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A Table of Integrals of the Error Function. II. Additions and Corrections*

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This is an extension of a compendium of indefinite and definite integrals of products of the error function with elementary or transcendental functions recently published by the authors.

Key words: Error functions; indefinite integrals; special functions.

1. Introduction

Since the present authors published an extensive compendium of integrals involving the error function [1],[1] numerous comments and suggestions have been received. In particular, we have been advised by P. I. Hadji (П. И. ХАДЖИ) of his publication in Russian [2], which contains formulas not included in [1]. Careful examination of [2] has revealed numerous mathematical and typographical errors. Bearing in mind also the inaccessibility of his report, the authors believe it to be appropriate to publish an extension of [1] based largely on corrected versions of Hadji's formulas but also containing new results. It should be noted that the authors have verified and/or corrected all of Hadji's formulas in addition to rederiving and checking all of the formulas in [1].

Throughout this article, we conform to the format and notation of [1], including section and equation numbers. A supplementary glossary is also included. Within each subsection, typographical errors, equivalent forms and errors detected in [1] will be listed under corrections.

2. Supplementary Glossary

$B(p, q)$	Beta Function	$\Gamma(p)\Gamma(q)/\Gamma(p+q)$
$\beta(x)$		$\frac{1}{2}[\psi(\frac{1}{2}+\frac{1}{2}x)-\psi(\frac{1}{2}x)]$
$C_n^\lambda(x)$	Gegenbauer Polynomial	
G	Catalan's Constant	0.915965594 . . .
$P_n^{(\alpha,\beta)}(x)$	Jacobi Polynomial	
$T_n(x)$	Chebyshev Polynomial	

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[1] Figures in brackets indicate the literature references at the end of this paper.

3.* Integrals of Products of Error Functions with Other Functions

3.1. Combination of Error Function with Powers

Additions

(20) $$\int_a^\infty \operatorname{erf}(x)x^{-p}dx=\frac{1}{p-1}\left[\frac{\operatorname{erf}(a)}{a^{p-1}}+\frac{1}{\sqrt{\pi}}\frac{e^{-a^2/2}}{a^{p/2}}W_{-p/4,\,(p-2)/4}(a^2)\right], \qquad p>1$$

(21) $$\int_0^\infty [\operatorname{erf}(ax+c)-\operatorname{erf}(bx+c)]\frac{dx}{x}=ln\left(\frac{a}{b}\right)\operatorname{erfc}(c)$$

(22) $$\int_0^\infty [b\operatorname{erf}(ax)-a\operatorname{erf}(bx)]\frac{dx}{x^2}=\frac{2ab}{\sqrt{\pi}}ln\left(\frac{b}{a}\right)$$

(23) $$\int_0^\infty [\sqrt{x}\operatorname{erf}(\sqrt{x})-\operatorname{erf}(x)]\frac{dx}{x^2}=\frac{1}{\sqrt{\pi}}(2-\gamma)$$

(24) $$\int_0^\infty \left[\frac{\operatorname{erf}(x)}{x}-\frac{2}{\sqrt{\pi}}\frac{1}{1+x^2}\right]\frac{dx}{x}=\frac{1}{\sqrt{\pi}}(2-\gamma)$$

(25) $$\int_0^u (u-x)^p\operatorname{erf}[\sqrt{(a/x)}]dx=\frac{u^{p+1}}{p+1}\left[1-\frac{\Gamma(p+2)}{\sqrt{\ \ \pi}}\left(\frac{u}{a}\right)^{1/4}e^{-a/(2u)}W_{-p-(5/4),\,1/4}\left(\frac{a}{u}\right)\right], \qquad p>-1$$

(26) $$\int_0^\infty [(b+ix)^{-p-1}-(b-ix)^{-p-1}]\operatorname{erf}(ax)dx=-\frac{2i}{p}(a\sqrt{2})^pe^{a^2b^2/2}D_{-p}(ab\sqrt{2}), \qquad p>0$$

(27) $$\int_0^\infty x[(b+ix)^{-p-1}+(b-ix)^{-p-1}]\operatorname{erf}(ax)dx$$

$$=\frac{2}{p(p-1)}(\alpha/b)^{p-1}e^{a^2b^2/2}[(p-1)\alpha D_{-p}(\alpha)-pD_{1-p}(\alpha)], \qquad p>1,\ \alpha=ab\sqrt{2}$$

(28) $$\int_{-\infty}^\infty \frac{\operatorname{erf}(ax)}{(x+b)^2}dx=2a\sqrt{\pi}e^{-a^2b^2}\operatorname{erfi}(ab)$$

(29) $$\int_0^\infty \frac{\operatorname{erfc}[a(1+x)^{1/4}]}{(1+x)^{3/4}}dx=\frac{4}{a\sqrt{\pi}}e^{-a^2}-4\operatorname{erfc}(a)$$

(30) $$\int \operatorname{erf}[\sqrt{(ax)}]\frac{x-b}{\sqrt{x}(x+b)^2}dx=-\frac{2\sqrt{x}}{x+b}\operatorname{erf}[\sqrt{(ax)}]-2\sqrt{(a/\pi)}e^{ab}E_1(ab+ax)$$

(31) $$\int_0^u x^{2\nu-1}(u^2-x^2)^{\mu-1}\operatorname{erf}(ax)dx$$

$$=\frac{a}{\sqrt{\pi}}u^{2\mu+2\nu-1}B(\mu,\nu+\tfrac{1}{2})_2F_2(\tfrac{1}{2},\nu+\tfrac{1}{2};\frac{3}{2},\mu+\nu+\tfrac{1}{2};-a^2u^2),\ \mu>0,\nu>-\tfrac{1}{2}$$

(32) $$\int_0^\infty \frac{x\operatorname{erf}(ax)}{(x^2+b^2)^2}dx=\frac{a}{2b}\sqrt{\pi}\,e^{a^2b^2}\operatorname{erf}c(ab)$$

*Section 3 corresponds to section 4 of reference [1].

(33) $$\int_0^1 \frac{x\,\mathrm{erf}\,(ax)}{(x^2+1)^2}\,dx = \frac{a}{4}\sqrt{\pi}\,e^{a^2}\,[1-\mathrm{erf}^2\,(a)] - \tfrac{1}{4}\,\mathrm{erf}\,(a)$$

(34) $$\int_1^\infty \frac{x\,\mathrm{erf}\,(ax)}{(x^2+1)^2}\,dx = \frac{a}{4}\sqrt{\pi}\,e^{a^2}[1-\mathrm{erf}\,(a)]^2 + \tfrac{1}{4}\,\mathrm{erf}\,(a)$$

(35) $$\int_0^\infty \frac{x\,\mathrm{erf}\,(ax)}{(x^2+b^2)^{p+1}}\,dx = \frac{a}{2}\,b^{1-2p}\,\frac{\Gamma(p-\frac{1}{2})}{\Gamma(p+1)}\,{}_1F_1(\tfrac{1}{2};\tfrac{3}{2}-p;a^2b^2)$$

$$+\frac{a^{2p}}{2\sqrt{\pi}}\,\frac{\Gamma(\frac{1}{2}-p)}{\Gamma(p+1)}\,{}_1F_1(p;p+\tfrac{1}{2};a^2b^2),\ p>0,\ p\neq n+\tfrac{1}{2}$$

(36) $$\int_0^\infty \frac{\mathrm{erf}\,(ax)}{(x^2+b^2)^{3/2}}\,dx = \frac{1}{b^2}\,[1-\exp\,(a^2b^2)\,\mathrm{erf}\,c\,(ab)]$$

(37) $$\int_0^\infty \frac{x\,\mathrm{erf}\,(ax)}{(x^2+b^2)^{3/2}}\,dx = \frac{a}{\sqrt{\pi}}\,\exp\,(a^2b^2/2)K_0\left(\frac{a^2b^2}{2}\right)$$

(38) $$\int_0^\infty \frac{x\,\mathrm{erf}\,[a\sqrt{(x^2+b^2)}]}{(x^2+b^2)^{3/2}}\,dx = \frac{1}{b}\,\mathrm{erf}\,(ab) + \frac{a}{\sqrt{\pi}}\,E_1(a^2b^2)$$

(39) $$\int_0^\infty \frac{x\,\mathrm{erf}\,[\sqrt{(ax)}]}{(x^2+4b^2)^{3/2}}\,dx = \frac{a\pi}{2\sqrt{2}}\,[J_{1/4}(ab)Y_{-1/4}(ab) - J_{-1/4}(ab)Y_{1/4}(ab)]$$

3.2. Combination of Error Function with Exponentials and Powers

Corrections

(10) Replace $(bz^n + nz^{n-1})$ in last expression by $(bz^{n-1} + (n-1)z^{n-2})$.

(12) Replace $(bz^n + nz^{n-1})$ in last expression by $\dfrac{1}{a\sqrt{\pi}}\,(bz^{n-1} + (n-1)z^{n-2})$.

(13) Replace $(2a)^{(k-n)/2}$ by $(2a^2)^{(k-n)/2}$.

Additions

(15) $$\int_0^\infty [\mathrm{erf}\,(e^{-ax}) - \mathrm{erf}\,(e^{-bx})]\,\frac{dx}{x} = \mathrm{erf}\,(1)\ln\left(\frac{b}{a}\right)$$

(16) $$\int_{-\infty}^\infty \mathrm{erf}\left[\frac{ae^{-ipx}}{(b\pm ix)^\alpha}\right]\frac{dx}{x^2+c^2} = \frac{\pi}{c}\,\mathrm{erf}\left[\frac{ae^{\mp pc}}{(b\pm c)^\alpha}\right],\ p>0,\ \alpha\geqslant 0,\ b\neq c \text{ for the lower sign}$$

(17) $$\int_{-\infty}^\infty \mathrm{erf}\left(\frac{a}{b\pm ix}\right)e^{-ipx}\,\frac{dx}{x^2+c^2} = \frac{\pi}{c}\,e^{\mp pc}\,\mathrm{erf}\left(\frac{a}{b\pm c}\right),\ p>0,\ b\neq c \text{ for the lower sign}$$

(18) $$\int_{-\infty}^\infty (ix)^{-p}\,\mathrm{erf}\,(ae^{-ibx})\,\frac{dx}{x^2+c^2} = \frac{\pi}{c^{p+1}}\,\mathrm{erf}\,(ae^{-bc}),\ |p|<1$$

(19) $$⨍_{-\infty}^\infty \mathrm{erf}\,(ae^{\pm ibx})\,\frac{dx}{x-c} = \pm\, i\pi\,\mathrm{erf}\,(ae^{\pm ibc})$$

(20) $$\int_0^\infty \frac{dx}{\sin h\frac{1}{2}\pi x}\left[\frac{\operatorname{erf}(1+ix)}{(1+ix)^2}-\frac{\operatorname{erf}(1-ix)}{(1-ix)^2}\right]=2i\operatorname{erf}(1)$$

3.3. Combination of Error Function with Exponentials of More Complicated Arguments

Corrections

(2) Equivalent form: $$\int_0^\infty \operatorname{erf}(ax)e^{-b^2x^2}dx=\frac{1}{b\sqrt{\pi}}\tan^{-1}\left(\frac{a}{b}\right)$$

(3) Remove condition "b may be complex" and replace by $\mathscr{R}(a^2)>\mathscr{R}(b^2)$

(8) For the form given, add the condition $\mathscr{R}(b^2)>\mathscr{R}(a^2)$

Equivalent form:

$$\int_0^\infty \operatorname{erf}(ax)e^{-b^2x^2}x^p dx=\frac{a}{b^{p+1}\sqrt{\pi}}\frac{\Gamma(1+\frac{1}{2}p)}{(a^2+b^2)^{\frac{1}{2}}}{}_2F_1\left(\tfrac{1}{2}-\tfrac{1}{2}p,\tfrac{1}{2};\tfrac{3}{2};\frac{a^2}{a^2+b^2}\right),$$

$$\mathscr{R}(a^2)>0,\ \mathscr{R}(b^2)>0,\ p>-2.$$

(16) $$\int_1^\infty \operatorname{erf}c\,(ax)e^{a^2x}x^{-3}dx=\frac{1}{2}(1+2a^2)e^{a^2}\operatorname{erf}c\,(a)-a\pi^{-1/2}$$

(35) Add the condition $\mathscr{R}(c^2)>\mathscr{R}(a^2)$

(36) Replace $K_b(a^2)$ by $K_b(\frac{1}{2}a^2)$

Additions

(39) $$\int x\exp(x^2)\operatorname{erf}(x)dx=\frac{1}{2}\exp(x^2)\operatorname{erf}(x)-\frac{1}{\sqrt{\pi}}x$$

(40) $$\int_{-a}^{a}\exp(x^2)\operatorname{erf}c(x)dx=\sqrt{\pi}\operatorname{erf}i\,(a)$$

(41) $$\int[\operatorname{erf}(ax)]^n\exp(-a^2x^2)dx=\frac{\sqrt{\pi}}{2a(n+1)}[\operatorname{erf}(ax)]^{n+1}$$

(42) $$\int[a\operatorname{erf}(bx)\exp(-a^2x^2)+b\operatorname{erf}(ax)\exp(-b^2x^2)]dx=\frac{\sqrt{\pi}}{2}\operatorname{erf}(ax)\operatorname{erf}(bx)$$

(43) $$2b^2\int\operatorname{erf}(ax+c)\exp(-b^2x^2)xdx$$

$$=\frac{a}{\sqrt{(a^2+b^2)}}\operatorname{erf}\left(x\sqrt{(a^2+b^2)}+\frac{ac}{\sqrt{(a^2+b^2)}}\right)\exp\left(-\frac{b^2c^2}{a^2+b^2}\right)-\operatorname{erf}(ax+c)\exp(-b^2x^2)$$

(44) $$\int x^{2n}\operatorname{erf}(ax)\exp(-a^2x^2)\,dx=\frac{(2n)!}{n!}\frac{\sqrt{\pi}}{(2a)^{2n+1}}A,$$

where

$$A=\frac{1}{2}\operatorname{erf}^2(ax)-\frac{2a}{\sqrt{\pi}}x\operatorname{erf}(ax)\exp(-a^2x^2)\sum_{k=0}^{n-1}\frac{k!}{(2k+1)!}(4a^2x^2)^k$$

$$-\frac{\exp(-2a^2x^2)}{\pi}\sum_{k=0}^{n-1}2^k\frac{k!k!}{(2k+1)!}e_k(2a^2x^2)$$

(45) $$\int x^{2n+1}\operatorname{erf}(ax)\exp(-a^2x^2)\,dx=\frac{n!}{a^{2n+2}}B,$$

where

$$B=2^{-3/2}\operatorname{erf}(ax\sqrt{2})\sum_{k=0}^{n}\binom{2k}{k}8^{-k}-\frac{1}{2}\operatorname{erf}(ax)e^{-a^2x^2}e_n(a^2x^2)$$

$$-\frac{ax}{\sqrt{\pi}}e^{-2a^2x^2}\sum_{k=1}^{n}\binom{2k}{k}8^{-k}\sum_{l=0}^{k-1}\frac{l!}{(2l+1)!}(8a^2x^2)^l$$

(46) $$2b^2\int x^{2n+1}\operatorname{erf}(ax)\exp(-b^2x^2)\,dx=2n\int x^{2n-1}\operatorname{erf}(ax)\exp(-b^2x^2)\,dx$$

$$-x^{2n}\operatorname{erf}(ax)\exp(-b^2x^2)+\frac{2a}{\sqrt{\pi}}\int x^{2n}\exp[-(a^2+b^2)x^2]\,dx$$

(47) $$\int\operatorname{erf}(ax)\exp(-a^2x^2)\frac{dx}{x^2}=-\frac{1}{x}\operatorname{erf}(ax)\exp(-a^2x^2)-\frac{a\sqrt{\pi}}{2}\operatorname{erf}^2(ax)-\frac{a}{\sqrt{\pi}}E_1(2a^2x^2)$$

(48) $$\int\exp\{-b^2[\operatorname{erf}(ax)]^p\}\exp(-a^2x^2)[\operatorname{erf}(ax)]^q\,dx$$

$$=\frac{\sqrt{\pi}}{2ap}b^{-2(1+q)/p}\gamma\left(\frac{q+1}{p},\,b^2[\operatorname{erf}(ax)]^p\right),\ q>-1,\ p>0$$

(49) $$\int\frac{e^{-a^2x^2}}{\operatorname{erf}(ax)}\,dx=\frac{\sqrt{\pi}}{2a}\ln[\operatorname{erf}(ax)]$$

(50) $$\int_0^\infty\operatorname{erf}(ax)e^{-b^2x^2}\sin hcx\,dx=\frac{\sqrt{\pi}}{2b}\exp[c^2/(4b^2)]\operatorname{erf}\left[\frac{ac}{2b\sqrt{(a^2+b^2)}}\right]$$

(51) $$\int_0^\infty\operatorname{erf}\left[a\sqrt{(x^2+b^2)}\right]e^{-c^2x^2}x\,dx=\frac{1}{2c^2}\operatorname{erf}(ab)+\frac{a}{2c^2}\frac{e^{b^2c^2}}{\sqrt{(a^2+c^2)}}\operatorname{erfc}\left[b\sqrt{(a^2+c^2)}\right]$$

(52) $$\int_0^\infty\operatorname{erf}\left[\sqrt{(ax)}\right]e^{-b/x}\frac{dx}{x^2}=\frac{1}{b}\left[1-e^{-2\sqrt{(ab)}}\right]$$

(53) $$\int_{-\infty}^\infty\operatorname{erf}(ae^{x/2})e^{x/2}\exp(-e^x)\,dx=\frac{2}{\sqrt{\pi}}\tan^{-1}a$$

(54) $$\int_0^\infty\left[\frac{a\operatorname{erf}(\alpha\exp[-ce^{ax}])}{1-e^{-ax}}-\frac{b\operatorname{erf}(\alpha\exp[-ce^{bx}])}{1-e^{bx}}\right]dx=\ln\left(\frac{b}{a}\right)\operatorname{erf}(\alpha e^{-c})$$

(55) $$\int_0^\infty\operatorname{erf}i(ax)e^{-b^2x^2}\frac{dx}{x}=\sin^{-1}\left(\frac{a}{b}\right),\ b\geqslant a$$

(56) $$\int_a^\infty e^{x^2}(x^2-a^2)^{-1/2}\,\mathrm{erf}\,c\,(x)\,dx=\frac{1}{2}e^{a^2/2}K_0(a^2/2)$$

(57) $$\int_0^\infty \mathrm{erf}\,i\,(ax)e^{-a^2x^2}\frac{x}{x^2+b^2}\,dx=\frac{\pi}{2}\,\mathrm{erf}\,c\,(ab)e^{a^2b^2}$$

3.4. Definite Integrals from Laplace Transforms Involving Erf $(\sqrt{ax})$

Corrections

(12) Replace erf $(\sqrt{ax})$ in integrand by erfc $(\sqrt{ax})$ and remove the condition $b>a$.

(13) First term inside square brackets in integrand should read $(2ax+1)xe^{ax}\,\mathrm{erf}\,c\,(\sqrt{ax})$ instead of $(2a^{1/2}x^{1/2}+1)xe^{ax}\,\mathrm{erf}\,(\sqrt{ax})$ and remove the condition $b>a$.

(14) Last term inside square brackets in integrand should read

$$-2(ax^3/\pi)^{1/2}(2ax+5) \text{ instead of } -2(a^3x^5/\pi)^{1/2}(2ax+5).$$

(15) Replace erfc $(\sqrt{cx})$ in integrand by erf $(\sqrt{cx})$ and add the condition $b>c$.

(16) First term inside square brackets in integrand should read

$$a^{1/2}e^{cx}\,\mathrm{erf}\,(\sqrt{cx}) \text{ instead of } a^{1/2}e^{cx}\,\mathrm{erf}\,(\sqrt{bx}).$$

(18) First term inside brackets in integrand should read

$$a(x/\pi)^{1/2}e^{-a^2/(4x)}.$$

Additions

(19) $$\int_0^\infty e^{-x}\frac{\mathrm{erf}\,(\sqrt{x})}{(1+e^{-x})^2}\,dx=1+(\sqrt{2}-1)\zeta\left(\frac{1}{2}\right)$$

(20) $$\int_0^\infty e^{-x}\,\mathrm{erf}\,(\sqrt{ax})\left[\frac{1-e^{-nx}}{1-e^{-x}}\right]\frac{dx}{\sqrt{x}}=\frac{2}{\sqrt{\pi}}\sum_{k=1}^{n}\frac{1}{\sqrt{k}}\tan^{-1}\left[\sqrt{(a/k)}\right]$$

(21) $$\int_0^\infty e^{-bx}\,\mathrm{erf}\,(\sqrt{ax})\,e^{-ncx}[\sin hcx]^n\frac{dx}{\sqrt{x}}$$
$$=\frac{2}{\sqrt{\pi}}\frac{1}{2^n}\sum_{k=0}^{n}(-1)^k\binom{n}{k}(b+2ck)^{-1/2}\tan^{-1}\left(\frac{a}{b+2ck}\right)^{1/2}$$

3.5. Combination of Error Function with Trigonometric Functions

Corrections

(10) For the form given $a<b$.

For $a>b$:

$$\int_0^\infty \mathrm{erf}\,(ax)\sin b^2x^2\,dx=\frac{1}{4b\sqrt{(2\pi)}}\left[\ln\left(\frac{a^2+b^2+ab\sqrt{2}}{a^2+b^2-ab\sqrt{2}}\right)+2\tan^{-1}\left(\frac{ab\sqrt{2}}{b^2-a^2}\right)+2\pi\right]$$

(11) Change the condition on p to $\mathscr{R}(p) > -2$.

(12) Change the condition on p to $\mathscr{R}(p) > -1$.

(13) $$\int_0^\infty \operatorname{erf}(ax) \sin bx \frac{dx}{x^2} = \frac{b}{2} E_1\left(\frac{b^2}{4a^2}\right) + a\sqrt{\pi} \operatorname{erf}\left(\frac{b}{2a}\right).$$

(16) $$\int_0^\infty \operatorname{erf}\left(\sqrt{\frac{a}{x}}\right) \sin bx dx = \frac{1}{b}[1 - e^{-A}\cos A], \qquad A = (2ab)^{1/2}$$

(17) $$\int_0^\infty \operatorname{erf}\left(\sqrt{\frac{a}{x}}\right) \cos bx dx = \frac{1}{b} e^{-A} \sin A, \qquad A = (2ab)^{1/2}$$

(24) $$\int_0^\infty \operatorname{erf} c(ax) x \sin x \sin hx dx = \frac{1}{2}\left[\frac{1}{a^2} \sin\left(\frac{1}{2a^2}\right) + \cos\left(\frac{1}{2a^2}\right) - 1\right]$$

Additions

(31) $$2i(b^2 + c^2) \int \operatorname{erf}(ax) e^{-bx} \sin cx dx = (b + ic)A_- - (b - ic)A_+,$$

where

$$A_\pm = \exp\left[\left(\frac{b \pm ic}{2a}\right)^2\right] \operatorname{erf}\left(ax + \frac{b \pm ic}{2a}\right) - \exp\left[-(b \pm ic)x\right] \operatorname{erf}(ax)$$

(32) $$\int \tan[\operatorname{erf}(ax)] e^{-a^2x^2} dx = -\frac{\sqrt{\pi}}{2a} \ln\{\cos[\operatorname{erf}(ax)]\}$$

(33) $$\int_0^\infty \operatorname{erf}(ax) e^{-b^2x^2} \sin cx dx = \frac{\sqrt{\pi}}{2b} e^{-c^2/(4b^2)} \operatorname{erf} i\left[\frac{ac}{2b(a^2 + b^2)^{1/2}}\right]$$

(34) $$\int_0^\pi \operatorname{erf}(a \sin x) \cos(2n+1)x\, dx = \int_0^\pi \operatorname{erf}(a \sin x) \sin 2nx\, dx = 0$$

(35) $$\int_0^{2\pi} \operatorname{erf}(a \sin x + b \cos x) dx = 0$$

(36) $$\int_0^{2\pi} \operatorname{erf}(ae^{ix}) \sin nx\, dx = \begin{cases} 0 & n = 2k \\ 2\sqrt{\pi} \dfrac{(-1)^k}{k!} \dfrac{a^{2k+1}}{2k+1} i & n = 2k+1 \end{cases}$$

(37) $$\int_0^{2\pi} \operatorname{erf}(ae^{ix}) \cos nx\, dx = \begin{cases} 0 & n = 2k \\ 2\sqrt{\pi} \dfrac{(-1)^k}{k!} \dfrac{a^{2k+1}}{2k+1} & n = 2k+1 \end{cases}$$

(38) $$\int_0^{\pi/2} \operatorname{erf}(a \tan x) \sin 2x\, dx = \sqrt{\pi}\, ae^{a^2} \operatorname{erf} c(a)$$

(39) $$\int_0^{\pi/2} \operatorname{erf} c(a \tan x) \cos 2x\, dx = \pi^{-1/2} ae^{a^2} E_1(a^2)$$

(40) $\int_0^1 \operatorname{erf}(ax) \cos[(2n+1)\cos^{-1} x](1-x^2)^{-1/2} dx$

$$= (-1)^n \frac{\Gamma\left(n+\frac{1}{2}\right)}{(2n+1)!} \frac{e^{-a^2/2}}{2a} M_{-1/2,\, n+1/2}(a^2)$$

(41) $a\int_0^\infty \operatorname{erf} c[b^{1/2}\{(x^2+c^2)^{1/2}-c\}^{1/2}] \sin ax dx = 1 - \frac{1}{\Delta}\left[\frac{1}{2}b(b+\Delta)\right]^{1/2} e^{-c(\Delta-b)}, \; \Delta = (a^2+b^2)^{1/2}$

(42) $$\int_0^\infty [\operatorname{erf}(\tan^{-1} ax) - \operatorname{erf}(\tan^{-1} bx)] \frac{dx}{x} = \ln\left(\frac{a}{b}\right) \operatorname{erf}\left(\frac{\pi}{2}\right)$$

3.6. Combination of Error Function With Logarithms and Powers

Corrections

(5) Replace $\frac{1}{2}z^2 \operatorname{erf}(az) \ln z$ on the R.H.S. by $\frac{1}{2}z^2 \operatorname{erf} c(az) \ln z$.

Additions

(10) $$\int_0^\infty \operatorname{erf}(ax) \ln(2 \pm 2\cos x) \frac{x}{x^2+c^2} dx = -\pi \ln(1 \pm e^{-c}) \operatorname{erf} i(ac)$$

(11) $$\int_0^1 \operatorname{erf}(a \ln x) \frac{x^{b-1}}{1+x^c} dx = \frac{1}{c} \beta\left(\frac{b}{c}\right) \operatorname{erf}\left(\frac{a}{2c}\right)$$

(12) $$\int_0^\infty \operatorname{erf}(a \ln x) \frac{x^{b-1}}{1+x^c} dx = \frac{1}{c}\left[\beta\left(\frac{b}{c}\right) - \beta\left(1-\frac{b}{c}\right)\right] \operatorname{erf}\left(\frac{a}{2c}\right), \qquad c > b$$

(13) $$\int_0^\infty \ln\left[\frac{c + d\operatorname{erf}(ax)}{c + d\operatorname{erf}(bx)}\right] \frac{dx}{x} = \ln\left(\frac{a}{b}\right) \ln\left(\frac{c+d}{c}\right)$$

(14) $$\int_0^{\pi/2} \operatorname{erf}(b \sin 2x) \ln(a \tan x) dx = \frac{1}{2}(\ln a) \operatorname{erf}(2b) \frac{[\Gamma(\frac{1}{2}a)]^2}{\Gamma(a)}$$

(15) $$\int_0^{\pi/2} \operatorname{erf}(a \ln \tan x)\, dx = 0$$

(16) $$\int_0^\infty \operatorname{erf}\left(\frac{ax}{x^2+b^2}\right) \ln\left(\frac{x}{b}\right) \frac{dx}{x} = 0$$

(17) $$\int_0^\infty \operatorname{erf}\left(\frac{x^p}{x^{2p}+b^{2p}}\right) \ln\left(\frac{x}{b}\right) \frac{dx}{x^2+b^2} = 0$$

(18) $$\int_0^\infty \operatorname{erf}(x^p + x^{-p}) \ln x \frac{dx}{x} = 0$$

(19) $$\int_0^\infty \operatorname{erf}(x^p + x^{-p}) \ln x \frac{dx}{1+x^2} = 0$$

(20) $$\int_0^\infty \frac{\operatorname{erf}(a \ln x)}{\ln x} \frac{dx}{1-x^2} = 0$$

(21) $$\int_0^\infty \operatorname{erf}(a \ln x) \frac{dx}{1+bx+x^2} = 0, \ |b| < 2$$

(22) $$2a^2 \int_0^\infty x \ln x \operatorname{erf} c\,(x)\, e^{-a^2x^2} dx$$

$$= -\frac{1}{2}\gamma - \ln\,[1+(1+a^2)^{1/2}] + \frac{1}{2}(a^2+1)^{-1/2}[\gamma + 2 \ln 2 + \ln\,(1+a^2)]$$

3.7. Combination of Two Error Functions

Corrections

(3) Last term on R.H.S. should read $+\frac{2b}{\sqrt{\pi}} E_1(2ab)$

(4) Last term on R.H.S. should read $-\frac{2b}{\sqrt{\pi}} E_1(2ab)$

(7) $$\int_0^1 \operatorname{erf}(x) \operatorname{erf}\,[(1-x^2)^{1/2}]x dx = (2e)^{-1}$$

(8) Remove the entire expression and replace by

$$12b^3\sqrt{\pi} \int_0^\infty x^2 \operatorname{erf} c\,(bx) \operatorname{erf} c\left(\frac{a}{2x}\right) dx = e^{-ab}(4+ab-a^2b^2) + a^3b^3E_1(ab).$$

(9) Add the condition $c > a$.

Additions

(10) $$\int \operatorname{erf}(ax) \operatorname{erf}(bx) dx = x \operatorname{erf}(ax) \operatorname{erf}(bx) - \frac{1}{ab\sqrt{\pi}}(a^2+b^2)^{1/2} \operatorname{erf}\,[x\sqrt{(a^2+b^2)}]$$

$$+\frac{1}{a\sqrt{\pi}} \operatorname{erf}(bx) e^{-a^2x^2} + \frac{1}{b\sqrt{\pi}} \operatorname{erf}(ax) e^{-b^2x^2}$$

(11) $$(2n+1) \int x^{2n} \operatorname{erf}(ax) \operatorname{erf}(bx) dx$$

$$= x^{2n+1} \operatorname{erf}(ax) \operatorname{erf}(bx) - \frac{2b}{\sqrt{\pi}} \int x^{2n+1} \operatorname{erf}(ax) e^{-b^2x^2} dx - \frac{2a}{\sqrt{\pi}} \int x^{2n+1} \operatorname{erf}(bx) e^{-a^2x^2} dx$$

(12) $$\int x^{2n} \operatorname{erf}^2(ax) dx = \frac{x^{2n+1}}{2n+1} \operatorname{erf}^2(ax) - \frac{4}{2n+1} \frac{n!}{\sqrt{\pi}\, a^{2n+1}} A,$$

where

$$A=\frac{1}{2\sqrt{2}}\operatorname{erf}(ax\sqrt{2})\sum_{k=0}^{n}\binom{2k}{k}8^{-k}-\frac{1}{2}\operatorname{erf}(ax)e^{-a^2x^2}e_n(a^2x^2)$$

$$-\frac{ax}{\sqrt{\pi}}e^{-2a^2x^2}\sum_{k=0}^{n}\binom{2k}{k}8^{-k}\sum_{l=0}^{k-1}\frac{l!}{(2l+1)!}(8a^2x^2)^l$$

(13) $$\int x\operatorname{erf}^2(ax)dx=\left(\frac{x^2}{2}-\frac{1}{4a^2}\right)\operatorname{erf}^2(ax)+\frac{x}{a\sqrt{\pi}}\operatorname{erf}(ax)e^{-a^2x^2}+\frac{1}{2\pi a^2}e^{-2a^2x^2}$$

(14) $$\int x^{2n+1}\operatorname{erf}^2(ax)dx=\left[x^{2n+2}-\frac{2}{n!}\frac{(2n+1)!}{(2a)^{2n+2}}\right]\frac{\operatorname{erf}^2(ax)}{2n+2}+\frac{2}{(2a)^{2n+2}}\frac{(2n+1)!}{(n+1)!}B,$$

where

$$B=\frac{2ax}{\sqrt{\pi}}\operatorname{erf}(ax)e^{-a^2x^2}\sum_{k=0}^{n}\frac{k!}{(2k+1)!}(4a^2x^2)^k+\frac{e^{-2a^2x^2}}{\pi}\sum_{k=0}^{n}\frac{2^k k!k!}{(2k+1)!}e_k(2a^2x^2)$$

(15) $$\int_0^\infty \operatorname{erf}(ax)\operatorname{erf}(bx)\frac{dx}{x^2}=\frac{2}{\sqrt{\pi}}\left[a\ln\left(\frac{b+c}{a}\right)+b\ln\left(\frac{a+c}{b}\right)\right],\ c=(a^2+b^2)^{1/2}$$

(16) $$\int_0^\infty x^{p-1}\operatorname{erf}c(ax)\operatorname{erf}(bx)dx=\frac{2b}{a^{p+1}}\frac{\Gamma(1+p/2)}{\pi(p+1)}\,{}_3F_2\left(\frac{1}{2},\frac{p+1}{2},\frac{p+2}{2};\frac{3}{2},\frac{p+3}{2};-\frac{b^2}{a^2}\right),$$

$$a>b,\qquad p>-1$$

(17) $$\int_0^\infty \operatorname{erf}(ax)\operatorname{erf}(bx)e^{-c^2x^2}\frac{dx}{x^2}$$

$$=\frac{2}{\sqrt{\pi}}\left\{a\ln\left[\frac{b+\Delta}{(a^2+c^2)^{1/2}}\right]+b\ln\left[\frac{a+\Delta}{(b^2+c^2)^{1/2}}\right]-c\tan^{-1}\left(\frac{ab}{c\Delta}\right)\right\},$$

$$\Delta=(a^2+b^2+c^2)^{1/2}$$

(18) $$\int_0^\infty \operatorname{erf}(ax)\operatorname{erf}(bx)e^{-c^2x^2}dx=\frac{1}{c\sqrt{\pi}}\tan^{-1}\left(\frac{ab}{c\Delta}\right),\ \Delta=(a^2+b^2+c^2)^{1/2}$$

(19) $$\int_0^\infty \operatorname{erf}(ax)\operatorname{erf}(bx)e^{-c^2x^2}x\,dx=\frac{1}{c^2\pi}\left[\frac{b}{\Delta_2}\tan^{-1}\frac{a}{\Delta_2}+\frac{a}{\Delta_1}\tan^{-1}\frac{b}{\Delta_1}\right]$$

$$\Delta_1=(a^2+c^2)^{1/2},\ \Delta_2=(b^2+c^2)^{1/2}$$

(20) $$\int_0^\infty xe^{x^2}\ln x\operatorname{erf}c^2(x)dx=\frac{2}{\pi}\left[G-1-\frac{\gamma}{2}\left(1-\frac{\pi}{4}\right)\right]$$

(21) $$\int_0^\infty[1-\operatorname{erf}^3(x)]dx=\frac{6}{\pi}\left(\frac{2}{\pi}\right)^{1/2}\tan^{-1}(2^{-1/2})$$

(22) $$\int_0^\infty[1-\operatorname{erf}^4(x)]dx=\frac{12}{\pi}\left(\frac{2}{\pi}\right)^{1/2}\tan^{-1}(8^{-1/2})$$

Corrections

(12) Replace $\Gamma\left(-p-1, \dfrac{b^2}{4a^2}\right)$ on the R.H.S. by $\Gamma\left(-p-\dfrac{1}{2}, \dfrac{b^2}{4a^2}\right)$.

(13) Replace $W_{-p/2,\,p/2}\left(\dfrac{b^2}{8a^2}\right)$ by $W_{-p/2,\,p/2}\left(\dfrac{b^2}{4a^2}\right)$;

replace $b^{p+1/2}/a^{3/2p+1}$ by b^p/a^{2p+1};

change the condition $-1<p<\frac{3}{2}$ to $-\frac{1}{2}<p<\frac{3}{2}$.

(19) Multiply the R.H.S. by $\dfrac{2}{\sqrt{\pi}}$;

change the condition on p to $-\frac{1}{2}<p<0$.

(20) Multiply the R.H.S. by $\dfrac{2}{\sqrt{\pi}}$

(21) Replace the condition $\lambda+\frac{1}{2}p>0$ by $\lambda+\frac{1}{2}p>-\frac{1}{2}$

Additions

(22) $$\int_0^\infty J_1[a\,\mathrm{erf}\,(x)]e^{-x^2}dx=\frac{\sqrt{\pi}}{2a}\,[1-J_0(a)]$$

(23) $$\int_0^\infty J_0[a\sqrt{\{1-\mathrm{erf}^2\,(x)\}}]e^{-x^2}dx=\frac{\sqrt{\pi}}{2a}\sin a$$

(24) $$\int_0^\infty K_0(ax)\,\mathrm{erf}\,(bx)x\,dx=\frac{1}{8b^2}\,e^c[K_1(c)-K_0(c)],\quad c=a^2/(8b^2)$$

(25) $$\int_0^\infty e^{-x^2}I_0(x^2)\,\mathrm{erf}\,c(ax)dx=(8\pi)^{-1/2}\ln\left[1+\frac{2}{a^2}+\frac{2}{a^2}(1+a^2)^{1/2}\right]$$

(26) $$\int_0^\infty x^{2p-2}e^{-x^2}K_\nu(x^2)\,\mathrm{erf}\,(x)dx=\frac{\Gamma(p+\nu)\Gamma(p-\nu)}{2^p\Gamma\left(p+\frac{1}{2}\right)}\,{}_3F_2\left(p+\nu,\,p-\nu,\,\frac{1}{2};\,p+\frac{1}{2},\,\frac{3}{2};\,-\frac{1}{2}\right),$$

$$p>|\nu|$$

(27) $$\int_0^\infty x^{2\lambda}J_{2\nu}\left(\frac{a}{x}\right)\mathrm{erf}\,(bx)\,dx$$

$$=\left(\frac{a}{2}\right)^{2\nu}\frac{b^{2\nu-2\lambda-1}\,\Gamma(\lambda-\nu+1)}{\sqrt{\pi}(2\nu-2\lambda-1)\Gamma(2\nu+1)}\,{}_1F_3\left(\nu-\lambda-\frac{1}{2};\,2\nu+1,\nu-\lambda,\nu-\lambda+\frac{1}{2};\,\frac{a^2b^2}{4}\right)$$

$$+\left(\frac{a}{2}\right)^{2\lambda+2}\frac{b\Gamma(\nu-\lambda-1)}{\sqrt{\pi}\,\Gamma(\nu+\lambda+2)}\,{}_1F_3\left(\frac{1}{2};\,\frac{3}{2},\,\lambda-\nu+2,\,\lambda+\nu+2;\,\frac{a^2b^2}{4}\right),$$

$$\nu>\lambda>-\frac{5}{4}$$

Corrections

(3) Replace $\operatorname{si}(2p\,x)$ in integrand by $\dfrac{1}{a}\operatorname{si}\left(\dfrac{x}{a}\right)$.

(5) Replace $e^{a^2/4}$ on R.H.S. by $e^{a^2/2}$;

change second index of W from $(1+2\mu)/4$ to $(1+2\nu)/4$;

change argument of W from a^2 to $a^2/2$;

change the condition $\mu < \nu$ to $\mu > \nu$.

(6) Add the condition $\nu > -\frac{1}{2}$.

(7) Add the condition $\nu > -\frac{1}{2}$.

(9) Change the R.H.S. to

$$\frac{1}{\sqrt{\pi}}\frac{\Gamma(b)}{p^{2b-1}}(2a-2b+1)^{-1}\,{}_2F_1(a, a-b+\tfrac{1}{2}; a-b+\tfrac{3}{2}; -1), \qquad a+\tfrac{1}{2} > b$$

(10) Change the R.H.S. to

$$\frac{1}{\sqrt{\pi}}\frac{\Gamma(b)}{p^{2b-1}}(p/q)^{2a}(2a-2b+1)^{-1}\,{}_2F_1\left(a, a-b+\tfrac{1}{2}; a-b+\tfrac{3}{2}; -\frac{p^2}{q^2}\right), \qquad a+\tfrac{1}{2} > b;\ q^2 \geqslant p^2$$

(11) Change the R.H.S. to

$$\frac{1}{\sqrt{\pi}}\frac{p}{q^{\nu+2}}\Gamma\left(1+\frac{\nu}{2}\right)\frac{\Gamma(b)\Gamma(a-1-\frac{1}{2}\nu)}{\Gamma(a)\Gamma(b-1-\frac{1}{2}\nu)}\,{}_3F_2\left(\tfrac{1}{2}, 1+\frac{\nu}{2}, 2-b+\frac{\nu}{2}; \tfrac{3}{2}, 2-a+\frac{\nu}{2}; -\frac{p^2}{q^2}\right), \qquad \nu > -2;\ q^2 \geqslant p^2$$

(16) Change the condition on p to $p > 0$.

(18) Integrand should contain $\operatorname{erfc}\left(\sqrt{\dfrac{a}{2}}\,x\right)$ instead of $\operatorname{erfc}\left(\dfrac{a}{\sqrt{2}}\,x\right)$; coefficient on R.H.S. should read

$$2^{\nu/2+1}a^{-3/2}\Big/\left[\left(1+\frac{\nu}{2}\right)\Gamma\left(1-\frac{\nu}{2}\right)\right] \text{ instead of } 2^{\nu/2+2}a^{-3/2}/(\nu\pi)$$

(20) Replace $p^{\nu+1/2}$ on the R.H.S. by $a^{\nu+1/2}$

(21) First term inside ${}_3F_2$ expression should read

$$-\lambda+\mu+\tfrac{1}{2} \text{ instead of } \lambda+\mu+\tfrac{1}{2};$$

argument of ${}_3F_2$ should read $+a$ instead of $-a$;

add the condition $a \leqslant 1$; $a=1$, $\lambda > \frac{1}{2}p$.

(22) $$\int_0^1 T_{2n+1}(x)\ \mathrm{erf}\ (ax)(1-x^2)^{-1/2}dx=(-1)^n\frac{\Gamma(n+\frac{1}{2})}{2a(2n+1)!}e^{-a^2/2}M_{-1/2,\ n+1/2}(a^2)$$

(23) $$\int_{-\infty}^{\infty}\{T_n[\mathrm{erf}\ (x)]\}^2e^{-x^2}dx=\sqrt{\pi}\left(\frac{2n^2-1}{4n^2-1}\right)$$

(24) $$\int_0^1 P_n(1-2x^2)\ \mathrm{erf}\ (ax)dx$$

$$=\left(\frac{a}{2}\right)^{2n+1}\frac{1}{(2n+1)\Gamma(n+\frac{3}{2})}\,{}_2F_2(n+\tfrac{1}{2},n+1;n+\tfrac{3}{2},2n+2;-a^2)$$

(25) $$\int_0^1 P_{2n+1}(x)\ \mathrm{erf}\ (ax)\frac{dx}{\sqrt{x}}=(-1)^{n+1}\frac{1}{2a\sqrt{(a\pi)}}\frac{\Gamma(n+\frac{1}{2})}{\Gamma(2n+\frac{5}{2})}e^{-a^2/2}M_{-3/4,\ n+3/4}(a^2)$$

(26) $$\int_0^1 (1-x^2)\ P_{2n+1}^{(p,p)}(x)\ \mathrm{erf}\ (ax)dx$$

$$=\frac{(-1)^n}{2\sqrt{\pi}}\frac{\Gamma(n+\frac{1}{2})}{(2n+1)!}\frac{\Gamma(2n+p+2)}{\Gamma(2n+p+\frac{5}{2})}\frac{e^{-a^2/2}}{a^{p+3/2}}M_{-p/2-3/4,\ n+p/2+3/4}(a^2),\quad p>-1$$

(27) $$\int_0^{\infty} x^{\lambda-1}(1-x^2)^{-\mu/2}p_\nu^\mu(x)\ \mathrm{erf}\ (ax)dx$$

$$=\frac{2^{\mu-\lambda}a\Gamma(\lambda+1)}{\Gamma(1+\frac{1}{2}\lambda-\frac{1}{2}\mu-\frac{1}{2}\nu)\Gamma(\frac{3}{2}+\frac{1}{2}\lambda-\frac{1}{2}\mu+\frac{1}{2}\nu)}\,{}_3F_3\left(\tfrac{1}{2},\frac{\lambda+1}{2},\frac{\lambda+2}{2};\tfrac{3}{2},\frac{2+\lambda-\mu-\nu}{2},\frac{3+\lambda-\mu-\nu}{2};-a^2\right),$$

$$\lambda>-1;\ \mu<1$$

(28) $$\int \mathrm{erf}\ (ax)H_n(ax)dx=[2a(n+1)]^{-1}\left\{\mathrm{erf}\ (ax)H_{n+1}(ax)+\frac{2}{\sqrt{\pi}}H_n(ax)e^{-a^2x^2}\right\}$$

(29) $$\int_0^{\infty}e^{-x^2}H_{2n+1}(x)\ \mathrm{erf}\ (x)dx=(-1)^n\frac{(2n)!}{2^{n+1/2}n!}$$

(30) $$\int_0^{\infty}e^{-x^2}H_{2n}(x)\ \mathrm{erf}\ (x)xdx=(-1)^{n+1}\frac{(2n)!}{2^{n+3/2}n!}\frac{2n+1}{2n-1}$$

(31) $$\int_0^{\infty}e^{-x^2}H_n(x)H_{n+2m+1}(x)\ \mathrm{erf}\ (x)dx=(-1)^m\frac{(2m+2n+2)!}{(m+n+1)!}\frac{1}{2^{m+n+3/2}}\frac{1}{2m+1}$$

(32) $$\int_0^{\infty}x^{p-1}H_{2n}(ax)\ \mathrm{erfc}\,(bx)dx=(-1)^n\frac{(2n)!}{n!}\frac{\Gamma(\frac{1}{2}+\frac{1}{2}p)}{pb^p\sqrt{\pi}}\,{}_3F_2\left(-n,\frac{p}{2},\frac{p+1}{2};\frac{1}{2},\frac{p+2}{2};\frac{a^2}{b^2}\right),$$

$$p>0;\ b^2>a^2$$

449-764 OL - 72 - 5

(33) $\int_0^1 (1-x^2)^{p-1/2}\, C_{2n+1}^p(x)\,\text{erf}\,(ax)\,dx$

$$=\frac{(-1)^n}{2^{2p}}\frac{a^{2n+1}}{(2n+1)!}\frac{\Gamma(n+\frac{1}{2})}{\Gamma(p)}\frac{\Gamma(2n+2p+1)}{\Gamma(2n+p+2)}\,{}_1F_1(n+\tfrac{1}{2};\,2n+p+2;\,-a^2)$$

$$p>0$$

(34) $$\int_0^\infty e^{-a^2x^2/4}D_{2n+1}(ax)\,\text{erf}\,(bx)dx=(-1)^n\frac{(2n-1)!!}{a}\left(\frac{2b^2}{a^2+2b^2}\right)^{n+1/2}$$

(35) $$\int_0^\infty e^{-x^2/4}\left[D_{2p}(x)-D_{2p}(-x)\right]\,\text{erf}\,(x)dx=\left(\frac{4}{3}\right)^p\sqrt{(2/\pi)}\,\sin p\pi\,\Gamma(p),\ p>0$$

(36) $\int_0^\infty x^{2p-1}e^{-x^2/4}D_{2q}(x)\,\text{erf}\,(\tfrac{1}{2}x)dx$

$$=2^{q-p-1/2}\frac{\Gamma(2p+1)}{\Gamma(p-q+1)}\,{}_3F_2\,(\tfrac{1}{2},\,p+\tfrac{1}{2},\,p+1;\,\tfrac{3}{2},\,p-q+1;\,-\tfrac{1}{2}),\ p>-\tfrac{1}{2}$$

(37) $\int_0^\infty x^{2n+1}e^{-x^2/2}\Gamma_n^{n+1/2}\,(x^2/2)\,\text{erf}\,(x)dx$

$$=\frac{1}{\sqrt{\pi}}\left(\frac{2}{3}\right)^{n+1/2}\frac{2^n}{n!}\frac{\Gamma(2n+3/2)}{(n+\frac{1}{2})}\,{}_2F_1\,(-n,\,n+\tfrac{1}{2};\,n+\tfrac{3}{2};\,\tfrac{2}{3})$$

(38) $\int_0^\infty L_n^\alpha(a^2x^2)\,\text{erf}\,c(bx)x^{p-1}dx$

$$=\binom{\alpha+n}{n}\frac{\Gamma(\frac{1}{2}+\frac{1}{2}p)}{pb^p\sqrt{\pi}}\,{}_3F_2\left(-n,\,\frac{p}{2},\,\frac{p+1}{2};\,\alpha+1,\,\frac{p+2}{2};\,\frac{a^2}{b^2}\right),\ \alpha>-1;\,p>0;\ b^2>a^2$$

(39) $$\int_{-\infty}^\infty\frac{[\text{erf}\,(ae^{i\pi x})-\text{erf}\,(ae^{-i\pi x})]}{\Gamma(\alpha+x)\Gamma(\beta-x)\,\sin\pi x}dx=i\,\frac{2^{\alpha+\beta-1}}{\Gamma(\alpha+\beta-1)}\,\text{erf}\,(a),\ \alpha+\beta>1$$

(40) $\int_0^\infty x^{p-1}\,\text{erf}\,c(bx)\gamma(\alpha,\,a^2x^2)dx$

$$=\frac{a^{2\alpha}\Gamma(\alpha+\frac{1}{2}+\frac{1}{2}p)}{\alpha\sqrt{\pi}\,(p+2\alpha)b^{p+2\alpha}}\,{}_3F_2\left(\alpha,\,\alpha+\tfrac{1}{2}p,\,\alpha+\tfrac{1}{2}+\tfrac{1}{2}p;\,\alpha+1,\,\alpha+1+\tfrac{1}{2}p;\,-\frac{a^2}{b^2}\right),\qquad p+2\alpha>0;\ b^2>a^2$$

(41) $\int_0^\infty \text{erf}\,(ax)\,{}_2F_1(\alpha+1,\,\beta+1+3/2;\,-c^2x^2)xdx$

$$=\frac{1}{4\alpha\beta a^2}\left(\frac{a}{c}\right)^{\alpha+\beta+1}e^{a^2/(2c^2)}W_{(1-\alpha-\beta)/2,\,(\alpha-\beta)/2}\left(\frac{a^2}{c^2}\right),\ \alpha>0;\ \beta>0$$

(42) $\int_0^\infty x^{4p}e^{-a^2x^2}{}_1F_1\left(\frac{1}{2}-2p;\ 2p+1;\ a^2x^2\right)\ \mathrm{erf}\ (bx)dx$

$$=\frac{1}{2a\sqrt{\pi}}\Gamma(2p)\left[\frac{b^2}{a^2(a^2+b^2)}\right]^{2p}{}_2F_1\left(\frac{1}{2}-2p,\ 2p;\ 2p+1;\ \frac{b^2}{a^2+b^2}\right),$$

$$p>0$$

(43) $\int_0^\infty e^{x^2}x^{2\beta-2}{}_1F_1\ (\alpha;\ \beta;\ -\gamma^2x^2)\ \mathrm{erf}\ c(x)dx$

$$=\frac{\Gamma(2\beta-1)\Gamma(1+\alpha-\beta)}{2^{2\beta-1}\Gamma(\frac{1}{2}+\alpha)}\,{}_2F_1\ (\alpha,\ \beta-\tfrac{1}{2};\ \alpha+\tfrac{1}{2};\ 1-\nu^2),$$

$$\gamma^2<2;\ 1+\alpha>\beta>\tfrac{1}{2}$$

(44) $\int_0^\infty e^{-x^2}\Psi(\alpha,\ \beta;\ \gamma^2x^2)\ \mathrm{erf}\ i(x)xdx$

$$=\frac{1}{2\gamma^3}\frac{\Gamma(\alpha-\frac{1}{2})\Gamma(5/2-\beta)}{\Gamma(\alpha)\Gamma(\alpha-\beta+2)}\,{}_2F_1\left(1,\frac{5}{2}-\beta;\ \alpha-\beta+2;\ 1-\frac{1}{\gamma^2}\right),$$

$$\alpha>\tfrac{1}{2};\ \gamma^2>\tfrac{1}{2};\ \beta<\tfrac{5}{2}$$

(45) $\int_0^\infty e^{-x/2}x^{p-1}W_{\alpha,\beta}(x)\ \mathrm{erf}\ (a^{1/2}\,x^{1/2})\,dx$

$$=2\sqrt{(a/\pi)}\ \frac{\Gamma(1+\beta+p)\Gamma(1-\beta+p)}{\Gamma(3/2-\alpha+p)}\,{}_3F_2\ (\tfrac{1}{2},\ 1+\beta+p,\ 1-\beta+p;\ \tfrac{3}{2},\ \tfrac{3}{2}-\alpha+p;\ -a),$$

$$p+1>|\beta|$$

4. References

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