Orthogonal Decompositions of Tensor Spaces*

Stephen Pierce
Institute for Basic Standards, National Bureau of Standards, Washington, D.C. 20234

(September 23, 1969)

Let \( V \) be an \( n \)-dimensional vector space over the complex numbers. Let \( H \) be a subgroup of \( S_m \), the symmetric group on \( \{1, \ldots, m\} \), and let \( W = \bigotimes_1^n V \) be the tensor product of \( V \) with itself \( m \) times.

In this note we give an orthogonal direct sum decomposition of \( W \) in terms of the system of inequivalent irreducible characters of \( H \).

AMS Subject Classifications: Primary, 1580; Secondary, 2080.

Key words: Irreducible character; symmetry class of tensors; symmetry operator; tensor product.

1. Introduction

Let \( V \) be an \( n \)-dimensional vector space over the field of complex numbers. Let \( W = \bigotimes_1^n V \) be the tensor product of \( V \) with itself \( m \) times. For \( \sigma \in S_m \), the symmetric group on \( \{1, \ldots, m\} \), define the permutation operator \( P(\sigma) : W \to W \) by \( P(\sigma)v_1 \otimes \ldots \otimes v_m = v_{\sigma(1)} \otimes \ldots \otimes v_{\sigma(m)} \), \( v_1, \ldots, v_m \in V \), where \( \theta = \sigma^{-1} \). It is easy to check [1] that \( P(\sigma) \) is linear and that \( P(\sigma)P(\tau) = P(\sigma\tau) \), \( \sigma, \tau \in S_m \). Any linear combination \( T = \sum_{\sigma \in S_m} a(\sigma)P(\sigma) \) is called a symmetry operator and the range of \( T \) is called a symmetry class of tensors.

In [4], Weyl expressed \( W \) as a direct sum of symmetry classes. The corresponding symmetry operators were determined from the idempotent generators of the minimal right ideals of the group ring of \( S_m \). In this paper we obtain an orthogonal direct sum decomposition of \( W \) into symmetry classes with respect to the inner product defined below.

Let \( (\cdot, \cdot) \) be an inner product on \( V \). Define an inner product on \( W \) by

\[
(x_1 \otimes \ldots \otimes x_m, y_1 \otimes \ldots \otimes y_m) = \prod_{i=1}^m (x_i, y_i), x_i, y_i \in V. \tag{1}
\]

**Theorem 1:** Let \( H \) be a subgroup of \( S_m \) of order \( h \). Let \( \chi_1, \ldots, \chi_k \) be the complete system of inequivalent irreducible characters on \( H \), with \( \chi_i \) having degree \( r_i \), \( i = 1, \ldots, k \). Define \( T_{\chi_i} : W \to W \) by

\[
T_{\chi_i} = \frac{r_i}{h} \sum_{\sigma \in H} \chi_i(\sigma)P(\sigma), \quad i = 1, \ldots, k.
\]

Let \( V_{\chi_i}^m(H) \) be the range of \( T_{\chi_i} \). Then with respect to the inner product (1), \( W \) is the orthogonal direct sum of the symmetry classes \( V_{\chi_i}^m(H) \):

\[
W = \bigoplus_{i=1}^k V_{\chi_i}^m(H). \tag{2}
\]

---

*This work was done (1968–1969) while the author was a National Academy of Sciences-National Research Council Postdoctoral Research Associate at the National Bureau of Standards, Washington, D.C.

1 Figures in brackets indicate the literature references at the end of this paper.
In section 3, we discuss some of the difficulties involved in constructing a suitable basis for $V_{\chi}^{m}(H)$. This problem has been dealt with [1] when $\chi$ is linear, but little has been done otherwise. We attempt to show the extent to which the methods of [1] will apply when $\chi$ is of higher degree.

2. Proof of Theorem 1

To establish (2), it suffices to show that

a. $T_{x_{i}}$ is hermitian,

b. $T_{x_{i}}T_{x_{j}} = \delta_{ij}T_{x_{i}}$,

c. $\sum_{i=1}^{k} T_{x_{i}} = \text{identity}.$

If "*" denotes the adjoint with respect to the inner product (1), then $(P(\sigma))^{*} = P(\sigma^{-1})$. Since $\chi_{i}(\sigma) = \chi_{i}(\sigma^{-1})$, $T_{x_{i}}^{*} = T_{x_{i}}$. We now compute

$$T_{x_{i}}T_{x_{j}} = \frac{r \delta_{ij}}{h^{2}} \sum_{\sigma, \tau \in H} \chi_{i}(\sigma)\chi_{j}(\tau)P(\sigma\tau) = \frac{r \delta_{ij}}{h^{2}} \sum_{\mu \in \chi_{i}} P(\mu) \sum_{\sigma \in H} \chi_{i}(\sigma)\chi_{j}(\sigma^{-1}\mu).$$

The orthogonality relations for characters [3, p. 16] now imply that

$$T_{x_{i}}T_{x_{j}} = \frac{r \delta_{ij}}{h^{2}} \sum_{\mu \in \chi_{i}} P(\mu) \frac{\delta_{ij}h\chi_{j}(\mu)}{r} = \delta_{ij}T_{x_{i}}.$$

Let $e$ be the identity in $H$. Then

$$\sum_{i=1}^{k} T_{x_{i}} = \sum_{i=1}^{k} \frac{r_{i}}{h} \sum_{\sigma \in H} \chi_{i}(\sigma)P(\sigma)$$

$$= \frac{1}{h} \sum_{\sigma \in H} P(\sigma) \sum_{i=1}^{k} \chi_{i}(e)\chi_{i}(\sigma).$$

Again, the orthogonality relations imply that

$$\sum_{i=1}^{k} T_{x_{i}} = \frac{1}{h} \sum_{\sigma \in H} P(\sigma)\delta_{\epsilon,\sigma}h$$

$$= P(e),$$

the identity transformation on $W$. This proves (3).

We note that Weyl’s decomposition of $W$ is orthogonal with respect to the above inner product only when $m = 2$.

3. Bases for Symmetry Classes

As above, $H$ is a subgroup of $S_{m}$, $\chi$ is an irreducible character on $H$ of degree $r$, and $T_{x}$ and $V_{\chi}^{m}(H)$ are the associated symmetry operator and symmetry class. If $x_{1} \otimes \ldots \otimes x_{m}$ is a decomposable tensor in $W$, set

$$T_{x_{1}} \otimes \ldots \otimes x_{m} = x_{1} \otimes \ldots \otimes x_{m}.$$

The tensor $x_{1} \otimes \ldots \otimes x_{m}$ is called a decomposable element of $V_{\chi}^{m}(H)$. Now if $v_{1}, \ldots, v_{n}$ is a basis of $V$, the set of $n^{m}$ tensors $v_{a_{1}} \otimes \ldots \otimes v_{a_{m}}$, $1 \leq a_{i} \leq n$ is a basis of $W = \bigoplus_{i=1}^{n} V$. Thus there is a basis
of \( V^\chi(H) \) consisting of decomposable elements. In [1], Marcus and Minc give a construction for such a basis when \( \chi \) is linear, i.e., \( r = 1 \). Let \( \Gamma_{m,n} \) be the set of all \( m \times n \) sequences \( \alpha = (\alpha_1, \ldots, \alpha_m) \), \( 1 \leq \alpha_i \leq n \). We write \( \alpha \sim \beta \) if \( \alpha^\sigma = (\alpha_{\sigma(1)}, \ldots, \alpha_{\sigma(m)}) = \beta \) for some \( \sigma \in H \). Clearly \( \sim \) is an equivalence relation on \( \Gamma_{m,n} \). Choose lexicographically the lowest representative from each class determined by \( \sim \). Call this set of representatives \( \Delta \). For each \( \alpha \in \Delta \), let \( H_\alpha \) be the subgroup of all \( \sigma \in H \) such that \( \alpha^\sigma = \alpha \). Let \( \Delta \) be the set of all \( \alpha \in \Delta \) such that \( \sum_{\sigma \in H_\alpha} \chi(\sigma) \neq 0 \). If \( v_1, \ldots, v_n \) is a basis of \( V \), then the tensors

\[
v_\alpha = v_{\alpha} \ast \ldots \ast v_{\alpha_m}, \quad \alpha \in \Delta \tag{4}
\]

are a linearly independent set in \( V^\chi(H) \). Moreover, if \( \chi \) is linear, the tensors (4) are a basis of \( V^\chi(H) \). (For details, see [1].) Note that \( \Delta \) depends only on \( H \) and \( \chi \), and if \( \chi \) is linear \( v_\alpha^* = \chi(\sigma)v_\alpha^* \) for all \( \sigma \in H, \alpha \in \Delta \) (see [1]). Thus in the linear case one can conveniently determine matrix representations of certain linear transformations on \( V^\chi(H) \). Using these properties, Marcus, Minc and Newman (see, e.g., [1], [2]) have been able to prove a large class of inequalities for determinants, permanents, and other multilinear matrix functions. Thus it would be useful to find a basis of \( V^\chi(H) \) when degree \( \chi > 1 \). The following result demonstrates some obstacles.

**Theorem 2:** If \( v_1, \ldots, v_n \) is a basis of \( V \), the tensors

\[
v_\alpha^*, \quad \alpha \in \Delta \tag{5}
\]

are a basis of \( V^\chi(H) \) if and only if \( \chi \) is linear. Moreover, if \( |\Delta| \) is the cardinality of \( \Delta \) then

\[
\dim V^\chi(H) \geq 2|\Delta| \tag{6}
\]

if \( \chi \) is not linear.

**Proof.** We may assume \( v_1, \ldots, v_n \) is an orthonormal basis of \( V \). The procedure in [1] still applies to show that the tensors (5) are linearly independent. For \( \sigma, \tau \in H \) and \( \alpha, \beta \in \Delta \), we compute

\[
(v_\alpha^*, v_\beta^*) = (T_xv_{\alpha(1)} \otimes \ldots \otimes v_{\alpha(m)}, T_xv_{\beta(1)} \otimes \ldots \otimes v_{\beta(m)})
\]

\[
= (T_xv_{\alpha(1)} \otimes \ldots \otimes v_{\alpha(m)}, v_{\beta(1)} \otimes \ldots \otimes v_{\beta(m)}),
\]

because \( T_x \) is idempotent hermitian. Hence

\[
(v_\alpha^*, v_\beta^*) = \frac{r}{h} \sum_{\rho \in H} \chi(\rho) \prod_{t=1}^m (v_{\alpha_{\rho^{-t}}(1), v_{\beta(t)}})
\]

\[
= \frac{r}{h} \sum_{\rho \in H} \chi(\rho) \prod_{t=1}^m (v_{\alpha_{\rho^{-t}}(1), v_{\beta(t)}}). \tag{7}
\]

Since \( \alpha, \beta \in \Delta \), the product in (7) is zero unless \( \alpha = \beta \) and \( \sigma \rho^{-1}\tau^{-1} \in H_\alpha \).

Thus

\[
(v_\alpha^*, v_\beta^*) = \delta_{\alpha, \beta} \frac{r}{h} \sum_{\rho \in H_{\alpha^\sigma \tau^{-1}}} \chi(\rho)
\]

\[
= \delta_{\alpha, \beta} \frac{r}{h} \sum_{\mu \in H_\alpha} \chi(\sigma \tau^{-1} \mu). \tag{8}
\]

43
Thus we are finished if we can show that for each $\alpha \in \Delta$ there is a $\sigma \in H$ such that $v_\alpha^*$ and $v_{\alpha \sigma}$ are linearly independent. Suppose this were false for some $\alpha \in \Delta$. Then

$$v_{\alpha \sigma}^* = \eta(\sigma)v_\alpha^*, \ \sigma \in H,$$

where $\eta(\sigma)$ is a scalar. From (8) and (9),

$$\eta(\sigma) \eta(\tau)(v_\alpha^*, v_\alpha^*) = (v_{\alpha \sigma}^*, v_{\alpha \tau}^*)$$

$$= \frac{r}{h} \sum_{\mu \in \mathcal{H}_\alpha} \chi(\sigma \tau^{-1} \mu)$$

$$= (v_{\alpha \sigma}, v_\alpha^*)$$

$$= \eta(\sigma^{-1})(v_\alpha^*, v_\alpha^*).$$

(10)

Since $v_\alpha^* \neq 0$, (10) implies that $\eta$ is a character on $H$ of degree 1. Thus for all $\sigma \in H$

$$\sum_{\mu \in \mathcal{H}_\alpha} \chi(\sigma \mu) = \frac{h}{r} (v_{\alpha \sigma}, v_\alpha^*)$$

$$= \frac{h}{r} \eta(\sigma)(v_\alpha^*, v_\alpha^*)$$

$$= \eta(\sigma) \sum_{\mu \in \mathcal{H}_\alpha} \chi(\mu).$$

(11)

Multiply both sides of (11) by $\eta(\sigma^{-1})$ and sum on $\sigma$, obtaining

$$0 \neq \sum_{\sigma \in \mathcal{H}} \sum_{\mu \in \mathcal{H}_\alpha} \chi(\mu)$$

$$= \sum_{\sigma \in \mathcal{H}} \eta(\sigma^{-1}) \sum_{\mu \in \mathcal{H}_\alpha} \chi(\sigma \mu)$$

$$= \sum_{\mu \in \mathcal{H}_\alpha} \sum_{\sigma \in \mathcal{H}} \eta(\sigma^{-1}) \chi(\sigma \mu).$$

(12)

As long as the degree of $\chi$ is greater than 1, the orthogonality relations imply that the right side of (12) is zero. This establishes (6) and thus Theorem 2 is proved.

4. References


(Paper 74B1–316)