 \) from \( A_1 \), \( d_2 \) from \( A_2 \), and \( d_3 \) from \( A_3 \), we call these the distance coordinates of \( P \) with respect to \( A_1A_2A_3 \), and write

\[
P = [d_1, d_2, d_3],
\]

or, when more convenient,

\[
P = <d_1^2, d_2^2, d_3^2>.
\]

For clarity, \( d_k = |P, A_k| \) for all \( k = 1, 2, 3 \).

Note that the triangle \( PA_2A_3 \) has side lengths \( a_1, d_2, d_3 \) so that

\[
16\Delta^2 b_1^2 = F(a_1^2, d_2^2, d_3^2)
\]

with similar equations for the other \( b_k \). Also note that \( P \) is the radical center of circles with centers \( A_1, A_2, A_3 \) and radii \( d_1, d_2, d_3 \) respectively.

Now consider the general case of circles with centers \( A_1, A_2, A_3 \) and radii \( r_1, r_2, r_3 \). Denote their radical center by

\[
P_0 = [\delta_1, \delta_2, \delta_3].
\]

In what follows there is no loss of generality in assuming that \( P_0 \) is in the interior of \( A_1A_2A_3 \). Near \( A_1 \) we have
Let $\alpha_1$ denote the interior angle at $A_1$. Then $|H_2, H_3| = \delta_1 \sin \alpha_1$. The formula for $\cos \alpha_1$ in the triangle $A_1H_2H_3$ is

\[
2 \left( \frac{a_2^2 + r_1^2 - r_3^2}{2a_2} \right) \left( \frac{a_3^2 + r_1^2 - r_2^2}{2a_3} \right) \cos \alpha_1 = \left( \frac{a_2^2 + r_1^2 - r_3^2}{2a_2} \right)^2 + \left( \frac{a_3^2 + r_1^2 - r_2^2}{2a_3} \right)^2 - (\delta_1 \sin \alpha_1)^2.
\]

In order to simplify this equation we shall need a few definitions and formulas. Define

\[
c_1 = a_2^2 + a_3^2 - a_1^2
\]

(7)

\[
c_2 = a_1^2 + a_3^2 - a_2^2
\]

\[
c_3 = a_1^2 + a_2^2 - a_3^2.
\]

Note that

\[
c_1 = 2a_2^2 - c_3 = 2a_3^2 - c_2
\]

(8)

and that

\[
2a_2a_3 \cos \alpha_1 = c_1.
\]

(9)

Since

\[
a_2a_3 \sin \alpha_1 = 2\Delta,
\]

(10)

we have

\[
c_1^2 + 16\Delta^2 = 4a_2^2a_3^2.
\]

(11)

Now return to eq (6). Multiply through by $4a_2^2a_3^2$, and use eqs (7) through (11) to simplify. We get $16\Delta^2f(r_1^2, r_2^2, r_3^2) = 16\Delta^2(r_1^2 - \delta_1^2)$, where

\[
16\Delta^2f(x_1, x_2, x_3) = \sum_{k=1}^{3} a_k^2 c_k x_k - a_1^2a_2^2a_3^2
\]

\[
-\frac{1}{2} \{c_1(x_2 - x_3)^2 + c_2(x_1 - x_3)^2 + c_3(x_1 - x_2)^2\}.
\]

Generally:

\[
f(r_1^2, r_2^2, r_3^2) = r_k^2 - \delta_k^2, \quad k = 1, 2, 3.
\]

(12)

If $f(r_1^2, r_2^2, r_3^2) = 0$ then $r_k = \delta_k$ for all $k = 1, 2, 3$ so that $P_0 = \{r_1, r_2, r_3\}$.

Now notice that

\[
\sum_{k=1}^{3} a_k^2 c_k = F(a_1^2, a_2^2, a_3^2) = 16\Delta^2,
\]

(13)

The circle with diameter $A_1P_0$ goes through $H_1$ and $H_3$. Thus $H_2H_3$ is a chord with opposite angle $\alpha_1$. 

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so that

\[(15) \quad f(x_1 - t, x_2 - t, x_3 - t) = f(x_1, x_2, x_3) - t \quad \text{all } t.\]

Let \(x_k = r_k^t\) for all \(k = 1, 2, 3\) and let \(t = f(r_1^t, r_2^t, r_3^t)\). Since (13) implies \(x_k - t = \delta_k^t\) for all \(k = 1, 2, 3\), we have \(f(\delta_1^t, \delta_2^t, \delta_3^t) = 0\). As we pointed out before, any point \(P = [d_1, d_2, d_3]\) can be a radical center. The above shows that \(f(d_1^t, d_2^t, d_3^t) = 0\).

Combining this with our remarks following eq (13), we have proved the first part of the following theorem:

**Theorem.** Let \(d_1, d_2, d_3\) be any real nonnegative numbers. Then there is a point \(P = [d_1, d_2, d_3]\) if and only if \(f(d_1^t, d_2^t, d_3^t) = 0\) (\(f\) defined in (12)). In that case \(P = (f_1(d_1^t, d_2^t, d_3^t), f_2(d_1^t, d_2^t, d_3^t), f_3(d_1^t, d_2^t, d_3^t))\) where \(f_k(x_1, x_2, x_3) = \partial f(x_1, x_2, x_3)/\partial x_k\) for all \(k = 1, 2, 3\).

For clarity, we write out the \(f_k:\)

\[(16) \quad 16\Delta^2 f_1(x_1, x_2, x_3) = a_1^2 c_1 + c_3(x_2 - x_1) + c_2(x_3 - x_1)\]
\[(17) \quad 16\Delta^2 f_2(x_1, x_2, x_3) = a_2^2 c_2 + c_3(x_1 - x_2) + c_1(x_3 - x_2)\]
\[(18) \quad 16\Delta^2 f_3(x_1, x_2, x_3) = a_3^2 c_3 + c_2(x_1 - x_3) + c_1(x_2 - x_3).\]

Note that (16) implies

\[(19) \quad f_1(x_1, x_2, x_3) + f_2(x_1, x_2, x_3) + f_3(x_1, x_2, x_3) = 1.\]

Also note that

\[(20) \quad f_k(x_1 - t, x_2 - t, x_3 - t) = f_k(x_1, x_2, x_3) \quad \text{all } t; \ k = 1, 2, 3.\]

In the process of proving the last part of the theorem, we shall need the following computations. The locus of points with \(b_3 = 1\) is a line through \(A_3\) parallel to \(A_1A_2\). Naturally if \(P = (b_1, b_2, b_3)\) then \(b_1 + b_2 = 0\) (see (3)). Suppose \(b_1 \leq 0\).

The area of \(PA_2A_3\) is \(\Delta \cdot | - b_2|\). It is also \(\frac{1}{2} a_1d_3\sin \alpha_2\).

Therefore

\[(19) \quad d_3 = a_3b_2.\]

Using the cosine formula in that triangle yields

\[(20) \quad d_2^2 = d_3^2 + a_1^2 - 2a_1d_3\cos \alpha_2\]
\[(21) \quad = a_1^2b_2^2 + a_2^2 - b_2c_2.\]

Using the cosine formula in the triangle \(A_1A_3P\) we have

\[(22) \quad d_1^2 = d_3^2 + a_2^2 - 2a_2d_3\cos (\alpha_2 + \alpha_3)\]

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Now use eq (18) with each $x_k = d_k^2$ and $t = d_3^2 = a_3^2b_2^2$. For $k = 1$ we have (using (20) and (21)):

$$16\Delta_x f_1(d_1^2, d_2^2, d_3^2) = 16\Delta_x f_1(a_2^2 + b_2c_1, a_1^2 - b_2c_2, 0)$$

$$= a_1^2c_1 + c_3(a_1^2 - a_2^2 - b_2(c_1 + c_2)) - c_2(a_2^2 + b_2c_1)$$

(22)

(using (16)). Since $a_1^2c_1 + c_3(a_1^2 - a_2^2) = c_2a_3^2$ (note $c_1 + c_3 = 2a_2^2$, $c_2 + c_3 = 2a_1^2$), and since

$$c_1c_2 + c_1c_3 + c_2c_3 = 16\Delta_x$$

$$(c_1 + c_2 = 2a_3^2$$, then add the three symmetric formulas, and use eq (14), we have

$$f_1(d_1^2, d_2^2, d_3^2) = -b_2 = b_1$$

as desired.

Similar computations prove $f_2(d_1^2, d_2^2, d_3^2) = b_2$ and (as a check), $f_3(d_1^2, d_2^2, d_3^2) = 1 = b_3$. The case $b_2 \leq 0$ is handled similarly to prove the last part of the theorem for the case $b_3 = 1$.

Return to eq (12) and solve $f(x_1, x_2, x_3) = 0$ for $x_1$. An intermediary stage is the equation

$$16\Delta_x \{f_1(x_1, x_2, x_3)\}^2 = 2a_1^2(x_2 + x_3) + 2x_2x_3 - a_1^2 - x_2^2 - x_3^2.$$  

The r.h.s. is recognized to be $F(a_1^2, x_2, x_3)$. Let $x_k = d_k^2$ for all $k = 1, 2, 3$, and use (4), to get $\{f_1(d_1^2, d_2^2, d_3^2)\}^2 = b_1^2$. Generally

$$f_k(d_1^2, d_2^2, d_3^2) = \pm b_k \quad k = 1, 2, 3$$

Set $f_k = f_k(d_1^2, d_2^2, d_3^2)$ for all $k = 1, 2, 3$. Equations (3) and (17), showing $\Sigma b_k = \Sigma f_k = 1$, imply that we cannot have $f_k = -b_k$ for all $k$. Suppose $f_3 = b_3$. Then

$$f_1 + f_2 = 1 - f_3 = 1 - b_3 = b_1 + b_2.$$  

If $f_1 = b_1$ then $f_2 = b_2$, and conversely. The only open case is $f_1 = -b_1$, $f_2 = -b_2$. This implies $b_1 + b_2 = 0$, whence $b_3 = 1$. We have already covered this case, so the proof of the theorem is complete.

An interesting implication for $P_0$ is immediate. Use eq (18) with $x_k = r_k^2$ for all $k = 1, 2, 3$ and $t = f(r_1^2, r_2^2, r_3^2)$ as before. The result is $f_k(\delta_{1k}^2, \delta_{2k}^2, \delta_{3k}^2) = f_k(r_1^2, r_2^2, r_3^2)$ for all $k = 1, 2, 3$. In other words the area coordinates for the radical center of three circles with centers at $A_1, A_2, A_3$ and radii $r_1, r_2, r_3$ respectively are given by

$$(24) \quad P_0 = (f_1(r_1^2, r_2^2, r_3^2), f_2(r_1^2, r_2^2, r_3^2), f_3(r_1^2, r_2^2, r_3^2)).$$

Of course, the distance coordinates are given by

$$(25) \quad P_0 = <r_1^2 - f(r_1^2, r_2^2, r_3^2), r_2^2 - f(r_1^2, r_2^2, r_3^2), r_3^2 - f(r_1^2, r_2^2, r_3^2)>.$$  

If we wish to find the (0 to 8) circles simultaneously tangent to the three circles used above, we can do so through $f$, to obtain four quadratic equations whose solutions solve the problem. The point is that a circle of radius $r$ which is simultaneously tangent to all three circles has a center
with each $\epsilon_k = \pm 1$ depending on whether the tangency is “outside” ($\epsilon = 1$) or “inside” ($\epsilon = -1$). Simplifying

$$f((r_1 + \epsilon_1 r)^2, (r_2 + \epsilon_2 r)^2, (r_3 + \epsilon_3 r)^2) = 0,$$

we get a quadratic equation in $r$ (with constant term $f(r_1^2, r_2^2, r_3^2)$). If $r$ is a negative root of that equation, we simply “assign” $-r$ to $-\epsilon_1, -\epsilon_2, -\epsilon_3$ since $\epsilon_k r = (-\epsilon_k)(-r)$, $k = 1, 2, 3$.

Thus we can cover all solutions with just four triples of $\epsilon$’s, no two of which are negatives of each other.

We end this note with a list of formulas connecting the area and distance coordinates of a point

$$P = (b_1, b_2, b_3) = [d_1, d_2, d_3].$$

The formulas are given without proof, but are easily derived, with extensive use of the formula for the distance between $P$ and $P' = (b'_1, b'_2, b'_3)$:

$$2 |P, P'|^2 = \sum_{k=1}^{3} c_k(b_k - b'_k)^2.$$  

First we complete the connection between the coordinates, begun in the last formula of the theorem, with

$$d_1^2 = a_1^2 b_2^2 + c_1 b_2 b_3 + a_2^2 b_3^2$$

$$d_2^2 = a_2^2 b_1^2 + c_2 b_1 b_3 + a_1^2 b_3^2$$

$$d_3^2 = a_3^2 b_1^2 + c_3 b_1 b_2 + a_1^2 b_2^2$$

or

$$2d_k^2 = (1 - 2b_k)c_k + \sum_{n=1}^{3} b_k^2 c_n$$  

Let $R$ denote the circumradius of $A_1A_2A_3$, and $\rho_P$ the distance from $P$ to the circumcenter. The latter has area coordinates $a_k^2 c_k/16\Delta^2$, $k = 1, 2, 3$. Also $4\Delta R = a_1 a_2 a_3$.

Define

$$G_P = R^2 - \rho_P^2.$$  

Then $G_P$ can be found using only the $b_k$:

$$G_P = a_1^2 b_2 b_3 + a_2^2 b_1 b_3 + a_3^2 b_1 b_2$$

$$-\frac{1}{2} \sum_{k=1}^{3} (b_k - b_k^2)c_k;$$

or only the $d_k$:

$$G_P = R^2 - (c_1(d_2 - d_3)^2 + c_2(d_1 - d_3)^2 + c_3(d_1 - d_2)^2)/32\Delta^2$$

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or symmetric combinations:

\[
G_p = \sum_{k=1}^{3} b_k d_k^2
\]

\[
= \sum_{k=1}^{3} (a_k^2 - 2d_k^2 + b_k^2 c_k)/4
\]

\[
= \sum_{k=1}^{3} (a_k^2 - 2d_k^2 + b_k c_k)/6
\]

\[
= \sum_{k=1}^{3} (a_k^2 - d_k^2 - b_k a_k^2)/3
\]

\[
= 2 \left( b_1 a_2 a_3 + b_2 a_1 a_3 + b_3 a_1 a_2 - \sum_{k=1}^{3} a_k^2 d_k^2 \right) / \sum_{k=1}^{3} a_k^2
\]

\[
= \left[ b_1 b_2 b_3 \sum_{k=1}^{3} a_k^2 + \sum_{k=1}^{3} b_k^2 d_k^2 \right] / 2 (b_1 + b_2 + b_3)
\]

(36) \quad = (2a_1^2 a_2^2 a_3^2 - \sum a_k^2 c_k d_k^2)/16\Delta^2;

or symmetric combinations:

\[
\\text{except at a vertex). Other relations are}
\]

\[
\begin{align*}
\text{(43)} & \quad d_i^2 = b_2 a_3^2 + b_3 a_2^2 - G_p \quad \text{etc.} \\
\text{(44)} & \quad b_1 c_i = 2G_p - a_i^2 + d_i^2 + d_3^2 \quad \text{etc.} \\
\text{(45)} & \quad (1 - b_1) G_p = b_1 d_1^2 + b_2 b_3 a_1^2 \quad \text{etc.} \\
\text{(46)} & \quad 2a_2^2 a_3^2 b_1 = c_i G_p + a_2^2 d_2^2 + a_3^2 d_3^2 - a_1^2 d_1^2 \quad \text{etc.}
\end{align*}
\]

The pedal triangle of \( P \), which has side lengths \( a_k d_k/2R \), \( k = 1, 2, 3 \), has area \( \Delta |G_p|/4R^2 \); i.e.,

\[
\begin{align*}
\text{(47)} & \quad 16\Delta^2 G_p^2 = F(a_1^2 d_1^2, a_2^2 d_2^2, a_3^2 d_3^2) \\
\text{(48)} & \quad = 4a_2^2 a_3^2 d_2^2 d_3^2 - (a_2^2 d_2^2 + a_3^2 d_3^2 - a_1^2 d_1^2)^2 \quad \text{etc.}
\end{align*}
\]

If \( P \) does not lie on the triangle \( A_1 A_2 A_3 \) then the lines \( A_1 P, A_2 P, A_3 P \) intersect the circle of radius \( \rho_p \), concentric with the circumcircle, in \( P \) and in points \( A'_1, A'_2, A'_3 \) respectively which have (opposite) side lengths \( \lambda_p |b_k| d_k \) for \( k = 1, 2, 3 \); \( \lambda_p = 4\Delta p /d_1 d_2 d_3 \). The area of \( A'_1 A'_2 A'_3 \) is \( \Delta \lambda_p^2 |b_1 b_2 b_3| \). Thus

\[
\begin{align*}
\text{(49)} & \quad 16\Delta^2 (b_1 b_2 b_3)^2 = F(b_1^2 d_1^2, b_2^2 d_2^2, b_3^2 d_3^2).
\end{align*}
\]

Finally, suppose \( d_k^2 = g_k(t) \), a differentiable function of \( t \), for \( k = 1, 2, 3 \). The \( b_k \) will also be differentiable functions of \( t \) (by the last statement of the theorem), and \( b'_k \) will denote the derivative. We have
(50) \[ \sum_{k=1}^{3} b_k g'_k(t) = 0 \]

and

(51) \[ g'_1(t) + c_1 b'_1 = g'_2(t) + c_2 b'_2 = g'_3(t) + c_3 b'_3 = \sum_{k=1}^{3} c_k b_k b'_k. \]

(Paper 76B3&4-367)