Normal Subgroups of the Modular Group*

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A number of results on the normal subgroup structure of the classical modular group is announced. A typical result is that a normal subgroup of square-free index is necessarily of genus 1, apart from 4 exceptions.

Key words: Genus; index; modular group; normal subgroups.

1. Introduction

In this note we summarize the results of some work on the normal subgroups of the classical modular group \( \Gamma \), which is a continuation of the work begun in [1] and [4]. We may regard \( \Gamma \) as the free product of a cyclic group of order 2 and a cyclic group of order 3; \( \Gamma = \{ x \} \ast \{ y \}, \ x^2 = y^3 = 1 \). The number of normal subgroups of \( \Gamma \) of index \( \mu \) will be denoted by \( N(\mu) \). If \( G \) is any subgroup of \( \Gamma \), \( G' \) will denote its commutator subgroup, and \( G^p \) the fully invariant subgroup of \( G \) generated by the \( p \)th powers of the elements of \( G \). The level of \( G \) is the least positive integer \( n \) such that \((xy)^n \in G\). If \( G \) is a normal subgroup of index \( \mu \geqslant 6 \) and \( n \) is its level, then the genus of \( G \) is given by

\[
g = 1 + \mu(n - 6)/12n,
\]

and the number of parabolic classes of \( G \) by

\[
t = \mu/n.
\]

Except for the groups \( \Gamma, \Gamma^2, \) or \( \Gamma^3 \), the index of a normal subgroup is a multiple of 6.

The commutator subgroup \( \Gamma' \) of \( \Gamma \) is a free group of rank 2, freely generated by

\[
a = xyxy^2, \quad b = xy^2xy.
\]

The normal subgroups of \( \Gamma \) of genus 1 (alternatively, of level 6) have been completely described in [5]. Any such subgroup \( G \) lies between \( \Gamma' \) and \( \Gamma'' \) and may be described uniquely by the triplet of integers \( (p, m, d) \), where \( p > 0, \ 0 \leqslant m \leqslant d-1, \ m^2 + m + 1 \equiv 0 \mod d \). \( G \) is of index \( 6dp^2 \) in \( \Gamma \) and consists of all words \( w \) of \( \Gamma' \) satisfying

\[
e_a(w) \equiv 0 \mod p, \quad e_b(w) \equiv me_a(w) \mod dp,
\]

where \( e_a(w), e_b(w) \) are the respective exponent sums in \( a \) and \( b \) of \( w \).

We also let \( G_{k,m} \) be the intersection of all normal subgroups of \( \Gamma \) containing

\[
(xy)^{mk}, \quad (yx)^k(xy)^{-k}.
\]

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1 Figures in brackets indicate the literature references at the end of this paper.
Going over to the representation of $\Gamma$ as $LF(2, Z)$, we define the principal congruence subgroup $\Gamma(n)$ as the totality of elements $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \epsilon \Gamma$ such that

$$a = d = \pm 1 \mod n, \quad b = c = 0 \mod n.$$ 

2. The Results

We now state the principal results obtained. Throughout this section $G$ is a normal subgroup of $\Gamma$ of index $\mu$, level $n$, genus $g$, and having $t$ parabolic classes.

1) Suppose that $\mu$ is square-free. Then either $G = \Gamma$, $\Gamma^2$, $\Gamma^3$, or $\Gamma(2)$, or else $G$ is of genus 1 and every prime divisor of $\mu/6$ is $\equiv 1 \mod 3$.

2) Define $f(\mu)$ as 1 if there is a normal subgroup $G$ of index $\mu$ with solvable quotient group $\Gamma/G$, and 0 otherwise. Then

$$\lim_{x \to \infty} \frac{1}{x} \sum_{\mu \leq x} f(\mu) = 0.$$ 

3) Let $p$ be a prime, $p = -1 \mod 3$, and suppose that $p > r$. Then there is no normal subgroup of $\Gamma$ of index $pr$.

4) If $\Gamma/G$ is nilpotent then it is abelian, and $G$ must be $\Gamma$, $\Gamma^2$, $\Gamma^3$, or $\Gamma'$.

5) Let $p$ be a prime, $p = 1 \mod 12$. Then there are no normal subgroups of $\Gamma$ having $2p$ parabolic classes.

6) Let $p$ be a prime $> 84$, $p = -1 \mod 3$; and let $n$ be any positive integer. Then there are no normal subgroups of $\Gamma$ of genus $1 + p^n$.

7) Let $p$ be a prime $> 5$, and suppose that $\mu = 6p^2$. Then $G$ must be one of the following groups:

   (i) $\Gamma(2)^p \Gamma(2)'$.
   (ii) $(p, 0, 1)$.
   (iii) $(1, m_1, p^2), (1, m_2, p^2)$, where $p = 1 \mod 3$ and $m_1, m_2$ are the solutions of $m^2 + m + 1 \equiv 0 \mod p^2$.

Thus

$$N(6p^2) = 3 + (p/3), \quad p \text{ prime, } p > 5.$$ 

8) Let $p$ be a prime $> 11$. Then $N(12p^2) = 0$.

9) Let $p$ be a prime $> 11$. Then the only normal subgroup of $\Gamma$ of index $12p^3$ is $\Gamma(3)^p \Gamma(3)'$.

10) $N(72) = 2, N(78) = 2, N(84) = 0, N(90) = 0$.

11) There is just one normal subgroup of $\Gamma$ of genus 2: namely $G_{4,2}$.

12) The normal subgroups of $\Gamma$ with $t$ parabolic classes, $t \leq 5$, are the following:

   \begin{align*}
   t = 1 & : \Gamma, \Gamma^2, \Gamma^3, \Gamma'. \\
   t = 2 & : \text{none.} \\
   t = 3 & : \Gamma(2), (1, 1, 3). \\
   t = 4 & : (2, 0, 1), \Gamma(3), G_{3,4}. \\
   t = 5 & : \text{none.}
   \end{align*}

3. Some Remarks

Perhaps the most striking results are the first two. A generalization of (2) with a precise estimate for the density function is in course of publication ([2]). As for (1), we note that if $G$ is any
finite group of square-free order generated by elements $x, y$ such that $x^2 = y^3 = 1$, then $(xy)^6 = 1$.
This is so since the second commutator subgroup $G''$ is necessarily $\{1\}$ (p. 148 of [3]), and

$$a = xyxy^2 \in G', \quad b = xy^2xy \in G',$$

$$(xy)^6 = ab^{-1}a^{-1}b \in G''.$$ The result (1) is now an easy consequence.

The other results are of varying degrees of difficulty, but generally present no special problems.

### 4. References


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