Partitions — A Survey*

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A historical survey of some aspects of the theory of partitions is given here.

Key words: Congruences; generating functions; identities; partitions; recurrence formulas.

1. Introduction

1.1. Unrestricted Partitions

As far back as 1669, Leibnitz asked John Bernoulli, if he had investigated the number of ways in which a given number can be expressed as a sum of two or more integers. This was necessarily the problem of partitioning a given number into a specified number of parts.

A partition of a positive integer \( n \), is a mode of expressing it as a sum of one or more positive integers, the order in which the summands occur being irrelevant. The parts are usually arranged according to size, one way or the other. Thus, 5 can be expressed as a sum in the following seven ways:

\[
5; 1 + 4; 2 + 3; 1 + 2 + 2; 1 + 1 + 3; 1 + 1 + 1 + 2; \text{ and } 1 + 1 + 1 + 1 + 1.
\]

These are all the partitions of 5. Since there is no restriction of any kind on the number or size of parts, we call these the unrestricted partitions of 5. In all that follows, the number of unrestricted partitions of any given positive integer \( n \) will be denoted by \( p(n) \) and we shall take \( p(0) = 1 \).

In this survey, we shall give a short resumé of the advances that have been made in recent years, in dealing with problems on partitions of integers. The bibliography includes papers on partitions of vectors also but we do not propose to deal with the topic here.

1.2. Restricted Partitions

Given a set \( A = \{a_1, a_2, a_3, \ldots \} \),

where the \( a \)'s are positive integers, one may be required to find the number of ways in which a given integer can be expressed as a sum of the \( a \)'s, repetitions being allowed or not. This will be an example of restricted partitions. As other examples, we can mention: (1) Partitions into distinct parts; (2) Partitions into odd parts; (3) Partitions into a specified number of parts; and (4) Partitions into a specified number of distinct parts. In fact, the restrictions that can be placed on the size or on the number of parts or both, are too many to enumerate. Very recently, I had to consider, for example, the number \( g(n, m, h, k) \) of partitions of \( n \) into exactly \( k \) summands, each \( \leq m \), just \( h \) (and any \( h \)) of the positive integers \( \leq m \) being used as summands, in any partition. Among several interesting results, I found that

\[
\sum_{n=1}^{\infty} g(n, m, h, k) = \binom{k-1}{h-1} \binom{m}{h}.
\]

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It has been brought to date with the help of Professor M. S. Cheema of the University of Arizona, Tucson.

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We shall denote by \( p(n, k) \) the number of partitions of \( n \) into exactly \( k \) summands; by \( q(n) \) the number of partitions of \( n \) into distinct parts; and by \( q(n, k) \) the number of partitions of \( n \) into exactly \( k \) distinct parts.

### 1.3. Decompositions

We have already stated that in a partition the order in which the summands appear is irrelevant. When the order of appearance of summands is relevant, the partition is called an ordered partition or a decomposition. (Some writers call it a composition.) We shall, however, invariably use the word decomposition in this sense.

The number of decompositions is usually easier to find than the number of partitions. The total number of decompositions of \( n \) is, in fact, given by \( 2^{n-1} \). This is easily proved as follows:

The decompositions of \( n \) can be classified under two heads:

1. Those in which the first part is 1; and
2. Those in which the first part is \( > 1 \).

Removing 1 (the first part), from each decomposition of \( n \) of the first kind, we obtain all the different decompositions of \( (n - 1) \), each once. Reducing by 1 the first part in the decompositions of the second kind, we again get all the decompositions of \( (n - 1) \) as before. Hence, the number of decompositions of \( n \), is twice the number of decompositions of \( (n - 1) \) and the result follows readily by induction.

### 1.4. Perfect Partitions

A partition of \( n \) is said to be perfect when it contains just one partition of every number up to \( n \). Thus, \( 1 + 2 + 2 + 2 \) is a perfect partition of 7, because every number up to 7, can be expressed uniquely as a sum by using the summands: 1, 2, 2, 2. Other such partitions of 7 are: \( 1 + 1 + 1 + 1 + 1 + 1 + 1; 1 + 1 + 1 + 4; 1 + 2 + 4 \).

### 1.5. Plane and Solid Partitions

The summands in a partition of \( n \) are sometimes arranged in the form of something like a matrix, so that the elements in each row and in each column are in a descending order, though not necessarily strictly. Such an arrangement gives a plane partition of \( n \). Thus

\[
\begin{array}{ccc}
3 & 3 & 2 \\
2 & 2 & 1 \\
1 & 1 & 1
\end{array}
\]

is a plane partition of 18. A plane partition is also called a rowed partition. A \( k \)-rowed partition of \( n \) includes all partitions of \( n \) with \( k \) or fewer number of rows. Thus the three-rowed partitions of 4 are:

\[
\begin{array}{cccc}
4; 31; 22; 211; 1111; 3; 2; 21; 111; 11; 2; 11.
\end{array}
\]

The number of \( k \)-rowed partitions of \( n \) is denoted by \( t_k(n) \).

In a solid partition the summands are arranged in three-dimensional space as they are arranged in a two dimensional space in the case of a plane partition. The elements are arranged in descending order of magnitude in each of the three principal directions.

### 2. Generating Functions

A convenient way of representing a partition of \( n \), in which one occurs \( h_1 \) times, two \( h_2 \) times, three \( h_3 \) times and so on, is to write

\[
n = h_1^{1} h_2^{2} h_3^{3} \ldots n^{h_n},
\]
where $h$'s are nonnegative integers, and

$$h_1 + 2h_2 + 3h_3 + \ldots + nh_n = n.$$ 

Now it would not be difficult to see that $p(n)$ is the coefficient of $x^n$ in the product:

$$(1 + x + x^2 + \ldots + x^h + \ldots + x^n)(1 + x^2 + x^4 + \ldots + x^{2h_2} + \ldots + x^{2[n/2]})$$

$$(1 + x^3 + x^6 + \ldots + x^{3h_3} + \ldots + x^{3[n/3]}) \ldots (1 + x^n);$$

because a one-one correspondence can be set up between the partition:

$$1^{h_1} 2^{h_2} 3^{h_3} \ldots n^{h_n}$$

of $n$ and the term:

$$x^{h_1} x^{2h_2} x^{3h_3} \ldots x^{nh_n}$$

of the product noted above. Using the algebra of formal power series, we can write

$$\prod_{r=1}^{\infty} (1-x^r)^{-1} = 1 + \sum_{n=1}^{\infty} p(n)x^n.$$ 

The function:

$$f(x) = \prod_{r=1}^{\infty} (1-x^r)^{-1}$$

is called the generating function for $p(n)$. In the same way, one can show that the generating function for $q(n)$ is:

$$g(x) = \prod_{r=1}^{\infty} (1+x^r).$$

In general, the generating function for the number of partitions of $n$ into members of set

$$A = \{a_1, a_2, a_3, \ldots \}$$

where $a$'s are positive integers which may be considered to be arranged in a strictly ascending order of magnitude, is

$$f_A(x) = \prod_{a_j \in A} (1-x^{a_j})^{-1},$$

when repetitions are allowed, and

$$g_A(x) = \prod_{a_j \in A} (1+x^{a_j}),$$

when repetitions are not allowed. Moreover, if $u_A(n)$ and $v_A(n)$ denote the numbers of partitions of $n$ into an even number of distinct $a$'s and an odd number of distinct $a$'s respectively, then the coefficient of $x^n$ in the product:

$$\prod_{a_j \in A} (1-x^{a_j})$$

is $u_A(n) - v_A(n)$.

The number of partitions of $n$ into exactly $k$ summands each taken from set $A$, is the coefficient of $z^k x^n$ in

$$\prod_{a_j \in A} (1-zx^{a_j})^{-1}$$
when repetitions are allowed, and in
\[
\prod_{a_j \in A} (1 + z a_j)
\]
when repetitions are not allowed. In particular, when the set \( A \) consists of all the positive integers, the generating function for \( p(n, k) \) is found to be
\[
x^k \prod_{r=1}^{k} (1 - x^r)^{-1}.
\]
Similarly the generating function for \( q(n, k) \) can be shown to be
\[
x^k \prod_{r=1}^{(k+1)/2} (1 - x^r)^{-1}.
\]
If
\[n^h_1 n^2 h_2 \ldots n^b h_b \ldots\]
be a perfect partition of \( n \), then we must have
\[
(1 + x + x^2 + \ldots + x^{h_1})(1 + x^2 + x^4 + \ldots + x^{2h_2})(1 + x^3 + \ldots + x^{3h_3})
\]
\[
\ldots \ldots \ldots (1 + x^{nh_n}) = 1 + x + x^2 + \ldots + x^n.
\]
Hence the number of perfect partitions of \( n \), is the same as the number of solutions of the equations:
\[
h_1 + 2h_2 + 3h_3 + \ldots + nh_n = n, \quad (h_1 + 1) (h_2 + 1) (h_3 + 1) \ldots (h_n + 1) = n + 1;
\]
in nonnegative integers. This can be shown to be the same as the number of ordered factorizations of \( (n + 1) \) without unit factors. Thus for \( n = 7 \), we have
\[
n + 1 = 8 = 8; 2.4; 4.2; 2.2.2;
\]
and 7 has exactly four perfect partitions as has already been seen.

Many theorems in partitions can be easily proved by a judicious use of generating functions. For example, we have
\[
g(x) = g(x) f(x) / f(x),
\]
\[
= \prod_{r=1}^{\infty} (1 + x^r) \prod_{r=1}^{\infty} (1 - x^r)^{-1} / \prod_{r=1}^{\infty} (1 - x^r)^{-1},
\]
\[
= \prod_{r=1}^{\infty} (1 - x^{2r}) / \prod_{r=1}^{\infty} (1 - x^r),
\]
\[
= \{ (1 - x) (1 - x^2) (1 - x^3) \ldots (1 - x^{2r-1}) \ldots \}^{-1}.
\]
This shows that the number of partitions of \( n \) into distinct parts is the same as the number of partitions of \( n \) into odd parts, repetitions allowed. As another example of this type of interpretation, we notice that formally we have
\[
(1 - x) (1 + x) (1 + x^2) (1 + x^4) \ldots \ldots (1 + x^{2m}) \ldots = 1.
\]
Hence
\[
(1 + x)(1 + x^2)(1 + x^4) \ldots (1 + x^{2m}) \ldots = (1 - x)^{-1} = 1 + x + x^2 + x^3 + \ldots + x^n + \ldots.
This readily shows that the number of partitions of \( n \) into distinct powers of 2 is just one, i.e., every number can be expressed as a sum of distinct powers of 2 in a unique way.

The generating function for \( t_k(n) \), the number of \( k \)-rowed partitions of \( n \), is:

\[
\prod_{r=1}^{\infty} \frac{1}{1-x^r} - \min(k, r).
\]

In 1964, Cheema and Gordon gave a combinatorial proof of this result and obtained some congruence properties of \( t_2(n) \) and \( t_3(n) \). Recently Gandhi has obtained some congruences for \( t_k(n) \).

3. Some Important Identities

3.1. Euler's Identity

A serious study of the subject of partitions started with Euler. In 1742, Euler gave the important expansion:

\[
\prod_{r=1}^{\infty} (1-x^r) = 1 - x - x^2 + x^3 + x^7 - x^{12} - x^{15} + \ldots + (-1)^k \{x^{k(3k-1)/2} + x^{k(3k+1)/2}\} + \ldots ,
\]

a result which he could not, at first, prove. Using his formula he calculated a table of values of \( p(n) \) for \( n \leq 59 \). I made use of this identity in extending my table of values of \( p(n) \) from \( n = 300 \) to \( n = 1000 \). In 1830, A. M. Legendre noted that Euler’s identity implies that every number which is not pentagonal, can be partitioned into an even number of distinct parts as often as it can be partitioned into an odd number of distinct integers, while the pentagonal number \( k(3k+1)/2 \) can be partitioned into an even number of distinct parts once oftener or once fewer times than into an odd number of distinct parts according as \( k \) is even or odd. Franklin used Ferrers graphs to give a remarkable proof of Euler’s identity.

3.2. Jacobi's Identity

Jacobi gave fundamental applications of elliptic functions to the study of the theory of partitions. In 1829, he proved that

\[
\prod_{r=1}^{\infty} (1-x^{2r})(1+zx^{2r-1})(1+z^{-1}x^{2r-1}) = 1 + x(z + z^{-1}) + x^4(z^2 + z^{-2}) + x^9(z^3 + z^{-3}) + \ldots .
\]

He then deduced the following relations:

\[
\prod_{r=1}^{\infty} (1-x^{2r})(1+x^{2r-1})^2 = \sum_{-\infty}^{\infty} x^{r^2};
\]

\[
\prod_{r=1}^{\infty} (1-x^{2r})(1-x^{2r-1})^2 = \sum_{-\infty}^{\infty} (-1)^r x^{r^2};
\]

\[
2x^{1/4} \prod_{r=1}^{\infty} (1-x^{2r})(1+x^{2r})^2 = \sum_{-\infty}^{\infty} x^{(2r+1)^2/4}
\]

and finally

\[
\prod_{r=1}^{\infty} (1-x^{2r})^3 = \sum_{r=0}^{\infty} (-1)^r (2r+1)x^{r^2+r}.
\]
The last one of these identities is of particular importance in the theory of partitions and has been used repeatedly by Gandhi and others. Euler's identity can be shown to be a particular case of Jacobi's.

In Jacobi's identity, replace $x^2$ by $uv$, $z^2$ by $u/v$, and change $u$ to $-u$ and $v$ to $-v$, to obtain the identity:

$$
\prod_{r=1}^{\infty} (1-u^r v^r) (1-u^{-r} v^{-r}) = \sum_{r=1}^{\infty} (-1)^r u^{r(r+1)/2} v^{r(r-1)/2}.
$$

In 1964, Cheema interpreted this combinatorially in the form of the

**Theorem:** The excess of the number of partitions of $(n, m)$ into even number of parts of the type $(a, a-1), (b-1, b), (c, c)$ over those into odd number of parts, distinct in each case, is $(-1)^r$ or zero, according as $(n, m)$ is or is not of the type $(r(r+1)/2, r(r-1)/2)$.

He also gave a graphical proof of this result.

In fact, identities involving theta functions are equivalent to combinatorial theorems involving partitions under two different conditions.

In 1966, Sudler gave two enumerative proofs of Jacobi's identity.

### 3.3. Cauchy's Identities

In 1843, A. Cauchy, besides proving Euler's identity, showed that

$$
\prod_{r=0}^{n-1} (1+z x^r) = 1 + \frac{1-x^n}{1-x} z + \frac{(1-x^n)(x-x^n)}{(1-x)(1-x^2)} z^2 + \ldots
$$

$$
+ \frac{(1-x^n)(x-x^n)(x^2-x^n)}{(1-x)(1-x^2)(1-x^3)} z^3 + \ldots (1-x^n) z^n;
$$

and

$$
\prod_{r=0}^{n-1} (1-z x^r)^{-1} = 1 + \frac{1-x^n}{1-x} z + \frac{(1-x^n)(1-x^{n+1})}{(1-x)(1-x^2)} z^2 + \frac{(1-x^n)(1-x^{n+1})(1-x^{n+2})}{(1-x)(1-x^2)(1-x^3)} z^3 + \ldots + \frac{(1-x^n)(1-x^{n+1})(1-x^{n+2})(1-x^{n+3})}{(1-x)(1-x^2)(1-x^3)(1-x^4)} z^4 + \ldots
$$

These identities lead to the generating functions for $p(n, k)$ and $q(n, k)$ given already. As a special case of Cauchy's identities, we have the following identities of Euler:

$$
\prod_{r=0}^{\infty} (1+z x^r) = 1 + \sum_{r=1}^{\infty} \frac{x^{r(r-1)/2} z^r}{(1-x)(1-x^2) \ldots (1-x^r)},
$$

$$
\prod_{r=0}^{\infty} (1-z x^r)^{-1} = 1 + \sum_{r=1}^{\infty} \frac{z^r}{(1-x)(1-x^2) \ldots (1-x^r)}.
$$

### 3.4. Gordon's Identities

Basil Gordon has given the identities:

$$
\prod_{r=1}^{\infty} (1-x^r)(1-yx^r)(1-y^{-1}x^{-r-1})(1-y^2x^{2r-1})(1-y^{-2}x^{-2r-1}) = \sum_{r=-\infty}^{\infty} x^{r(3r+1)/2} (y^{3r}-y^{-3r-1}).
$$

(A proof of this identity was given by Mordell in 1962.)

$$
\prod_{r=1}^{\infty} (1-x^r)^3(1-x^{2r-1})^2 = \sum_{r=-\infty}^{\infty} (6r+1) x^{r(3r+1)/2}.
$$

(This was also proved by Bailey.)
3.5. Ramanujan's Identities

Ramanujan stated without proof the two remarkable identities:

(1) \( p(4) + p(9)x + p(14)x^2 + \ldots = 5\{f(x)\}^6/\{f(x^5)\}^5; \)

(2) \( p(5) + p(12)x + p(19)x^2 + \ldots = 7\{f(x)\}^4/\{f(x^7)\}^3 + 49x \cdot \{f(x)\}^8/\{f(x^7)\}^7. \)

These identities lead to the well known congruences of Ramanujan for the moduli 5, 5², 7, 7²; which we consider a little later. In 1950, using formal power series, D. Kruyswijk obtained the two Ramanujan identities.

3.6. Rogers-Ramanujan Identities

The following two identities, first found by Rogers in 1894, were rediscovered by Ramanujan about 1913.

\[
\prod_{r=0}^{\infty} \left\{ (1-x^{5r+1})(1-x^{5r+4}) \right\}^{-1} = 1 + \sum_{r=1}^{\infty} \frac{x^r}{(1-x)(1-x^2) \ldots (1-x^r)}
\]

\[
\prod_{r=0}^{\infty} \left\{ (1-x^{5r+2})(1-x^{5r+3}) \right\}^{-1} = 1 + \sum_{r=1}^{\infty} \frac{x^{r(r+1)}}{(1-x)(1-x^2) \ldots (1-x^r)}
\]

Rogers-Ramanujan identities have received considerable attention. Several proofs of these identities have been given, besides two by Rogers himself. An ingenious proof was given by J. M. Dobbie in 1962. In 1954, Henry L. Alder gave a generalization of these identities. A combinatorial interpretation of the identities was given by Gordon in 1961. B. Gordon and G. E. Andrews have generalized these identities in different directions.

3.7. Generalizations of \( p(n) \)

In 1951, Gupta considered the function:

\[
J(x) = \prod_{r=1}^{\infty} (1-x^r)^{-r^{j-1}}.
\]

This is a generalization of the generating function \( f(x) \) for \( p(n) \). He also considered the function:

\[
J(x, m) = \prod_{r=1}^{m} (1-x^r)^{-r^{j-1}}.
\]

In 1960, O. Kolberg gave another generalization of \( f(x) \) viz,

\[
P(x, j) = \{f(x)\}^{-j} = 1 + \sum_{n=1}^{\infty} p_j(n)x^n.
\]

For positive values of \( j \), Newman has shown that \( p_j(n) \) is a polynomial in \( j \) of degree \( n \). Gupta has expressed \( p_j(n) \) in terms of combinatorial functions. Wright has obtained an estimate for \( p_j(n) \) for large \( n \). Gandhi has congruences for \( p_j(n) \) from which results due to Ramanathan and Lahiri and also those concerning the parity of \( p(n) \) follow as special cases.

Kolberg proved that

\[
P(x^2, 5)P(-x, -2) = \sum_{m=-\infty}^{\infty} (3m+1)x^{m(3m+2)}
\]
and
\[ P(x, 5)P(x^2, -2) = \sum_{m=-\infty}^{\infty} (6m + 1)x^{m(3m+1)/2}. \]

In 1953, L. Carlitz derived Newman's formula:
\[ P(x, 6)P(x^5, -1) = \sum_{m=0}^{\infty} p_5(5m)x^m. \]

4. Recursion Formulas

In extending to \( n = 300 \), the partition table which MacMahon and Ramanujan had computed up to \( n = 200 \), I adopted a new procedure. Denoting by \( r(n, m) \), the number of partitions of \( n \) with one part equal to \( m \) and other parts \( \geq m \), I showed that

\[ r(n, m) = r(n-m, m) + r(n+1, m+1). \]

This recursion formula enabled me to compute a double entry table for the values of \( r(n, m) \).

The work was very much reduced with the help of the following relations:

\[ r(m, m) = 1; \]
\[ r(m+j, m) = 0, \quad 0 < j < m; \]
\[ r(2m+j, m) = 1, \quad 0 \leq j < m; \]
\[ r(3m+j, m) = 2 + \lfloor j/2 \rfloor, \quad 0 \leq j < m; \]

and
\[ r(4m+j, m) = 3 + \lfloor (m+j)/2 \rfloor + \lfloor j(j+6)/12 \rfloor, \quad 0 \leq j < m. \]

The result for \( r(5m+j, m) \) is rather complicated. Moreover, for \( 0 \leq k < 4 \),
\[ r(4m+k, > m) = m+2 + \lfloor k/2 \rfloor + \lfloor (m+k+3)(m+k-3)/12 \rfloor. \]

Since \( p(n) = r(n+1, 1) \), my table provided a table of partitions as well. The generating function for \( r(n, m) \) will be readily seen to be:

\[ x^m/(1-x^m)(1-x^{m+1})(1-x^{m+2}) \ldots \]

In 1958, G. Palama evaluated \( r(n, > \lfloor n/5 \rfloor) \).

The recursion formula for \( p(n, k) \) is:
\[ p(n, k) = p(n-1, k-1) + p(n-k, k). \]

This gives:
\[ p(n, 1) = 1; \]
\[ p(n, 2) = \lfloor n/2 \rfloor; \]
\[ p(n, 3) = \lfloor (n^2+3)/12 \rfloor. \]
Formulas for \( p(n, k), k \leq 12, \) are given in the introduction to the Royal Society Tables of Partitions. In 1942, I obtained a formula expressing \( r(n, m) \) in terms of \( p(n, m) \). It gave

\[
r(n + 2m + 1, m + 1) = p(n, 1) + p(n - 2m, 2) + p(n - 4m, 3) + \ldots + p(n - jm, j + 1),
\]

with \( j = \lfloor (n - 1)/(m + 1) \rfloor \).

Very recently, using elementary methods, J. Intrator has proved that the number of partitions of any natural number \( n \) into exactly \( k \) summands, is given by a polynomial of degree exactly \( (k - 1) \) in \( n \), the first \( \lfloor (k + 1)/2 \rfloor \) coefficients of which are independent of \( n \), while the others depend on the residue of \( n \) modulo the least common multiple of the integers: 1, 2, 3, \ldots, \( k \). This is an improvement on earlier results. Actually the coefficient of \( n^j \) in the polynomial depends on the residue of \( n \) modulo the least common multiple of the integers: 1, 2, 3, \ldots, \( \lfloor k/(j+1) \rfloor \); \( j \leq (k - 1) \).

Let \( p^*(n, k) \) denote the number of partitions of \( n \) into at most \( k \) summands. Then, we have

\[
p^*(n, k) = p^*(n, k - 1) + p^*(n - k, k).
\]

In fact, \( p^*(n, k) = p(n + k, k) \).

In 1955, Lothar Berg considering solutions of the form:

\[
t^{-ng_k(t)}
\]

for the above recurrence, derived many known results for \( p^*(n, k) \) and for \( p(n) \). In particular, he showed, in a very simple way, that

\[
p(n) < C \exp(2C \sqrt{n}) / \sqrt{n}, \ C^2 = \pi^2/6.
\]

5. Graphical Methods

5.1. Ferrers Graphs

An idea that has been very useful in the study of the theory of partitions, is that of Ferrers graphs. The partition: 6 + 2 + 1 of 9, is represented by means of dots—six in the first row, two in the second and one in the third, as in the following diagram:

\[
\begin{array}{ccccccc}
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\end{array}
\]

If we read this graph in columns, we get the partition: 3 + 2 + 1 + 1 + 1 + 1 of 9. The two partitions are said to be conjugate. Some partitions are self-conjugate too. Such, for example, is the partition: 5 + 4 + 2 + 2 + 1 of 14. Corresponding to every self-conjugate partition of \( n \), we have a partition of \( n \) into odd distinct parts and conversely. Hence, the number of self-conjugate partitions of \( n \) is the same as the number of partitions of \( n \) into odd distinct parts. This and several other theorems in partitions are easily proved by means of graphs. For example, the following two can be mentioned:

1. The number of partitions of \( n \) into exactly \( k \) summands, is the same as the number of partitions of \( n \) into summands the largest of which is \( k \).

2. The number of partitions of \( n \) into odd parts is the same as the number of partitions of \( n \) into distinct parts. (This has already been proved with the help of generating functions.)

5.2. Decompositions of \( n \)

Closely connected with Ferrers graphs, is the following method of representing a decomposition of \( n \).
Take a straight line $ON$ of $n$ units in length. If

$$n = n_1 + n_2 + n_3 + \ldots + n_k,$$

be a decomposition of $n$ into $k$ parts, then taking along the line $ON$;

$$0A_1 = n_1, A_1A_2 = n_2, A_2A_3 = n_3, \ldots, A_{k-1}N = n_k;$$

we get

as a graphic representation of the said decomposition. Conversely, given the graphic representation, the decomposition can be written out. It will be readily seen that the number of decompositions of $n$ into exactly $k$ summands, is the same as the number of ways in which exactly $(k-1)$ lattice points can be marked strictly between 0 and N. Since there are $(n-1)$ lattice points between 0 and N, in all; this can be done in $\binom{n-1}{k-1}$ ways. This gives the number of decompositions of $n$ into $k$ parts. The total number of decompositions of $n$ is given by

$$\sum_{k=1}^{n} \binom{n-1}{k-1} = 2^{n-1}.$$ 

Since a partition of $n$ into $k$ parts, cannot give rise to more than $k!$ decompositions (and that too only if the $k$ parts are distinct), it follows that

$$p(n, k) \geq \frac{1}{k!} \binom{n-1}{k-1} \geq q(n, k).$$

We might mention here that the number of decompositions of $n$ into at most $k$ summands, is the number of solutions of the equation:

$$x_1 + x_2 + x_3 + \ldots + x_k = n,$$

in nonnegative integers $x$. This is the same as the number of ways in which $n$ like objects can be distributed among $k$ persons when there is no restriction as to the number of objects any of them may receive.

The method of representation described above can be extended to a $j$-dimensional space. Gupta has thus obtained generalizations of the above inequality for partitions of $j$-partite numbers.

### 6. Congruences

#### 6.1. Ramanujan's Conjecture

About 1919, Ramanujan proved that for any nonnegative $m$,

$$p(5m+4) = 0 \pmod{5}, \quad p(25m+24) = 0 \pmod{25};$$

$$p(7m+5) = 0 \pmod{7}, \quad p(49m+47) = 0 \pmod{49}. $$

He further stated that

$$p(11m+6) = 0 \pmod{11}.$$
All these results are included in his famous conjecture:

$$\text{If } p = 5, 7, \text{ or } 11 \text{ and } (24n - 1) \equiv 0 \pmod{p^a}, \ a \geq 1;$$

then also $$p(n) \equiv 0 \pmod{p^a}.$$ 

This astounding conjecture held good till, using my table of values of $$p(n)$$ for $$n \leq 300$$, S. Chowla found that the conjecture failed for $$n = 243$$. For this value of $$n$$,

$$24n - 1 = 5831 \equiv 0 \pmod{7^3},$$

while

$$p(243) = 13397 \ 82593 \ 44888 \equiv 0 \pmod{7^2},$$

but

$$\neq 0 \pmod{7^3}.$$ 

Later it was found that the conjecture failed also when $$n = 586$$.

This discovery led to a lot of work particularly by D. H. Lehmer, who using Hardy-Ramanujan and Rademacher Series (discussed later in this survey), computed values of $$p(n)$$ for certain large values of $$n$$. He thus found that

$$p(599) \equiv 0 \pmod{5^4}, \quad p(721) \equiv 0 \pmod{11^3};$$

$$p(1224) \equiv 0 \pmod{5^4}, \quad p(2052) \equiv 0 \pmod{11^3};$$

$$p(2474) \equiv 0 \pmod{5^5}, \quad p(14031) \equiv 0 \pmod{11^4}.$$ 

These results are in conformity with Ramanujan’s conjecture.

In 1948, D. B. Lahiri found three new congruences:

$$p(49m + r) \equiv 0 \pmod{49} \text{ for } r = 19, 33, 40;$$

and announced two more:

$$p(125m + r) \equiv 0 \pmod{125} \text{ for } r = 74, 124.$$

In 1952, from a study of Ramanujan’s manuscript, J. M. Rushforth proved that

$$p(121m + 116) \equiv 0 \pmod{121};$$

and

$$p(49m + r) \equiv 0 \pmod{49} \text{ for } r = 19, 33, 40, 47.$$ 

In 1960, Professor M. Newman showed that for $$(m, 30) = 1$$.

$$p((167m^2 + 1)/24) \equiv 0 \pmod{5}.$$ 

These were all results of the Ramanujan type.

In the meantime, G. N. Watson had proved Ramanujan’s conjecture completely for powers of 5. For powers of 7, he proved the following modification of the conjecture:

If

$$24n - 1 \equiv 0 \pmod{7^b},$$

then

$$p(n) \equiv 0 \pmod{7^d} \quad \text{where } d = \left\lceil \frac{(b+2)}{2} \right\rceil.$$
Finally, A. O. L. Atkin has essentially settled the problem by proving Ramanujan’s conjecture for powers of 11. The full truth with regard to the conjecture can now be stated in the form of the

**Theorem:** If $24n - 1 \equiv 0 \pmod{5^a 7^b 11^c}$, where $a, b, c$ are nonnegative integers, then

$$p(n) \equiv 0 \pmod{5^a 7^b 11^c}$$

with $d = \lceil (b + 2)/2 \rceil$.

Watson’s method was based on modular equations. Atkin followed the method of J. Lehner, which had enabled Lehner to prove Ramanujan’s conjecture for 11, 11², 11³, in 1943, 1950.

Mention must also be made of the method which Professor H. Rademacher developed in 1942, for the investigation of the Ramanujan identities with respect to the moduli $5^a$, 7, 7², and 13a.

Very recently, A. O. L. Atkin and J. N. O’Brien have made a valuable contribution to the literature on the subject by giving some properties of $p(n)$ modulo powers of 13.

### 6.2. The Parity of $p(n)$

In 1920, MacMahon had shown that

$$p(n) \equiv \sum_{t} p(t) \pmod{2}$$

where $t$ runs through the positive integral values given by the relation:

$$8t = 2n - j(j + 1), \quad j \geq 0.$$ 

It is rather strange that nothing better should be known about the parity of $p(n)$ than MacMahon’s congruence.

In 1946, making use of the fact that Ramanujan’s function $\tau(n)$ is odd if and only if $n$ is an odd square, I obtained MacMahon’s congruence.

### 6.3. Congruences for $p_j(n)$

Newman, Kolberg, Ramanathan, Gandhi and several others have studied the function $p_j(n)$ defined in section 3.7. In 1956, Newman gave extensive tables of values of $p_j(n)$ for $j \leq 16$.

For $j = 2, 4, 6, 8, 10, 14, 26$ and a prime $p > 3$ such that

$$j(p + 1) \equiv 0 \pmod{24},$$

M. Newman proved that

$$p_j(np + k) = (-p)^{(j-2)/2}p_j(n/p),$$

where $k = j(p^2 - 1)/24$ and we take $p_j(\alpha) = 0$ if $\alpha$ is not a nonnegative integer. He also showed that

$$p_9(3n + 2) = 9p_9(n/3);$$

and that $p_j(n)$ vanishes for arbitrarily long strings of consecutive values of $n$,when $j = 2, 4, 6, 8, 10, 14$ or 26.

Ramanathan showed that

If $24n + j \equiv 0 \pmod{5^a}$, $j = 16, 21, 26 \pmod{30},$

then

$$p_j(n) \equiv 0 \pmod{5^b}$$

where $b = \lceil (a + 1)/2 \rceil$. 
6.4. Rank of a Partition

In 1954, Atkin and Swinnerton-Dyer proved some conjectures of Dyson, who had discovered, in 1944, empirically a remarkable combinatorial method of dividing the partitions of \((5n+4)\) and \((7n+5)\) into five and seven equal classes respectively. Dyson defined the rank of a partition as the largest part minus the number of parts in the partition. The partitions of a given number could then be divided into classes according to their ranks. In the case of numbers of the forms \((5n+4)\) and \((7n+5)\), these classes had equal number of elements in each class.

Atkin and Swinnerton-Dyer showed that for \(q=5, 7,\) or \(11\), and \(0 < b < q;\)

\[
\sum_{n=0}^{\infty} p(qn+b)y^n
\]

is congruent modulo \(q\) to a simple infinite product.

In 1958, Atkin and Hussain obtained an identity for

\[
\sum_{n=0}^{\infty} p(11n+6)y^n.
\]

6.5. Sylvester's Theorem

In 1945, D. P. Banerjee proved Sylvester's theorem, which states that \(p(n) - Q(n)\) is even, where \(Q(n)\) denotes the number of partitions of \(n\) into odd distinct parts.

6.6. Newman's Conjecture

In 1960, M. Newman conjectured that for every pair of natural numbers \(m\) and \(r\), there are infinitely many natural numbers \(n\), such that

\[p(n) \equiv r \pmod{m}.
\]

He proved the conjecture for \(m=5, 13, 65\) and some other composite moduli. Kolberg and Subbarao have proved it for \(m=2\). In this connection, the following two results of Newman might be of interest:

Neither of the congruences:

\[p(13n-7) \equiv r \pmod{13},\]

and

\[p(13^2n-7) \equiv r \pmod{13},\]

can hold for all sufficiently large \(n\) and a fixed \(r\).

7. Some Inequalities

We have already shown that

\[k!q(n, k) \leq \left(\frac{n-1}{k-1}\right) \leq k!p(n, k).
\]

Since,

\[q(n, k) = p(n-\frac{1}{2}k(k-1), k),\]

we must have

\[\left(\frac{n+\frac{1}{2}k(k-1)-1}{k-1}\right) \geq k!p(n, k) \geq \left(\frac{n-1}{k-1}\right).
\]

(Originally, I had made use of the recursion formula for \(p(n, k)\) to prove this inequality.)
The inequality implies that
\[ p(n, k) \sim \frac{1}{k!} \left( \frac{n-1}{k-1} \right), \text{ for } k = o(n^{1/3}). \]

Erdos and Lehner had proved this result in 1941, by a rather lengthy method. They had in fact proved that if \( p^*(n, k) \) denotes the number of partitions of \( n \) into at most \( k \) summands, then
\[ p^*(n, k)/p(n) \sim \exp \left( -C^{-1}e^{-Cx} \right), \]
where (here and in all that follows), \( C = \pi/\sqrt{6}; \) and \( x \) is given by the relation:
\[ CK = \sqrt{n} \log \sqrt{n} + Cx \sqrt{n}. \]

They gave a similar result for partitions into distinct parts.

As an analogue of this remarkable result of Erdos and Lehner, I showed that
\[ p(n, k)/p(n) \sim tn^{-1/2} \exp \left( -t/C \right). \]
where
\[ t = \lim_{n,k \to \infty} \sqrt{n} \exp \left( -Ckn^{-1/2} \right). \]

In 1957, S. M. Luthra obtained an asymptotic formula for
\[ \sum_{k=1}^{n} kp(n, k)/p(n). \]

The case of partitions of \( n \) into \( k \) distinct parts was also considered.

In 1950, Auluck, Singwi, and Agarwala gave another asymptotic result of some interest:

If \( w(n) \) denotes the number of partitions of \( n \) into integers each of which occurs in the partition only an odd number of times, then for a certain constant \( \gamma \),
\[ w(n) \sim \frac{\gamma}{2\pi n} \exp \left( 2\gamma \sqrt{n} \right). \]

If \( p(n, A) \) denotes the number of partitions of \( n \) into members of the set:
\[ A = \{1, a_1, a_2, a_3, \ldots, a_m\}, \]
then, I proved in 1955, the inequality:
\[ (n+m) \leq p(n, A) \prod_{j=1}^{m} a_j \leq \left( n + \sum_{j=1}^{m} a_j \right). \]

This is a generalization of the inequality given earlier.

In 1954, writing
\[ \{(1-x)(1-x^2)(1-x^3)\ldots(1-x^r)}^{-1} = \sum_{j=0}^{\infty} c_j(r) (1-x)^{j-r}, \]
I showed that for a fixed \( j \) and large \( r \),
\[ c_j(r) \sim \{j!(r-2j)!4^j\}^{-1}. \]

This led to an asymptotic formula for \( p^*(n, r) \).
In 1956, Bateman and Erdos showed that the number of partitions of \( n \) into parts taken from a nonempty set \( A \) of positive integers, is strictly increasing for large \( n \), if and only if \( A \) contains more than one element and if when any element is removed from \( A \), the remaining elements have the greatest common divisor unity.

## 8. Conjecture of Auluck, Chowla, and Gupta

In 1942, Auluck, Chowla, and Gupta conjectured that \( p(n, k) \) has a unique maximum for any given \( n \), i.e., regarding \( n \) as fixed there exists an integer \( k_0 \) such that

\[
p(n, k) \geq p(n, k-1), \text{ for } k \leq k_0;
\]
and

\[
\leq p(n, k-1), \text{ for } k > k_0.
\]

This conjecture has not been completely proved so far.

In 1953, however, G. Szekeres proved that a maximum of \( p(n, k) \) occurs at

\[
k = \frac{\sqrt{n}}{C} + \frac{1}{C^2} \left( \frac{3}{2} + \frac{3}{2} L - \frac{1}{4} L^2 \right) - \frac{1}{2} + \mathcal{O} \left( \frac{\log^4 n}{\sqrt{n}} \right),
\]

where

\[
L = \log \left( \frac{\sqrt{n}}{C} \right).
\]

He deduced this result from his formula for \( p^*(n, k) \) true for bounded \((n/k^2)\). The formula itself was obtained by using the method of steepest descent.

In fact, the work of Szekeres opened up the possibility of obtaining the infinite series for \( p(n) \) without using the theory of elliptic modular functions. He sharpened the results of Erdos and Lehner regarding maximum of \( p(n, k) \) and \( q(n, k) \). It was shown that \( q(n, k) \) has a maximum at

\[
k = \frac{2\sqrt{3n} \log 2}{\pi} + \beta + \mathcal{O}(n^{-1/2});
\]

where \( \beta \) is a fixed constant.

In 1954, Haselgrove and Temperley developed a powerful method which made no use of Farey dissections and enabled them to prove that \( p(n, k) \) attains its greatest value for at most two consecutive values of \( k \) when \( n \) is large and fixed. This had been conjectured by Auluck, Chowla, and Gupta.

## 9. Hardy-Ramanujan and Rademacher Series

No account of partition theory can be even fairly complete without a mention of the great work of Hardy and Ramanujan on the subject. Using Farey dissections, they obtained an infinite series for \( p(n) \), the first few terms of which gave the value of \( p(n) \) exactly if we neglect the decimal part in the answer. This was astounding. Later on Lehmer showed that the Hardy-Ramanujan series was divergent and Hardy and Ramanujan had been fortunate in breaking at a point where the series gave a correct answer.

The Hardy-Ramanujan series gives

\[
p(n) = \frac{(12)^{1/2}}{(24n-1)u} \sum_{k=1}^{\sqrt{n}} A_k^*(n)(u-k) \exp \left( u/k \right) + \mathcal{O}(\log n/\sqrt{n});
\]

where

\[
u = \pi (24n-1)^{1/2}/6.
\]
A change in the path of integration enabled Rademacher to replace \((u-k) \exp (u/k)\) in the Hardy-Ramanujan series by

\[(u-k) \exp (u/k) + (u+k) \exp (-u/k)\]

This simple-looking change gave a convergent series for \(p(n)\).

The functions \(A_k^n(n)\) in the series for \(p(n)\) have been shown by Lehmer to be multiplicative in a certain sense. In 1956, Whiteman used Fourier series for the evaluation of these functions and Rademacher obtained the Selberg formula for \(A_k^n(n)\) by using a transformation. He also put Lehmer's theorems for the evaluation and factorization of \(A_k^n(n)\) in a form suitable for computation.

The method of Rademacher has been widely applied to partition problems of various types. For example, Lehner has obtained a convergent series for the number of partitions of a positive integer \(n\) into summands of the form: \(5m \pm a, a=1, 2\); and Hua for \(q(n)\). Hagis has not only derived Rademacher's formula for \(p(n)\) and the convergent series for \(q(n)\), but also a convergent series for \(q(n; t)\)—the number of partitions of \(n\) into odd summands, no summand appearing more than \(t\) times in a partition.

In 1941, Ingham used a Tauberian theorem to obtain the asymptotic formulas for \(p(n)\) and \(q(n)\). These are:

\[p(n) \sim (4n \sqrt{3})^{-1} \exp (\pi \sqrt{2n/3}),\]
\[q(n) \sim (4n^{3/2} 3^{1/4})^{-1} \exp (\pi \sqrt{n/3}).\]

In 1961, Iseki gave a shorter proof than Rademacher's for a functional equation which implies the transformation equation for Dedekind's modular function \(\eta(\tau)\). D. J. Newman has obtained the asymptotic formula for \(p(n)\) in an elementary and simple way. In 1962, he gave another proof for it.

In 1954, Petersson developed a function-theoretic method for finding an asymptotic formula for the partition function. This method is different from the Hardy-Ramanujan method.

## 10. Miscellaneous Results

Mention must be made of the work of O. P. Gupta and S. Luthra who, in 1955, gave a table for the number of partitions of \(n \leq 300\), into primes. I gave a table for the number of partitions into distinct primes.

In 1957, Takayoshi Mitsui used the powerful methods of Vinogradov and Hua to obtain an asymptotic formula for the number of partitions of \(n\) into \(k\)th powers of primes not exceeding \(m\). His results generalized those of Haselgrove and Temperley.

Before I conclude, I give some results which, I believe, will interest the reader.

(i) The number of partitions of \(n\) into distinct parts, the smallest being odd, is odd if and only if \(n\) is a square. This was given by N. J. Fine in 1948.

(ii) In 1958, Guy proved that the number of partitions of an integer into:

( a) odd parts greater than unity;

( b) unequal parts such that the two greatest parts differ by unity;

( c) unequal parts which are not powers of 2; are all equal.

(iii) In 1963, R. L. Graham showed that every integer greater than 77 can be partitioned into distinct positive integers whose reciprocals add to 1.

(iv) J. B. Kelly proved the following result in 1964: Given \(k \geq 3\), every \(n \geq N(k)\), can be partitioned into \(k\) parts in \((k-1)\) different ways so that the products of the integers in each of the \((k-1)\) partitions are equal.

(v) By far the most remarkable result regarding two-rowed partitions of \(n\), when the elements in rows strictly decrease, is that the number of such partitions is \(p(n)\), but it is only \(p(\lfloor n/2 \rfloor)\) when the parts are all odd.
This was proved by B. Gordon in 1962. Sudler gave a direct proof of this result in 1965. More recently B. Gordon and L. Houten obtained generating functions and asymptotic formulas for various restricted rowed and plane partitions.

11. Bibliography

Alder, Henry L.,


Andrews, George E.,


Apostol, Tom M.,


Arkin, Joseph,


Atkin, A. O. L.,


Atkinson, F. V.,


Auluck, F. C.,

Auluck, F. C., Singwi, K. S., and Agarwala, B. K.,


Avakumovic, Vojislav G.,


Bailey, W. N.,


Banerjee, D. P.,


Basu, N. M.,


Bateman, Paul T., and Erdős, Paul,


Bell, Eric Temple,

(2) A class of numbers connected with partitions, Amer. J. Math 45, 73–82 (1923).

Bennett, J. H.,


Bergmann, S.,


Bergmann, Lothar,


Bioche, Ch.,


Blakley, G. R.,


Blij, F. Vander,


Brauer, Alfred,


Brauer, A., and Seelbinder, B. M.,


Brigham, Nelson A.,


Browkin, J.,


Carlitz, Leonard,


Carlitz, L., and Riordan, John,


Chandra, R.,


Cheema, M. S.,

(1) Tables of partitions of Gaussian integers giving the number of partitions of \( n + im \), Nat. Inst. Sci. India, Mathematical Tables 1, (1956).
(3) The maxima of \( P_r(n, n_2) \), Math. Comp. 22, 199–200 (1968).

Cheema, M. S., and Gordon, Basil,


Cheema, M. S., and Gupta, H.,

Csorba, G.,

Dirac, G. A.,

Dobbie, J. M.,

Dutta, Mahadeb,

Dutta, Mahadeb, and Debnath, Lokenath,

Erdős, P.,

Erdős, P., and Lehner, J.,

Erdős, P., and Turan, P.,

Fekete, A. E.,
(1) Signature of partitions and divisors, published by the author, Memorial University of Newfoundland, St. John’s, Newfoundland, i + 31 pp. (1965).

Fielder, Daniel C.,
(1) Enumeration of partitions subject to limitations on size of members, Fibonacci Quart. 4, 209–216 (1966).

Fine, N. J.,
(1) On a system of modular functions connected with the Ramanujan identities, Tohoku Math. (2) 8, 149–164 (1956).

Ford, W. B.,

Franklin, F.,
(1) Sur le development du produit infinit, (1−x)(1−x²)(1−x³) . . . . , C.R. 92, 448–450 (1881).

Freiman, G. A.,

Gandhi, J. M.,
(2) Generalization of Ramanujan’s congruences p(5m+4) = 0 (mod 5) and p(7m+5) = 0 (mod 7), Monatsh. Math. 69, 389–392 (1965).

Gigli, D.,
(1) Sulle somme di n addendi diversi presi fra i numeri 1, 2, . . . , m, Palermo Rend 16, 280–285 (1902).
Glaisner, J.,
(1) On the number of partitions of a number into a given number of parts, Quart. J. **50**, 57–143 (1908).
(2) Formulae for partitions into given elements derived from Sylvester’s theorem, Quart. J. **40**, 275–348 (1909).
(3) Formulae for the number of partitions of a number into the elements 1, 2, . . ., n up to n = 9, Quart. J. **41**, 94–112 (1909).

Gleissberg, W.,

Glosel, K.,

Gordon, B.,

Gordon, B., and Houten, L.,

Gould, H. W.,
(1) Binomial coefficients, the bracket function, and compositions with relatively prime summands, Fibonacci Quart. **2**, 241–260 (1964).

Graham, R. L.,

Grosswald, E.,

Gupta, H.,
(9) On the maximum values of \( p_k(n) \) and \( \pi_k(n) \), J. Indian Math. Soc. (N.S.) **7**, 72–75 (1943).
(20) Partition of \( j \)-partite numbers, Math. Student **31** (1963); 179–186 (1964).

Gupta, H., Gwyther, C. E., and Miller, J. C. P.,

Gupta, O. P.,
Gupta, O. P., and Luthra, S.,

Gustin, W. E.,

Haberzetle, M.,

Hagis, P. Jr.,

Hahn, Hwa S.,

Hardy, G. H.,
(1) Ramanujan; Twelve lectures on subjects suggested by his life and work, Cambridge Univ. Press (1940), vii+236.

Hellund, E. J.,

Hua, Loo-Keng,

Husimi, H.,

Van Ijzeren, J.,
(1) Elementary properties of the partitions of the natural numbers, Mathematica, Zutphen B. 12, 115–118 (1944).

Ingham, A. E.,

Intrator, Jakub,

Iseki, Kaneshiro,

Iseki, Sho,

Kantz, G.,

Kelly, John B.,
Kempner, A. J.,


Kloosterman, H. D.,


Knopp, K.,


Knopp, M. I.,


Kolberg, O.,

(2) Identities involving partition functions $g(n)$ and $q_r(n)$. Math. Scand. 6, 80–86 (1958).

Koshti, M. E.,


Kreemar, W.,


Kruyswijk, D.,

(1) On some well known properties of the partition function $p(n)$ and Euler’s infinite product. Nieuw Arch. Wiskunde (2) 23, 97–107 (1950).

Lahiri, D. B.,


Landau, E.,

(2) Handbuch der Lehre von der Verteilung der Primzahlen, Bd. 1 und 2 (1909).

Larsen, O.,


Lehmer, D. H.,

(3) An application of Schlöfli’s modular equation to a conjecture of Ramanujan, Bull. Amer. Math. Soc. 44, 84–90 (1938).

Lehner, J.,

(2) Ramanujan identities involving the partition function for the moduli $11^q$, Amer. J. Math. 65, 492–520 (1943).
Lindenbaum, A.,

Livingood, J.,

Lorentz, G. C.,

Luckey, P.,

MacMahon, P. A.,
(2) Note on the parity of the number which enumerates the partitions of a number, Proc. Cambridge Phil. Soc. 20, 281–283 (1920–21).
(6) The parity of $p(n)$, the number of partitions of $n$, when $n \leq 1000$, J. London Math. Soc. 1, 225–226 (1926).

Majumdar, K. N.,

Makowski, A.,

Meinardus, G.,

Melzak, Z. A.,

Mian, A. M. and Chowla, S.,

Mordell, L. J.,

Motzkin, T. S.,
(1) Ordered and cyclic partitions, Riveon Lematematika 1, 61–67 (1947).

Nanda, V. S.,

Narayana, T. V. and Fulton, C. E.,

Newman, D. J.,
(1) The evaluation of the constant in the formula for the number of partitions of $n$, Amer. J. Math. 73, 599–601 (1951).
Newman, M.,

(5) Some theorems about $p_{r}(n)$, Canad. J. Math. 9, 68–70 (1957).

Nicol, C. A.,


Nicol, C. A., and Vandiver, H. S.,


Niven, I.,


Oderfeld, J., and Pleszczynska, E.,

(1) On some applications of partitions (Polish, Russian and English summaries) Zastos. Mat. 6, 189–198 (1962).

Ostmann, Hans-Heinrich,


Pagni, P.,


Palama, G.,


Parameswaran, S.,


Pennington, W. B.,


Petersson, H.,


Pleszczynska, E.,

(1) Recurrence formula for the number of limited partitions with different components, Zastos. Mat. 6, 305–307 (1961/1962).
Putter, J.,
(1) On a modular equation connected with partitions, Riveon Lematematika 3, 42–43 (1949).

Rademacher, H.,

Rademacher, H., and Whiteman, A.,

Rademacher, H., and Zuckerman, H. S.,

Raghavan, S.,
(1) Modular forms of degree n and representation by quadratic forms, Ann. of Math. (2) 70, 446–477 (1959).

Ramanathan, K. G.,

Ramanujan, S.,
(2) Collected papers (Cambridge Univ. Press 1927, xxxvi +355 pp).

Ricci, G.,

Rieger, G. J.,

Robertson, M. M.,

Rogers, L. J.,

Rogers, L. J., and Ramanujan, S.,

Roselle, D. P.,

Roth, K. F., and Szekeres, G.,

Rudert, W. S., and Lill, H. G.,
Rushforth, J. M.,


Sandham, H. F.,


Sapiro-Pjateckij,


Scheenfeld, L.,


Schrutka, L.,


Schur, I.,


Scorza, G.,

(1) Osservazioni varie sulla teoria delle sostituzioni e sulle partizioni di numeri interi in numeri interi, Palermo Rend. 36, 163–170 (1913).

Selberg, A.,


Shah, S. M.,

(1) An inequality for the arithmetical function \( g(x) \). J. Indian Math. Soc. 3, 316–318 (1939).

Shanks, D.,


Simons, W. H.,

(1) Congruences involving the partition function \( p(n) \), Bull. Amer. Math. Soc. 50, 883–892 (1944).

Slater, L. J.,


Stern, M. A.,


Sterneck, R.,

(3) Über ein Analogon zur additiven Zahlenentheorie, D. M. V. 12, 110–113 (1903).
(4) Geometrische Ableitung des Satzes Von De Morgan-Sylvester und seines Analogons für 4 Summanden, Palermo Rend. 32, 88–94 (1911).

Stohr, A.,


Straus, E. G.,

Subba Rao, M. V.,

Subrahmanya Sastri, V. V.,

Sudler, Culbreth, Jr.,

Sugai, I.,

Szekeres, G.,

Tanturri, A.,

Tietze, H.,

Todd, J. A.,

Vahlen, Th.,

Vaidya, A. M.,
(1) A formula for the partitions of $n$ into four parts. J. Gujarat Univ. 5, No. 1/2, 172–176 (1962).

Venkata Narayana, T.,

Vinogradov, I. M.,

Wall, H. S.,
Watson, G. N.,

Whitman, A. L.,

Wintner, A.,

Wright, E. M.,

Zeller, Chr.,

Ziaud Din, M.,

Zubrzycki, S.,
(1) A recursive formula for the number of restricted partitions. (Polish, Russian and English summaries.) Zastos Mat. 6, 231–234 (1962).

Zuckerman, H. S.,

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