JOURNAL OF RESEARCH of the National Bureau of Standards — B. Mathematical Sciences
Vol. 74B, No. 1, January–March 1970

Partitions — A Survey*

Hansraj Gupta**

(September 19, 1969)

A historical survey of some aspects of the theory of partitions is given here.

Key words: Congruences; generating functions; identities; partitions; recurrence formulas.

1. Introduction

1.1. Unrestricted Partitions

As far back as 1669, Leibnitz asked John Bernoulli, if he had investigated the number of ways in which a given number can be expressed as a sum of two or more integers. This was necessarily the problem of partitioning a given number into a specified number of parts.

A partition of a positive integer n, is a mode of expressing it as a sum of one or more positive integers, the order in which the summands occur being irrelevant. The parts are usually arranged according to size, one way or the other. Thus, 5 can be expressed as a sum in the following seven ways:

$$5;\ 1+4;\ 2+3;\ 1+2+2;\ 1+1+3;\ 1+1+1+2;\ \text{and}\ 1+1+1+1+1.$$

These are all the partitions of 5. Since there is no restriction of any kind on the number or size of parts, we call these the unrestricted partitions of 5. In all that follows, the number of unrestricted partitions of any given positive integer n will be denoted by $p(n)$ and we shall take $p(0)=1$.

In this survey, we shall give a short resumé of the advances that have been made in recent years, in dealing with problems on partitions of integers. The bibliography includes papers on partitions of vectors also but we do not propose to deal with the topic here.

1.2. Restricted Partitions

Given a set
$$A=\{a_1, a_2, a_3, \ldots \ldots\},$$

where the a's are positive integers, one may be required to find the number of ways in which a given integer can be expressed as a sum of the a's, repetitions being allowed or not. This will be an example of restricted partitions. As other examples, we can mention: (1) Partitions into distinct parts; (2) Partitions into odd parts; (3) Partitions into a specified number of parts; and (4) Partitions into a specified number of distinct parts. In fact, the restrictions that can be placed on the size or on the number of parts or both, are too many to enumerate. Very recently, I had to consider, for example, the number $g(n, m, h, k)$ of partitions of n into exactly k summands, each $\leq m$, just h (and any h) of the positive integers $\leq m$ being used as summands, in any partition. Among several interesting results, I found that

$$\sum_{n=1}^{\infty} g(n, m, h, k)=\binom{k-1}{h-1}\binom{m}{h}.$$

*An invited paper. This survey is based on an address given by the writer at a conference of the Indian Mathematical Society held at Madras in December 1963. It has been brought to date with the help of Professor M. S. Cheema of the University of Arizona, Tucson.

**Present address: 402 Mumfordganj, Allahabad 2. India.

We shall denote by $p(n, k)$ the number of partitions of n into exactly k summands; by $q(n)$ the number of partitions of n into distinct parts; and by $q(n, k)$ the number of partitions of n into exactly k distinct parts.

1.3. Decompositions

We have already stated that in a partition the order in which the summands appear is irrelevant. When the order of appearance of summands is relevant, the partition is called an ordered partition or a decomposition. (Some writers call it a composition.) We shall, however, invariably use the word decomposition in this sense.

The number of decompositions is usually easier to find than the number of partitions. The total number of decompositions of n is, in fact, given by 2^{n-1}. This is easily proved as follows:

The decompositions of n can be classified under two heads:

(1) Those in which the first part is 1; and

(2) Those in which the first part is >1.

Removing 1 (the first part), from each decomposition of n of the first kind, we obtain all the different decompositions of $(n-1)$, each once. Reducing by 1 the first part in the decompositions of the second kind, we again get all the decompositions of $(n-1)$ as before. Hence, the number of decompositions of n, is twice the number of decompositions of $(n-1)$ and the result follows readily by induction.

1.4. Perfect Partitions

A partition of n is said to be perfect when it contains just one partition of every number up to n. Thus, $1+2+2+2$ is a perfect partition of 7, because every number up to 7, can be expressed uniquely as a sum by using the summands: 1, 2, 2, 2. Other such partitions of 7 are: $1+1+1+1+1+1+1$; $1+1+1+4$; $1+2+4$.

1.5. Plane and Solid Partitions

The summands in a partition of n are sometimes arranged in the form of something like a matrix, so that the elements in each row and in each column are in a descending order, though not necessarily strictly. Such an arrangement gives a plane partition of n. Thus

$$\begin{array}{cccc} 3 & 3 & 2 & 1 \\ 2 & 2 & 2 & 1 \\ 1 & 1 & & \end{array}$$

is a plane partition of 18. A plane partition is also called a rowed partition. A k-rowed partition of n includes all partitions of n with k or fewer number of rows. Thus the three-rowed partitions of 4 are:

$$\begin{array}{cccccccccccc} 4; & 31; & 22; & 211; & 1111; & 3; & 2; & 21; & 111; & 11; & 2; & 11. \\ & & & & & 1 & 2 & 1 & 1 & 11 & 1 & 1 \\ & & & & & & & & & & 1 & 1 \end{array}$$

The number of k-rowed partitions of n is denoted by $t_k(n)$.

In a solid partition the summands are arranged in three-dimensional space as they are arranged in a two dimensional space in the case of a plane partition. The elements are arranged in descending order of magnitude in each of the three principal directions.

2. Generating Functions

A convenient way of representing a partition of n, in which one occurs h_1 times, two h_2 times, three h_3 times and so on, is to write

$$n = 1^{h_1} 2^{h_2} 3^{h_3} \ . \ . \ . \ n^{h_n},$$

where h's are nonnegative integers, and

$$h_1+2h_2+3h_3+\ .\ .\ .\ +nh_n=n.$$

Now it would not be difficult to see that $p(n)$ is the coefficient of x^n in the product:

$$(1+x+x^2+\ .\ .\ .\ +x^{h_1}+\ .\ .\ .\ +x^n)(1+x^2+x^4+\ .\ .\ .\ +x^{2h_2}+\ .\ .\ .\ +x^{2[n/2]})$$

$$(1+x^3+x^6+\ .\ .\ .\ +x^{3h_3}+\ .\ .\ .\ +x^{3[n/3]})\ .\ .\ .\ (1+x^n);$$

because a one-one correspondence can be set up between the partition:

$$1^{h_1}2^{h_2}3^{h_3}\ .\ .\ .\ n^{h_n}$$

of n and the term:

$$x^{h_1}x^{2h_2}x^{3h_3}\ .\ .\ .\ x^{nh_n}$$

of the product noted above. Using the algebra of formal power series, we can write

$$\prod_{r=1}^{\infty}(1-x^r)^{-1}=1+\sum_{n=1}^{\infty}p(n)x^n.$$

The function:

$$f(x)=\prod_{r=1}^{\infty}(1-x^r)^{-1}$$

is called the generating function for $p(n)$. In the same way, one can show that the generating function for $q(n)$ is:

$$g(x)=\prod_{r=1}^{\infty}(1+x^r).$$

In general, the generating function for the number of partitions of n into members of set

$$A=\{a_1,\ a_2,\ a_3,\ .\ .\ .\}$$

where a's are positive integers which may be considered to be arranged in a strictly ascending order of magnitude, is

$$f_A(x)=\prod_{a_j\epsilon A}(1-x^{a_j})^{-1},$$

when repetitions are allowed, and

$$g_A(x)=\prod_{a_j\epsilon A}(1+x^{a_j}),$$

when repetitions are not allowed. Moreover, if $u_A(n)$ and $v_A(n)$ denote the numbers of partitions of n into an even number of distinct a's and an odd number of distinct a's respectively, then the coefficient of x^n in the product:

$$\prod_{a_j\epsilon A}(1-x^{a_j})$$

is $u_A(n)-v_A(n)$.

The number of partitions of n into exactly k summands each taken from set A, is the coefficient of z^kx^n in

$$\prod_{a_j\epsilon A}(1-zx^{a_j})^{-1}$$

when repetitions are allowed, and in

$$\prod_{a_j \in A} (1+zx^{a_j})$$

when repetitions are not allowed. In particular, when the set A consists of all the positive integers, the generating function for $p(n, k)$ is found to be

$$x^k \prod_{r=1}^{k} (1-x^r)^{-1}.$$

Similarly the generating function for $q(n, k)$ can be shown to be

$$x^{k(k+1)/2} \prod_{r=1}^{k} (1-x^r)^{-1}.$$

If

$$1^{h_1} 2^{h_2} 3^{h_3} \ldots n^{h_n}$$

be a perfect partition of n, then we must have

$$(1+x+x^2+\ldots+x^{h_1})(1+x^2+x^4+\ldots+x^{2h_2})(1+x^3+\ldots+x^{3h_3})$$
$$\ldots (1+x^{nh_n}) = 1+x+x^2+\ldots+x^n.$$

Hence the number of perfect partitions of n, is the same as the number of solutions of the equations:

$$h_1+2h_2+3h_3+\ldots+nh_n=n, \qquad (h_1+1)(h_2+1)(h_3+1)\ldots(h_n+1)=n+1;$$

in nonnegative integers. This can be shown to be the same as the number of ordered factorizations of $(n+1)$ without unit factors. Thus for $n=7$, we have

$$n+1=8=8;\ 2.4;\ 4.2;\ 2.2.2;$$

and 7 has exactly four perfect partitions as has already been seen.

Many theorems in partitions can be easily proved by a judicious use of generating functions. For example, we have

$$\begin{aligned} g(x) &= g(x)f(x)/f(x), \\ &= \prod_{r=1}^{\infty}(1+x^r)\prod_{r=1}^{\infty}(1-x^r)^{-1} \Big/ \prod_{r=1}^{\infty}(1-x^r)^{-1}, \\ &= \prod_{r=1}^{\infty}(1-x^{2r}) \Big/ \prod_{r=1}^{\infty}(1-x^r), \\ &= \{(1-x)(1-x^3)(1-x^5)\ldots(1-x^{2r-1})\ldots\}^{-1}. \end{aligned}$$

This shows that the number of partitions of n into distinct parts is the same as the number of partitions of n into odd parts, repetitions allowed. As another example of this type of interpretation, we notice that formally we have

$$(1-x)(1+x)(1+x^2)(1+x^4)\ldots(1+x^{2^m})\ldots=1.$$

Hence

$$(1+x)(1+x^2)(1+x^4)\ldots(1+x^{2^m})\ldots=(1-x)^{-1}=1+x+x^2+x^3+\ldots+x^n+\ldots.$$

This readily shows that the number of partitions of n into distinct powers of 2 is just one, i.e., every number can be expressed as a sum of distinct powers of 2 in a unique way.

The generating function for $t_k(n)$, the number of k-rowed partitions of n, is:

$$\prod_{r=1}^{\infty} (1-x^r)^{-\min(k,r)}.$$

In 1964, Cheema and Gordon gave a combinatorial proof of this result and obtained some congruence properties of $t_2(n)$ and $t_3(n)$. Recently Gandhi has obtained some congruences for $t_k(n)$.

3. Some Important Identities

3.1. Euler's Identity

A serious study of the subject of partitions started with Euler. In 1742, Euler gave the important expansion:

$$\prod_{r=1}^{\infty} (1-x^r) = 1-x-x^2+x^5+x^7-x^{12}-x^{15}+\ldots+(-1)^k\{x^{k(3k-1)/2}+x^{k(3k+1)/2}\}+\ldots,$$

$$= \sum_{k=-\infty}^{\infty} (-1)^k x^{k(3k+1)/2};$$

a result which he could not, at first, prove. Using his formula he calculated a table of values of $p(n)$ for $n \leqslant 59$. I made use of this identity in extending my table of values of $p(n)$ from $n=300$ to $n=1000$. In 1830, A. M. Legendre noted that Euler's identity implies that every number which is not pentagonal, can be partitioned into an even number of distinct parts as often as it can be partitioned into an odd number of distinct integers, while the pentagonal number $k(3k+1)/2$ can be partitioned into an even number of distinct parts once oftener or once fewer times than into an odd number of distinct parts according as k is even or odd. Franklin used Ferrers graphs to give a remarkable proof of Euler's identity.

3.2. Jacobi's Identity

Jacobi gave fundamental applications of elliptic functions to the study of the theory of partitions. In 1929, he proved that

$$\prod_{r=1}^{\infty} (1-x^{2r})(1+zx^{2r-1})(1+z^{-1}x^{2r-1}) = 1+x(z+z^{-1})+x^4(z^2+z^{-2})+x^9(z^3+z^{-3})+\ldots.$$

He then deduced the following relations:

$$\prod_{r=1}^{\infty} (1-x^{2r})(1+x^{2r-1})^2 = \sum_{-\infty}^{\infty} x^{r^2};$$

$$\prod_{r=1}^{\infty} (1-x^{2r})(1-x^{2r-1})^2 = \sum_{-\infty}^{\infty} (-1)^r x^{r^2};$$

$$2x^{1/4} \prod_{r=1}^{\infty} (1-x^{2r})(1+x^{2r})^2 = \sum_{-\infty}^{\infty} x^{(2r+1)^2/4}$$

and finally

$$\prod_{r=1}^{\infty} (1-x^{2r})^3 = \sum_{r=0}^{\infty} (-1)^r (2r+1) x^{r^2+r}.$$

The last one of these identities is of particular importance in the theory of partitions and has been used repeatedly by Gandhi and others. Euler's identity can be shown to be a particular case of Jacobi's.

In Jacobi's identity, replace x^2 by uv, z^2 by u/v, and change u to $-u$ and v to $-v$, to obtain the identity:

$$\prod_{r=1}^{\infty}(1-u^r v^r)(1-u^r v^{r-1})(1-u^{r-1}v^r)=\sum_{-\infty}^{\infty}(-1)^r u^{r(r+1)/2}v^{r(r-1)/2}.$$

In 1964, Cheema interpreted this combinatorially in the form of the

THEOREM: *The excess of the number of partitions of* (n, m) *into even number of parts of the type* (a, a−1), (b−1, b), (c, c) *over those into odd number of parts, distinct in each case, is* $(-1)^r$ *or zero, according as* (n, m) *is or is not of the type* (r(r+1)/2, r(r−1)/2).

He also gave a graphical proof of this result.

In fact, identities involving theta functions are equivalent to combinatorial theorems involving partitions under two different conditions.

In 1966, Sudler gave two enumerative proofs of Jacobi's identity.

3.3. Cauchy's Identities

In 1843, A. Cauchy, besides proving Euler's identity, showed that

$$\prod_{r=0}^{n-1}(1+zx^r)=1+\frac{1-x^n}{1-x}z+\frac{(1-x^n)(x-x^n)}{(1-x)(1-x^2)}z^2+\ldots.$$
$$+\frac{(1-x^n)(x-x^n)(x^2-x^n)\ \ldots\ (x^{n-1}-x^n)}{(1-x)(1-x^2)(1-x^3)\ \ldots\ (1-x^n)}z^n;$$

and

$$\prod_{r=0}^{n-1}(1-zx^r)^{-1}=1+\frac{1-x^n}{1-x}z+\frac{(1-x^n)(1-x^{n+1})}{(1-x)(1-x^2)}z^2+\frac{(1-x^n)(1-x^{n+1})(1-x^{n+2})}{(1-x)(1-x^2)(1-x^3)}z^3+\ldots\ldots$$

These identities lead to the generating functions for $p(n, k)$ and $q(n, k)$ given already. As a special case of Cauchy's identities, we have the following identities of Euler:

$$\prod_{r=0}^{\infty}(1+zx^r)=1+\sum_{r=1}^{\infty}\frac{x^{r(r-1)/2}z^r}{(1-x)(1-x^2)\ \ldots\ (1-x^r)},$$

$$\prod_{r=0}^{\infty}(1-zx^r)^{-1}=1+\sum_{r=1}^{\infty}\frac{z^r}{(1-x)(1-x^2)\ \ldots\ (1-x^r)}.$$

3.4. Gordon's Identities

Basil Gordon has given the identities:

$$\prod_{r=1}^{\infty}(1-x^r)(1-yx^r)(1-y^{-1}x^{r-1})(1-y^2x^{2r-1})(1-y^{-2}x^{2r-1})=\sum_{r=-\infty}^{\infty}x^{r(3r+1)/2}(y^{3r}-y^{-3r-1}).$$

(A proof of this identity was given by Mordell in 1962.)

$$\prod_{r=1}^{\infty}(1-x^r)^3(1-x^{2r-1})^2=\sum_{r=-\infty}^{\infty}(6r+1)x^{r(3r+1)/2}.$$

(This was also proved by Bailey.)

3.5. Ramanujan's Identities

Ramanujan stated without proof the two remarkable identities:

(1) $p(4)+p(9)x+p(14)x^2+\dots=5\{f(x)\}^6/\{f(x^5)\}^5$;

(2) $p(5)+p(12)x+p(19)x^2+\dots=7\{f(x)\}^4/\{f(x^7)\}^3+49x\cdot\{f(x)\}^8/\{f(x^7)\}^7$.

These identities lead to the well known congruences of Ramanujan for the moduli 5, 5^2, 7, 7^2; which we consider a little later. In 1950, using formal power series, D. Kruyswijk obtained the two Ramanujan identities.

3.6. Rogers-Ramanujan Identities

The following two identities, first found by Rogers in 1894, were rediscovered by Ramanujan about 1913.

(1) $$\prod_{r=0}^{\infty}\{(1-x^{5r+1})(1-x^{5r+4})\}^{-1}=1+\prod_{r=1}^{\infty}\frac{x^{r^2}}{(1-x)(1-x^2)\dots(1-x^r)}$$

(2) $$\prod_{r=0}^{\infty}\{(1-x^{5r+2})(1-x^{5r+3})\}^{-1}=1+\prod_{r=1}^{\infty}\frac{x^{r(r+1)}}{(1-x)(1-x^2)\dots(1-x^r)}.$$

Rogers-Ramanujan identities have received considerable attention. Several proofs of these identities have been given, besides two by Rogers himself. An ingenious proof was given by J. M. Dobbie in 1962. In 1954, Henry L. Alder gave a generalization of these identities. A combinatorial interpretation of the identities was given by Gordon in 1961. B. Gordon and G. E. Andrews have generalized these identities in different directions.

3.7. Generalizations of $p(n)$

In 1951, Gupta considered the function:

$$J(x)=\prod_{r=1}^{\infty}(1-x^r)^{-rj-1}.$$

This is a generalization of the generating function $f(x)$ for $p(n)$. He also considered the function:

$$J(x,m)=\prod_{r=1}^{m}(1-x^r)^{-rj-1}.$$

In 1960, O. Kolberg gave another generalization of $f(x)$ viz,

$$P(x,j)=\{f(x)\}^{-j}=1+\prod_{n=1}^{\infty}p_j(n)x^n.$$

For positive values of j, Newman has shown that $p_j(n)$ is a polynomial in j of degree n. Gupta has expressed $p_j(n)$ in terms of combinatory functions. Wright has obtained an estimate for $p_j(n)$ for large n. Gandhi has congruences for $p_j(n)$ from which results due to Ramanathan and Lahiri and also those concerning the parity of $p(n)$ follow as special cases.

Kolberg proved that

$$P(x^2,5)P(-x,-2)=\sum_{m=-\infty}^{\infty}(3m+1)x^{m(3m+2)}$$

and

$$P(x, 5)P(x^2, -2) = \sum_{m=-\infty}^{\infty} (6m+1)x^{m(3m+1)/2}.$$

In 1953, L. Carlitz derived Newman's formula:

$$P(x, 6)P(x^5, -1) = \sum_{m=0}^{\infty} p_5(5m)x^m.$$

4. Recursion Formulas

In extending to $n=300$, the partition table which MacMahon and Ramanujan had computed up to $n=200$, I adopted a new procedure. Denoting by $r(n, m)$, the number of partitions of n with one part equal to m and other parts $\geqslant m$, I showed that

$$r(n, m) = r(n-m, m) + r(n+1, m+1).$$

This recursion formula enabled me to compute a double entry table for the values of $r(n, m)$. The work was very much reduced with the help of the following relations:

$$r(m, m) = 1;$$

$$r(m+j, m) = 0, \qquad 0 < j < m;$$

$$r(2m+j, m) = 1, \qquad 0 \leqslant j < m;$$

$$r(3m+j, m) = 2 + [j/2], \qquad 0 \leqslant j < m;$$

and

$$r(4m+j, m) = 3 + [(m+j)/2] + [j(j+6)/12], \; 0 \leqslant j < m.$$

The result for $r(5m+j, m)$ is rather complicated. Moreover, for $0 \leqslant k < 4$,

$$r(4m+k, > m) = m + 2 + [k/2] + [(m+k+3)(m+k-3)/12].$$

Since $p(n) = r(n+1, 1)$, my table provided a table of partitions as well. The generating function for $r(n, m)$ will be readily seen to be:

$$x^m/(1-x^m)(1-x^{m+1})(1-x^{m+2}) \; \ldots \ldots$$

In 1958, G. Palama evaluated $r(n, > [n/5])$.
The recursion formula for $p(n, k)$ is:

$$p(n, k) = p(n-1, k-1) + p(n-k, k).$$

This gives:

$$p(n, 1) = 1;$$

$$p(n, 2) = [n/2];$$

$$p(n, 3) = [(n^2+3)/12].$$

Formulas for $p(n, k)$, $k \leq 12$, are given in the introduction to the Royal Society Tables of Partitions. In 1942, I obtained a formula expressing $r(n, m)$ in terms of $p(n, m)$. It gave

$$r(n+2m+1, m+1) = p(n, 1) + p(n-m, 2) + p(n-2m, 3) + \ . \ . \ . + p(n-jm, j+1),$$

with $j = [(n-1)/(m+1)]$.

Very recently, using elementary methods, J. Intrator has proved that the number of partitions of any natural number n into exactly k summands, is given by a polynomial of degree exactly $(k-1)$ in n, the first $[(k+1)/2]$ coefficients of which are independent of n, while the others depend on the residue of n modulo the least common multiple of the integers: 1,2,3, . . ., k. This is an improvement on earlier results. Actually the coefficient of n^j in the polynomial depends on the residue of n modulo the least common multiple of the integers: $1, 2, 3, \ldots, [k/(j+1)]$; $j \leq (k-1)$.

Let $p^*(n, k)$ denote the number of partitions of n into at most k summands. Then, we have

$$p^*(n, k) = p^*(n, k-1) + p^*(n-k, k).$$

In fact, $p^*(n, k) = p(n+k, k)$.

In 1955, Lothar Berg considering solutions of the form:

$$t^{-n} g_k(t)$$

for the above recurrence, derived many known results for $p^*(n, k)$ and for $p(n)$. In particular, he showed, in a very simple way, that

$$p(n) < C \exp (2C \sqrt{n}) / \sqrt{n}, \ C^2 = \pi^2/6.$$

5. Graphical Methods

5.1. Ferrers Graphs

An idea that has been very useful in the study of the theory of partitions, is that of Ferrers graphs. The partition: $6+2+1$ of 9, is represented by means of dots—six in the first row, two in the second and one in the third, as in the following diagram:

$$\begin{array}{llllll} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & & & & \\ \cdot & & & & & \end{array}$$

If we read this graph in columns, we get the partition: $3+2+1+1+1+1$ of 9. The two partitions are said to be conjugate. Some partitions are self-conjugate too. Such, for example, is the partition: $5+4+2+2+1$ of 14. Corresponding to every self-conjugate partition of n, we have a partition of n into odd distinct parts and conversely. Hence, the number of self-conjugate partitions of n is the same as the number of partitions of n into odd distinct parts. This and several other theorems in partitions are easily proved by means of graphs. For example, the following two can be mentioned:

1. The number of partitions of n into exactly k summands, is the same as the number of partitions of n into summands the largest of which is k.

2. The number of partitions of n into odd parts is the same as the number of partitions of n into distinct parts. (This has already been proved with the help of generating functions.)

5.2. Decompositions of *n*

Closely connected with Ferrers graphs, is the following method of representing a decomposition of n.

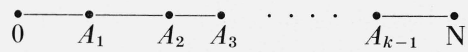
Take a straight line 0N of n units in length. If

$$n=n_1+n_2+n_3+\ . \ . \ . \ +n_k,$$

be a decomposition of n into k parts, then taking along the line 0N;

$$0A_1=n_1,\ A_1A_2=n_2,\ A_2A_3=n_3,\ . \ . \ . \ ,\ A_{k-1}N=n_k;$$

we get



as a graphic representation of the said decomposition. Conversely, given the graphic representation, the decomposition can be written out. It will be readily seen that the number of decompositions of n into exactly k summands, is the same as the number of ways in which exactly $(k-1)$ lattice points can be marked strictly between 0 and N. Since there are $(n-1)$ lattice points between 0 and N, in all; this can be done in $\binom{n-1}{k-1}$ ways. This gives the number of decompositions of n into k parts. The total number of decompositions of n is given by

$$\sum_{k=1}^{n}\binom{n-1}{k-1}=2^{n-1}.$$

Since a partition of n into k parts, cannot give rise to more than $k!$ decompositions (and that too only if the k parts are distinct), it follows that

$$p(n,k)\geqslant\frac{1}{k!}\binom{n-1}{k-1}\geqslant q(n,k).$$

We might mention here that the number of decompositions of n into at most k summands, is the number of solutions of the equation:

$$x_1+x_2+x_3+\ . \ . \ . \ +x_k=n,$$

in nonnegative integers x. This is the same as the number of ways in which n like objects can be distributed among k persons when there is no restriction as to the number of objects any of them may receive.

The method of representation described above can be extended to a j-dimensional space. Gupta has thus obtained generalizations of the above inequality for partitions of j-partite numbers.

6. Congruences

6.1. Ramanujan's Conjecture

About 1919, Ramaṇujan proved that for any nonnegative m,

$$p(5m+4)\equiv 0(\mathrm{mod}\ 5),\qquad p(25m+24)\equiv 0(\mathrm{mod}\ 25);$$

$$p(7m+5)\equiv 0(\mathrm{mod}\ 7),\qquad p(49m+47)\equiv 0(\mathrm{mod}\ 49).$$

He further stated that

$$p(11m+6)\equiv 0(\mathrm{mod}\ 11).$$

All these results are included in his famous conjecture:

$$\text{If } p=5,\ 7, \text{ or } 11 \text{ and } (24n-1) \equiv 0 (\bmod p^a),\ a \geqslant 1;$$

then also $p(n) \equiv 0 (\bmod p^a)$.

This astounding conjecture held good till, using my table of values of $p(n)$ for $n \leqslant 300$, S. Chowla found that the conjecture failed for $n=243$. For this value of n,

$$24n-1=5831 \equiv 0 (\bmod 7^3),$$

while

$$p(243)=13397 \quad 82593 \quad 44888 \equiv 0 (\bmod 7^2),$$

but

$$\not\equiv 0 (\bmod 7^3).$$

Later it was found that the conjecture failed also when $n=586$.

This discovery led to a lot of work particularly by D. H. Lehmer, who using Hardy-Ramanujan and Rademacher Series (discussed later in this survey), computed values of $p(n)$ for certain large values of n. He thus found that

$$p(599) \equiv 0 (\bmod 5^4), \qquad p(721) \equiv 0 (\bmod 11^3);$$
$$p(1224) \equiv 0 (\bmod 5^4), \qquad p(2052) \equiv 0 (\bmod 11^3);$$
$$p(2474) \equiv 0 (\bmod 5^5), \qquad p(14031) \equiv 0 (\bmod 11^4).$$

These results are in conformity with Ramanujan's conjecture.

In 1948, D. B. Lahiri found three new congruences:

$$p(49m+r) \equiv 0 (\bmod 49) \text{ for } r=19,\ 33,\ 40;$$

and announced two more:

$$p(125m+r) \equiv 0 (\bmod 125) \text{ for } r=74,\ 124.$$

In 1952, from a study of Ramanujan's manuscript, J. M. Rushforth proved that

$$p(121m+116) \equiv 0 (\bmod 121);$$

and

$$p(49m+r) \equiv 0 (\bmod 49) \text{ for } r=19,\ 33,\ 40,\ 47.$$

In 1960, Professor M. Newman showed that for $(m, 30)=1$.

$$p((167m^2+1)/24) \equiv 0 (\bmod 5).$$

These were all results of the Ramanujan type.

In the meantime, G. N. Watson had proved Ramanujan's conjecture completely for powers of 5. For powers of 7, he proved the following modification of the conjecture:

If
$$24n-1 \equiv 0 (\bmod 7^b),$$

then
$$p(n) \equiv 0 (\bmod 7^d) \text{ where } d=[(b+2)/2].$$

Finally, A. O. L. Atkin has essentially settled the problem by proving Ramanujan's conjecture for powers of 11. The full truth with regard to the conjecture can now be stated in the form of the

THEOREM: *If* $24n-1 \equiv 0 (\mathrm{mod}\ 5^a 7^b 11^c)$, *where* a, b, c *are nonnegative integers, then*

$$p(n) \equiv 0 \ (\mathrm{mod}\ 5^a\, 7^d\, 11^c) \text{ with } d = [(b+2)/2].$$

Watson's method was based on modular equations. Atkin followed the method of J. Lehner, which had enabled Lehner to prove Ramanujan's conjecture for 11, 11^2, 11^3, in 1943, 1950.

Mention must also be made of the method which Professor H. Rademacher developed in 1942, for the investigation of the Ramanujan identities with respect to the moduli 5^a, 7, 7^2, and 13^a.

Very recently, A. O. L. Atkin and J. N. O'Brien have made a valuable contribution to the literature on the subject by giving some properties of $p(n)$ modulo powers of 13.

6.2. The Parity of $p(n)$

In 1920, MacMahon had shown that

$$p(n) \equiv \sum_t p(t) \quad (\mathrm{mod}\ 2)$$

where t runs through the positive integral values given by the relation:

$$8t = 2n - j(j+1), \qquad j \geqslant 0.$$

It is rather strange that nothing better should be known about the parity of $p(n)$ than MacMahon's congruence.

In 1946, making use of the fact that Ramanujan's function $\tau(n)$ is odd if and only if n is an odd square, I obtained MacMahon's congruence.

6.3. Congruences for $p_j(n)$

Newman, Kolberg, Ramanathan, Gandhi and several others have studied the function $p_j(n)$ defined in section 3.7. In 1956, Newman gave extensive tables of values of $p_j(n)$ for $j \leqslant 16$.

For $j = 2, 4, 6, 8, 10, 14, 26$ and a prime $p > 3$ such that

$$j(p+1) \equiv 0 (\mathrm{mod}\ 24).$$

M. Newman proved that

$$p_j(np+k) = (-p)^{(j-2)/2} p_j(n/p),$$

where $k = j(p^2-1)/24$ and we take $p_j(\alpha) = 0$ if α is not a nonnegative integer. He also showed that

$$p_6(3n+2) = 9p_6(n/3);$$

and that $p_j(n)$ vanishes for arbitrarily long strings of consecutive values of n, when $j = 2, 4, 6, 8, 10, 14$ or 26.

Ramanathan showed that

If $24n + j \equiv 0 (\mathrm{mod}\ 5^a)$, $j \equiv 16, 21, 26 \ (\mathrm{mod}\ 30)$.

then $$p_j(n) \equiv 0 (\mathrm{mod}\ 5^b) \text{ where } b = [(a+1)/2].$$

6.4. Rank of a Partition

In 1954, Atkin and Swinnerton-Dyer proved some conjectures of Dyson, who had discovered, in 1944, empirically a remarkable combinatorial method of dividing the partitions of $(5n+4)$ and of $(7n+5)$ into five and seven equal classes respectively. Dyson defined the rank of a partition as the largest part minus the number of parts in the partition. The partitions of a given number could then be divided into classes according to their ranks. In the case of numbers of the forms $(5n+4)$ and $(7n+5)$, these classes had equal number of elements in each class.

Atkin and Swinnerton-Dyer showed that for $q=5$, 7, or 11, and $0<b<q$;

$$\sum_{n=0}^{\infty} p(qn+b)y^n$$

is congruent modulo q to a simple infinite product.

In 1958, Atkin and Hussain obtained an identity for

$$\sum_{n=0}^{\infty} p(11n+6)y^n.$$

6.5. Sylvester's Theorem

In 1945, D. P. Banerjee proved Sylvester's theorem, which states that $p(n)-Q(n)$ is even, where $Q(n)$ denotes the number of partitions of n into odd distinct parts.

6.6. Newman's Conjecture

In 1960, M. Newman conjectured that for every pair of natural numbers m and r, there are infinitely many natural numbers n, such that

$$p(n) \equiv r(\text{mod } m).$$

He proved the conjecture for $m=5$, 13, 65 and some other composite moduli. Kolberg and Subbarao have proved it for $m=2$. In this connection, the following two results of Newman might be of interest:

Neither of the congruences:

$$p(13n-7) \equiv r(\text{mod } 13),$$

and

$$p(13^2n-7) \equiv r(\text{mod } 13),$$

can hold for all sufficiently large n and a fixed r.

7. Some Inequalities

We have already shown that

$$k!q(n,k) \leqslant \binom{n-1}{k-1} \leqslant k!p(n,k).$$

Since,

$$q(n,k)=p(n-\tfrac{1}{2}k(k-1),k),$$

we must have

$$\binom{n+\frac{1}{2}k(k-1)-1}{k-1} \geqslant k!p(n,k) \geqslant \binom{n-1}{k-1}.$$

(Originally, I had made use of the recursion formula for $p(n, k)$ to prove this inequality.)

The inequality implies that

$$p(n, k) \sim \frac{1}{k!}\binom{n-1}{k-1}, \text{ for } k = o(n^{1/3}).$$

Erdos and Lehner had proved this result in 1941, by a rather lengthy method. They had in fact proved that if $p^*(n, k)$ denotes the number of partitions of n into at most k summands, then

$$p^*(n, k)/p(n) \sim \exp\,(-C^{-1}e^{-Cx}),$$

where (here and in all that follows), $C = \pi/\sqrt{6}$; and x is given by the relation:

$$Ck = \sqrt{n}\,\log\sqrt{n} + Cx\,\sqrt{n}.$$

They gave a similar result for partitions into distinct parts.

As an analogue of this remarkable result of Erdos and Lehner, I showed that

$$p(n, k)/p(n) \sim tn^{-1/2}\exp\,(-t/C).$$

where

$$t = \lim_{n, k \to \infty} \sqrt{n}\,\exp\,(-Ckn^{-1/2}).$$

In 1957, S. M. Luthra obtained an asymptotic formula for

$$\sum_{k=1}^{n} kp(n, k)/p(n).$$

The case of partitions of n into k distinct parts was also considered.

In 1950, Auluck, Singwi, and Agarwala gave another asymptotic result of some interest:

If $w(n)$ denotes the number of partitions of n into integers each of which occurs in the partition only an odd number of times, then for a certain constant γ,

$$w(n) \sim \frac{\gamma}{2\pi n}\exp\,(2\gamma\sqrt{n}).$$

If $p(n, A)$ denotes the number of partitions of n into members of the set:

$$A = \{1, a_1, a_2, a_3, \ldots, a_m\},$$

then, I proved in 1955, the inequality:

$$\binom{n+m}{m} \leqslant p(n, A)\prod_{j=1}^{m} a_j \leqslant \binom{n+\sum_{j=1}^{m} a_j}{m}.$$

This is a generalization of the inequality given earlier.

In 1954, writing

$$\{(1-x)(1-x^2)(1-x^3)\ \ldots\ (1-x^r)\}^{-1} = \sum_{j=0}^{\infty} c_j(r)(1-x)^{j-r},$$

I showed that for a fixed j and large r,

$$c_j(r) \sim \{j\,!\,(r-2j)\,!\,4^j\}^{-1}.$$

This led to an asymptotic formula for $p^*(n, r)$.

In 1956, Bateman and Erdos showed that the number of partitions of n into parts taken from a nonempty set A of positive integers, is strictly increasing for large n, if and only if A contains more than one element and if when any element is removed from A, the remaining elements have the greatest common divisor unity.

8. Conjecture of Auluck, Chowla, and Gupta

In 1942, Auluck, Chowla, and Gupta conjectured that $p(n, k)$ has a unique maximum for any given n, i.e., regarding n as fixed there exists an integer k_0 such that

$$p(n, k) \geqslant p(n, k-1), \text{ for } k \leqslant k_0;$$

and

$$\leqslant p(n, k-1), \text{ for } k > k_0.$$

This conjecture has not been completely proved so far.

In 1953, however, G. Szekeres proved that a maximum of $p(n, k)$ occurs at

$$k=\frac{\sqrt{n}}{C} L+\frac{1}{C^2}\left(\frac{3}{2}+\frac{3}{2} L-\frac{1}{4} L^2\right)-\frac{1}{2}+\mathbf{O}\left(\frac{\log^4 n}{\sqrt{n}}\right),$$

where

$$L=\log\ (\sqrt{n}/C).$$

He deduced this result from his formula for $p^*(n, k)$ true for bounded (n/k^2). The formula itself was obtained by using the method of steepest descent.

In fact, the work of Szekeres opened up the possibility of obtaining the infinite series for $p(n)$ without using the theory of elliptic modular functions. He sharpened the results of Erdos and Lehner regarding maximum of $p(n, k)$ and $q(n, k)$. It was shown that $q(n, k)$ has a maximum at

$$k=\frac{2\sqrt{3n}\ \log 2}{\pi}+\beta+\mathbf{O}(n^{-1/2});$$

where β is a fixed constant.

In 1954, Haselgrove and Temperley developed a powerful method which made no use of Farey dissections and enabled them to prove that $p(n, k)$ attains its greatest value for at most two consecutive values of k when n is large and fixed. This had been conjectured by Auluck, Chowla, and Gupta.

9. Hardy-Ramanujan and Rademacher Series

No account of partition theory can be even fairly complete without a mention of the great work of Hardy and Ramanujan on the subject. Using Farey dissections, they obtained an infinite series for $p(n)$, the first few terms of which gave the value of $p(n)$ exactly if we neglect the decimal part in the answer. This was astounding. Later on Lehmer showed that the Hardy-Ramanujan series was divergent and Hardy and Ramanujan had been fortunate in breaking at a point where the series gave a correct answer.

The Hardy-Ramanujan series gives

$$p(n)=\frac{(12)^{1/2}}{(24n-1)u}\sum_{k=1}^{[\sqrt{n}]} A_k^*(n)(u-k)\exp\ (u/k)+\mathbf{O}(\log n/\sqrt{n});$$

where

$$u=\pi(24n-1)^{1/2}/6.$$

A change in the path of integration enabled Rademacher to replace $(u-k)\exp(u/k)$ in the Hardy-Ramanujan series by

$$(u-k)\exp(u/k)+(u+k)\exp(-u/k).$$

This simple-looking change gave a convergent series for $p(n)$.

The functions $A_k^*(n)$ in the series for $p(n)$ have been shown by Lehmer to be multiplicative in a certain sense. In 1956, Whiteman used Fourier series for the evaluation of these functions and Rademacher obtained the Selberg formula for $A_k^*(n)$ by using a transformation. He also put Lehmer's theorems for the evaluation and factorization of $A_k^*(n)$ in a form suitable for computation.

The method of Rademacher has been widely applied to partition problems of various types. For example, Lehner has obtained a convergent series for the number of partitions of a positive integer n into summands of the form: $5m\pm a$, $a=1, 2$; and Hua for $q(n)$. Hagis has not only derived Rademacher's formula for $p(n)$ and the convergent series for $q(n)$, but also a convergent series for $q(n; t)$—the number of partitions of n into odd summands, no summand appearing more than t times in a partition.

In 1941, Ingham used a Tauberian theorem to obtain the asymptotic formulas for $p(n)$ and $q(n)$. These are:

$$p(n) \sim (4n\sqrt{3})^{-1}\exp(\pi\sqrt{2n/3}),$$

$$q(n) \sim (4n^{3/2}3^{1/4})^{-1}\exp(\pi\sqrt{n/3}).$$

In 1961, Iseki gave a shorter proof than Rademacher's for a functional equation which implies the transformation equation for Dedekind's modular function $\eta(\tau)$. D. J. Newman has obtained the asymptotic formula for $p(n)$ in an elementary and simple way. In 1962, he gave another proof for it.

In 1954, Petersson developed a function-theoretic method for finding an asymptotic formula for the partition function. This method is different from the Hardy-Ramanujan method.

10. Miscellaneous Results

Mention must be made of the work of O. P. Gupta and S. Luthra who, in 1955, gave a table for the number of partitions of $n \leqslant 300$, into primes. I gave a table for the number of partitions into distinct primes.

In 1957, Takayoshi Mitsui used the powerful methods of Vinogradov and Hua to obtain an asymptotic formula for the number of partitions of n into kth powers of primes not exceeding m. His results generalized those of Haselgrove and Temperley.

Before I conclude, I give some results which, I believe, will interest the reader.

(i) The number of partitions of n into distinct parts, the smallest being odd, is odd if and only if n is a square. This was given by N. J. Fine in 1948.

(ii) In 1958, Guy proved that the number of partitions of an integer into:

(a) odd parts greater than unity;

(b) unequal parts such that the two greatest parts differ by unity;

(c) unequal parts which are not powers of 2; are all equal.

(iii) In 1963, R. L. Graham showed that every integer greater than 77 can be partitioned into distinct positive integers whose reciprocals add to 1.

(iv) J. B. Kelly proved the following result in 1964: Given $k \geqslant 3$, every $n \geqslant N(k)$, can be partitioned into k parts in $(k-1)$ different ways so that the products of the integers in each of the $(k-1)$ partitions are equal.

(v) By far the most remarkable result regarding two-rowed partitions of n, when the elements in rows strictly decrease, is that the number of such partitions is $p(n)$, but it is only $p([n/2])$ when the parts are all odd.

This was proved by B. Gordon in 1962. Sudler gave a direct proof of this result in 1965. More recently B. Gordon and L. Houten obtained generating functions and asymptotic formulas for various restricted rowed and plane partitions.

11. Bibliography

Alder, Henry L.,

(1) The non-existence of certain identities in the theory of partitions and compositions, Bull. Amer. Math. Soc. **54,** 712–722 (1948).
(2) Generalizations of the Rogers-Ramanujan identities, Pacific J. Math. **4,** 161–168 (1954).

Andrews, George E.,

(1) A simple proof of Jacobi's triple product identity, Proc. Amer. Math. Soc. **16**, 333–334 (1965).
(2) An analytic proof of the Rogers-Ramanujan-Gordon identities, Amer. J. Math. **88**, 844–846 (1966).
(3) On generalization of Euler's partition theorem, Michigan Math. J. **13,** 491–498 (1966).
(4) On the theorems of Watson and Dragonette for Ramanujan's mock theta functions, Amer. J. Math. **88,** 454–490 (1966).
(5) Enumerative proofs of certain q-identities, Glasgow Math. J. **8,** 33–40 (1967).
(6) Partition theorems related to the Rogers-Ramanujan identities, J. Combinatorial Theory **2,** 422–430 (1967).
(7) Some new partition theorems, J. Combinatorial Theory **2,** 431–436 (1967).
(8) A generalization of a partition theorem of MacMahon, J. Combinatorial Theory **3,** 100–101 (1967).
(9) On Schur's second partition theorem, Glasgow Math. J. **8,** 127–132 (1967).
(10) A generalization of the Gollnitz-Gordon partition theorems, Proc. Amer. Math. Soc. **18,** 945–952 (1967).

Apostol, Tom M.,

(1) Generalized Dedekind sums and transformation formulae of certain Lambert series. Duke Math. J. **17,** 147–157 (1950).
(2) Asymptotic series related to the Partition Function, Ann. of Math. (2) **53,** 327–331 (1951).

Arkin, Joseph,

(1) An extension of a theorem of Ramanujan, Amer. Math. Monthly **73**, 1087–1090 (1966).

Atkin, A. O. L.,

(1) Some properties of Partitions II, Trans. Amer. Math. Soc. **89,** 184–200 (1958).
(2) A note on ranks and conjugacy of partitions, Quart. J. Math. Oxford Ser. (2) **17,** 335–338 (1966).
(3) Note on a paper of Cheema and Gordon, Duke Math. J. **34,** 57–58 (1967).
(4) Proof of a conjecture of Ramanujan, Glasgow Math. J. **8,** 14–32 (1967).
(5) Congruences for modular forms, IBM Blaricum Proceedings (1966) (in the press).

Atkin, A. O. L., and O'Brien, J. N.,

(1) Some properties of $p(n)$ and $c(n)$ modulo powers of 13, Trans. Amer. Math. Soc. **126,** 442–459 (1967).

Atkin, A. O. L., Bratley, P., Macdonald, I. G., and Mackay, J.,

(1) Some computations for m-dimensional partitions, Proc. Cambridge Phil. Soc. (in the press).

Atkin, A. O. L., and Swinnerton-Dyer, P.,

(1) Some properties of Partitions. Proc. London Math. Soc. (3) **4,** 84–106 (1954).

Atkinson, F. V.,

(1) A summation formula for $p(n)$, the partition function, J. London Math. Soc. **14,** 175–184 (1939).

Auluck, F. C.,

(1) An asymptotic formula for $p_k(n)$, J. Indian Math. Soc. (N.S.) **6,** 113–114 (1942).
(2) On some new types of partitions associated with generalized Ferrers graphs, Proc. Cambridge Philos. Soc. **47,** 679–686 (1951).
(3) On Partitions of bipartite numbers, Proc. Cambridge Philos. Soc. **49,** 72–83 (1953).

Auluck, F. C., Chowla, S., and Gupta, H.,

(1) On the maximum value of the number of partitions of n into k parts, J. Indian Math. Soc. (N.S.) **6,** 105–112 (1942).

Auluck, F. C., and Haselgrove, C. B.,

(1) On Ingham's Tauberian theorem for partitions, Proc. Cambridge Philos. Soc. **48,** 566–570 (1952).

372-985 O - 70 - 3

Auluck, F. C., Singwi, K. S., and Agarwala, B. K.,

(1) On a new type of partition, Proc. Nat. Inst. Sci. India, **16**, 147–156 (1950).

Avakumovic, Vojislav G.,

(1) Neuer Beweis eines Satzes von G. H. Hardy und S. Ramanujan über das asymptotische Verhalten der Zerfällungskoeffizeinten. Amer. J. Math. **62**, 877–880 (1940).

Bailey, W. N.,

(1) Identities of the Rogers-Ramanujan type. Proc. London, Math. Soc. (2) **50**, 1–10 (1948).
(2) On the simplification of some identities of the Rogers-Ramanujan type. Proc. London Math. Soc. (3) **1**, 217–221 (1951).
(3) A note on two of Ramanujan's formulae, Quart. J. Math. Oxford Ser. (2) **3**, 29–31 (1952).
(4) A further note on two of Ramanujan's formulae. Quart. J. Math. Oxford Ser. (2) **3**, 158–160 (1952).

Banerjee, D. P.,

(1) On a theorem in the theory of partition, Bull. Calcutta Math. Soc. **37**, 113–114 (1945).
(2) On some identities in partition functions. Proc. Nat. Acad. Sci. India Sect. A **29**, 356–359 (1960).

Basu, N. M.,

(1) A note on partitions, Bull. Calcutta Math. Soc. **44**, 27–30 (1952); corrections 142 (1952).

Bateman, Paul T., and Erdös, Paul,

(1) Monotonicity of partition functions, Mathematika **3**, 1–14 (1956).
(2) Partitions into primes, Publ. Math. Debrecen **4**, 198–200 (1956).

Bell, Eric Temple,

(1) Periodicities in the theory of partitions, Ann. of Math. **2**, **24**, 1–22 (1923).
(2) A class of numbers connected with partitions, Amer. J. Math **45**, 73–82 (1923).
(3) Recurrences for certain functions of partitions. Amer. J. Math. **55**, 667–670 (1933).

Bennett, J. H.,

(1) Partitions in more than one dimension, J. Roy. Statist. Soc. Ser. B. **18**, 104–112 (1956).

Bergmann, S.,

(1) Über eine Eigenschaft der additiven Zerfällungen der Zahlen. M **29**, 194–202 (1918).

Bergmann, Lothar,

(1) Über eine Differenzengleichung Aus Der Theorie Der Partitionen, Wiss. Z. Univ. Rostock Math. Nat. Reihe **5**, 269–278 (1955–56).

Bioche, Ch.,

(1) Sur les systemes de trois entiers de somme n, C. R. Soc. Math. France, 29–31 (1937).

Blakley, G. R.,

(1) Combinatorial remarks on partitions of a multipartite number, Duke Math. J. **31**, 335–340 (1964).

Blij, F. Vander,

(1) The Function $\tau(n)$ of S. Ramanujan, Math. Student **18**, 83–99 (1950).

Brauer, Alfred,

(1) On a problem of partitions, Amer. J. Math. **64**, 299–312 (1942).

Brauer, A., and Seelbinder, B. M.,

(1) On a problem of partitions II, Amer. J. Math. **76**, 343–346 (1954).

Brigham, Nelson A.,

(1) A general asymptotic formula for partition functions, Proc. Amer. Math. Soc. **1**, 182–191 (1950).
(2) On a certain weighted partition function, Proc. Amer. Math. Soc. **1**, 192–204 (1950).

Browkin, J.

(1) Sur les decompositions des nombres naturals en sommes de nombres premiers, Colloq. Math. **5**, 205–207 (1958).

de Bruijn, N. G.,

(1) On Mahler's partition problem, Nederl. Akad. Wetensch Proc. **51,** 659–669; Indagationes Math. **10,** 210–220 (1948).

Carlitz, Leonard,

(1) Note on some partition identities, Proc. Amer. Math. Soc. **4,** 530–534 (1953).
(2) Some theorems on generalized Dedekind sums, Pacific J. Math. **3,** 513–522 (1953).
(3) The reciprocity theorem for Dedekind sums, Pacific J. Math. **3,** 523–527 (1953).
(4) Note on some partition formulae, Quart. J. Math. Oxford Ser. (2), **4,** 168–172 (1953).
(5) A note on generalized Dedekind sums, Duke Math. J. **21,** 399–403 (1954).
(6) Dedekind sums and Lambert Series, Proc. Amer. Math. Soc. **5,** 580–584 (1954).
(7) Some partition formulas, Tohoku Math. J. (2), **6,** 149–154 (1954).
(8) Some Partition Formulas related to sums of Squares. Nieuw Arch. Wisk (3) **3,** 129–133 (1955).
(9) Note on a q-identity, Univ. Beograd. Publ. Elektrotehn Fak. Ser. Mat. Fiz. No. 78–83, 19–20 (1962).
(10) A note on the Jacobi Theta formula, Bull. Amer. Math. Soc. **68,** 591–592 (1962).
(11) Some q-identities related to the theta functions. Boll. Un. Mat. Ital. (3) **17,** 172–178 (1962).
(12) A problem in partitions, Duke Math. J. **30,** 203–213 (1963).
(13) Some generating functions, Duke Math. J. **30,** 191–201 (1963).
(14) A problem in partitions related to the Stirling numbers, Bull. Amer. Math. Soc. **70,** 275–278 (1964).
(15) A problem in partitions related to the Stirling numbers, Riv. Mat. Univ. Parma (2) **5,** 61–75 (1964).
(16) Some partition problems related to the Stirling numbers of the second kind, Acta Arith. **10,** 409–422 (1964–65).
(17) Generating functions and partition problems, Proc. Sympos. Pure Math. **VIII,** 144–169. Amer. Math. Soc. Providence, R. I. (1965).
(18) Combinatorial equivalents of certain theta formulas. Bol. Un. Mat. Ital. (3) **21,** 234–238 (1966).
(19) Rectangular arrays and plane partitions. Acta Arithmetica **13,** 29–67 (1967).

Carlitz, L., and Riordan, John,

(1) Enumeration of certain two-line arrays, Duke Math. J. **32,** 529–539 (1965).

Carlitz, L., and Roselle, D. P.,

(1) Restricted bipartite partitions, Pacific J. Math. **19,** 221–228 (1966).

Cassels, J. W. S.,

(1) On the representation of integers as the sums of distinct summands taken from a fixed set, Acta Sci. Math. Szeged **21,** 111–124 (1960).

Chandra, R.,

(1) Tables of Solid partitions, Proc. Nat. Inst. Sci. India Part A **26,** 134–139 (1960).

Chaundy, T. W.,

(1) The unrestricted plane partition, Quart. J. Math. **3,** 76–80 (1932).
(2) Plane partitions, Proc. London Math. Soc. (2) **35,** 14–22 (1933).

Cheema, M. S.,

(1) Tables of partitions of Gaussian integers giving the number of partitions of $n+im$, Nat. Inst. Sci. India, Mathematical Tables **1,** (1956).
(2) Vector partitions and combinatorial identities, Math. Comp. **18,** 414–420 (1964).

Cheema, M. S., and Gordon, Basil,

(1) Some remarks on two- and three-line partitions. Duke Math. J. **31,** 267–273 (1964).

Cheema, M. S., and Gupta, H.,

(1) The maxima of $P_r(n_1, n_2)$, Math. Comp. **22,** 199–200 (1968).

Cheema, M. S., and Haskell, C. T.,

(1) Multirestricted and rowed partitions, Duke Math. J. **34,** 443–452 (1967).

Chowla, S.,

(1) Congruence properties of partitions, J. London Math. Soc. **9,** 247 (1934).
(2) Solution of a problem of Erdös and Turan in additive-number theory, Proc. Nat. Acad. Sci. India Sect. A **14,** 1–2 (1944).
(3) Outline of a new method for proving results of elliptic function theory (such as identities of the Ramanujan-Rademacher-Zuckerman type), Proc. Lahore Phil. Soc. **7,** 3 pp. (1945).

Chowla, S., and Mian, M. A.,

(1) The differential equations satisfied by certain functions, J. Indian Math. Soc. (N.S.) **8,** 27–28 (1944).

Csorba, G.,

(1) Über die Partitionen der ganzen Zahlen, Math. Ann. **75,** 545–568 (1914).

Dirac, G. A.,

(1) Note on a problem in additive number theory, J. London Math. Soc. **26,** 312–313 (1951).

Dobbie, J. M.,

(1) A simple proof of some partition formulae of Ramanujan's, Quart. J. Math. Oxford Ser. (2) **6,** 193–196 (1955).
(2) A simple proof of the Rogers-Ramanujan identities, Quart. J. Math. Oxford Ser. (2) **13,** 31–34 (1962).

Dutta, Mahadeb,

(1) On new partition of numbers, Rend. Sem. Mat. Univ. Padova **25,** 138–143 (1956).
(2) On new partitions of numbers, Bull. Calcutta Math. Soc. **49,** 221–224 (1957).

Dutta, Mahadev, and Debnath, Lokenath,

(1) On new partitions of numbers II, Bull. Calcutta Math. Soc. **51,** 77–78 (1959).

Erdös, P.,

(1) On an elementary proof of some asymptotic formulas in the theory of partitions, Ann. of Math. (2) **43,** 437–450, (1942).
(2) On a problem of Sidon in additive Number Theory and on some related problems, Addendum. J. London Math. Soc. **19,** 208 (1944).
(3) On some asymptotic formulas in the theory of partitions. Bull. Amer. Math. Soc. **52,** 185–188 (1946).
(4) On a problem of Sidon in additive number theory, Acta. Sci. Math. Szeged **15,** 255–259 (1954).
(5) On the representation of large integers as sums of distinct summands taken from a fixed set, Acta. Arith. **7,** 345–354 (1961–62).

Erdös, P., and Lehner, J.,

(1) The distribution of the number of summands in the partitions of a positive integer, Duke Math. J. **8,** 335–345 (1941).

Erdös, P., and Turan, P.,

(1) On a problem of Sidon in additive number theory and some related problems, J. London Math. Soc. **16,** 212–215 (1941).

Fekete, A. E.,

(1) Signature of partitions and divisors, published by the author, Memorial University of Newfoundland, St. John's, Newfoundland, i + 31 pp. (1965).

Fielder, Daniel C.,

(1) Enumeration of partitions subject to limitations on size of members, Fibonacci Quart. **4,** 209–216 (1966).

Fine, N. J.,

(1) On a system of modular functions connected with the Ramanujan identities, Tohoku Math. J. (2) **8,** 149–164 (1956).

Ford, W. B.,

(1) Two theorems on the partition of numbers, Amer. Math. Monthly **38,** 183–184 (1931).

Franklin, F.

(1) Sur le development du produit infinie, $(1-x)(1-x^2)(1-x^3)$ . . . , C.R. **92,** 448–450 (1881).

Freiman, G. A.,

(1) An elementary method of solution of problems on the partition of numbers into an unbounded number of summands, Trudy Moskov Mat. Obsc. **4,** 113–124 (1955).

Gandhi, J. M.,

(1) Congruences for $p_r(n)$ and Ramanujan's τ function. Amer. Math. Monthly. **70,** 265–274 (1963).
(2) Generalization of Ramanujan's congruences $p(5m+4) \equiv 0 \pmod 5$ and $p(7m+5) \equiv 0 \pmod 7$, Monatsh. Math. **69,** 389–392 (1965).
(3) Some congruences for k-line partition of a number, Amer. Math. Monthly **74,** 179–181 (1967).

Gigli, D.,

(1) Sulle somme di n addendi diversi presi fra i numeri 1, 2, . . . m, Palermo Rend **16,** 280–285 (1902).

Glaisher, J.,

(1) On the number of partitions of a number into a given number of parts, Quart. J. **50,** 57–143 (1908).
(2) Formulae for partitions into given elements derived from Sylvester's theorem, Quart. J. **40,** 275–348 (1909).
(3) Formulae for the number of partitions of a number into the elements 1, 2, . . ., n up to $n=9$. Quart. J. **41,** 94–112 (1909).

Gleissberg, W.,

(1) Über einen Satz Von Herrn J. Schur, MZ **28,** 372–382 (1928).

Glosel, K.,

(1) Über die Zerfällung der ganzen Zahlen, Mh. Math. U. Phys. **7,** 133–141, 290 (1896).

Gollnitz, H.,

(1) Partitionen mit differenzenbedingungen, J. Reine Angew. Math. **225,** 154–190 (1967).

Gordon, B.,

(1) Some identities in combinatorial analysis, Quart. J. Math. Oxford Ser. (2) **12,** 285–290 (1961).
(2) A combinatorial generalization of Rogers-Ramanujan identities, Amer. J. Math. **83,** 393–399 (1961).
(3) Two new representations of partition function. Proc. Amer. Math. Soc. **13,** 869–873 (1962).
(4) Two theorems on multipartite partitions, J. London Math. Soc. **38,** 459–464 (1963).

Gordon, B., and Houten, L.,

(1) Notes on plane partitions, I, J. Combinatorial Theory **4,** 72–80 (1968).
(2) Notes on plane partitions, II, J. Combinatorial Theory **4,** 81–99 (1968).

Gould, H. W.,

(1) Binomial coefficients, the bracket function, and compositions with relatively prime summands, Fibonacci Quart. **2,** 241–260 (1964).

Graham, R. L.,

(1) A theorem on partitions. J. Austral. Math. Soc. **3,** 435–441 (1963).

Grosswald, E.,

(1) Some theorems concerning partitions. Trans. Amer. Math. Soc. **89,** 113–128 (1958).
(2) Correction and addition to 'Some theorems concerning partitions', Trans. Amer. Math. Soc. **95,** 190 (1960).
(3) Partitions into prime powers, Michigan Math. J. **7,** 97–122 (1960).
(4) Results, new and old, in the theory of partitions (Spanish) Rev. Un. Mat. Argentina **20,** 40–57 (1962).
(5) Elementary proofs in the theory of partitions. Math. Z. **81,** 52–61 (1963).

Gupta, H.,

(1) A table of partitions, Proc. London Math. Soc. (2) **39,** 142–149 (1935).
(2) On a conjecture of Ramanujan, Proc. Indian Acad. Sci. A. **4,** 625–629 (1936).
(3) On partitions of n, J. London Math. Soc. **11,** 278–280 (1936).
(4) A table of partitions (II), Proc. London Math. Soc. (2) **42,** 546–549 (1937).
(5) Tables of Partitions, Madras, Indian Math. Soc., vi+81 (1939).
(6) A formula in partitions, J. Indian Math. Soc. (N.S.) **6,** 115–117 (1942).
(7) An inequality in partitions, J. Univ. of Bombay **11,** 16–18 (1942).
(8) On an asymptotic formula in partitions, Proc. Indian Acad. Sci. Sect–A **16,** 101–102 (1942).
(9) On the maximum values of $p_k(n)$ and $\pi_k(n)$, J. Indian Math. Soc. (N.S.) **7,** 72–75 (1943).
(10) A note on the parity of $p(n)$, J. Indian Math. Soc. (N.S.) **10,** 32–33 (1946).
(11) An asymptotic formula in partitions. J. Indian Math. Soc. (N.S.) **10,** 73–76 (1946).
(12) A generalization of the partition function. Proc. Nat. Inst. Sci. India **17,** 231–238 (1951).
(13) On a generating function in partition theory, Proc. Nat. Inst. Sci. India **20,** 582–586 (1954).
(14) Partitions in general, Res. Bull. Panjab Univ., No. 67, 31–38 (1955).
(15) Partitions into distinct primes, Proc. Nat. Inst. Sci. India Part A **21.** 185–187 (1955).
(16) Partitions in terms of combinatory functions, Res. Bull. Panjab Univ., No. 94, 153–159 (1956).
(17) Certain averages connected with partitions. Res. Bull. Panjab Univ., No. 124, 427–430 (1957).
(18) On the partition of j-partite numbers. Proc. Nat. Inst. Sci. India Part A **27,** 579–587 (1961).
(19) An inequality for $P(N_j, k)$, Res. Bull. Panjab Univ. (N.S.) **13,** 173–178 (1962).
(20) Partition of j-partite numbers, Math. Student **31** (1963); 179–186 (1964).

Gupta, H., Gwyther, C. E., and Miller, J. C. P.,

(1) Tables of partitions, Royal Society Mathematical Tables, London, 1958 (reprinted 1962).

Gupta, O. P.,

(1) Partitions into exactly k distinct primes, Res. Bull. Panjab Univ. **107,** 283–290 (1957).

Gupta, O. P., and Luthra, S.,

(1) Partitions into primes, Proc. Nat. Inst. Sci. India, Part A. **21,** 181–184 (1955).

Gustin, W. E.,

(1) An operational characterization of certain partition polynomials. Proc. Amer. Math. Soc. **3,** 31–35 (1925).

Haberzetle, M.,

(1) On some partition function. Amer. J. Math. **63,** 589–599 (1941).

Hagis, P. Jr.,

(1) A problem on partitions with prime modulus $p \geqslant 3$. Trans. Amer. M. S. **102,** 30–62 (1962).
(2) Partitions into odd summands, Amer. J. Math. **85,** 213–222 (1963).
(3) On a class of partitions with distinct summands, Trans. Amer. Math. Soc. **112,** 401–415 (1964).
(4) Partitions into odd and unequal parts, Amer. J. Math. **86,** 317–324 (1964).
(5) A correction of some theorems on partitions, Trans. Amer. Math. Soc. **118,** 550 (1965).
(6) On the partitions of an integer into distinct odd summands. Amer. J. Math. **87,** 867–873 (1965).
(7) Partitions with odd summands—some comments and corrections. Amer. J. Math. **87,** 218–220 (1965).
(8) Some theorems concerning partitions into odd summands. Amer. J. Math. **88,** 664–681 (1966).

Hahn, Hwa S.,

(1) On the relative growth of differences of partition functions, Pacific J. Math. **14,** 93–106 (1964).

Hardy, G. H.,

(1) Ramanujan; Twelve lectures on subjects suggested by his life and work, Cambridge Univ. Press (1940), vii+236.

Hellund, E. J.,

(1) On the unrestricted partitions of a positive integer, Amer. Math. Monthly **42,** 91–93 (1935).

Hua, Loo-Keng,

(1) On the number of partitions of a number into unequal parts, Trans. Amer. Math. Soc. **51,** 194–201 (1942).

Husimi, H.,

(1) Partitio numerorum as occurring in a problem of nuclear physics, Proc. Phys. Math. Soc. Japan (3) **20,** 912–925 (1938).

Van Ijzeren, J.,

(1) Elementary properties of the partitions of the natural numbers, Mathematica, Zutphen B. 12, 115–118 (1944).

Ingham, A. E.,

(1) A Tauberian theorem for partitions. Ann. of Math. (2) **42,** 1075–1090 (1941).

Intrator, Jakub,

(1) Partitions I., Czechoslovak Math. J. **18** (93), 16–24 (1968).

Iseki, Kaneshiro,

(1) Ein Theorem der Zahlentheorie, Tohoku Math. J. **48,** 60–63 (1941).
(2) A proof of a transformation formula in the theory of partitions, J. Math. Soc. Japan **4,** 14–26 (1952).

Iseki, Sho,

(1) Some transformation equations in the theory of partitions, Proc. Japan Acad. **34,** 131–135 (1958).
(2) A partition function with some congruence condition. Amer. J. Math. **81,** 939–961 (1959).
(3) A generalization of a functional equation related to the theory of partitions, Duke Math. J. **27,** 95–110 (1960).
(4) On some partition function, J. Math. Soc. Japan **12,** 81–88 (1960).
(5) A proof of a functional equation related to the theory of partitions, Proc. Amer. Math. Soc. **12,** 502–505 (1961).
(6) Partitions in certain arithmetic progressions, Amer. J. Math. **83,** 243–264 (1961).

Kantz, G.,

(1) Zerfällung einer Zahl in Summanden, Deutsche Math. **5,** 476–481 (1941).

Kelly, John B.,

(1) Partitions with equal products, Proc. Amer. Math. Soc. **15,** 987–990 (1964).

Kempner, A. J.,

(1) The development of "partitio numerorum" with particular reference to the work of Messrs. Hardy, Littlewood and Ramanujan, Amer. Math. Monthly **30,** Part I 354–369; Part II 416–425 (1923).

Kloosterman, H. D.,

(1) Partitions, Euclides Groningen 21, 67–77 (1946).

Knopp, K.,

(1) Asymptotische formeln der additiven, Zahlentheorie Schriften Konigsbg **2,** 45–74 (1925).

Knopp, M. I.,

(1) Construction of a class of modular functions and forms, Pacific J. Math. **11,** 275–293; Part II, 661–678 (1961).

Kolberg, O.,

(1) Some identities involving the partition function. Math. Scand. **5,** 77–92 (1957).
(2) Identities involving partition functions $q(n)$ and $q_0(n)$, Math. Scand. **6,** 80–86 (1958).
(3) Note on the parity of partition function, Math. Scand. **7,** 377–378 (1959).
(4) Congruences involving the partition function for the moduli 17, 19, and 23, Univ. Bergen Arbok Naturvit Rekke (1959) No. 15, 10 pp. (1960).
(5) An elementary discussion of certain modular forms. Univ. Bergen Arbok Naturvit Rekke 1959 No. 16, 26 pp. (1960).
(6) The coefficients of $j(\tau)$ modulo powers of 3, Arbok Univ. Bergen Mat. Natur. Ser. 1962, No. 16, 7 pp.
(7) Congruences for the coefficients of the modular invariant $J(\tau)$. Math. Scand. **10,** 173–181 (1962).

Koshti, M. E.,

(1) Partitions and the method of differences, Math. Student **32,** 83–85 (1964).

Krecmar, W.,

(1) Sur les proprietes de la divisibilite d'une fonction additive. Bull. Acad. Sci. URSS **7,** 763–800 (1933).

Kruyswijk, D.,

(1) On some well known properties of the partition function $p(n)$ and Euler's infinite product. Nieuw Arch. Wiskunde (2) **23,** 97–107 (1950).

Lahiri, D. B.,

(1) On a type of series involving the partition function with applications to certain congruence relations. Bull. Calcutta Math. Soc. **38,** 125–132 (1946).
(2) Some non-Ramanujan congruence properties of the partition function, Proc. Nat. Inst. Sci. India **14,** 337–338 (1948).
(3) Further non-Ramanujan congruence properties of the partition function. Sci. and Culture **14,** 336–337 (1949).

Landau, E.,

(1) Bemerkungen zu Herrn D. N. Lehmer's Abhandlung in Bd. 22 dieses Journals, S. 293–335, Amer. J. Math. **26,** 209–322 (1904).
(2) Handbuch der Lehre von der verteilung der Primzahlen, Bd. 1 und 2 (1909).

Larsen, O.,

(1) Auf wieviel Arten Kann ein beliebiger Betrag in Kleingeld gewechselt werden, (i) Math Tidskr. A. Kobenhavn, 45–62 (1947). (ii) Math. Tidskr. A. Kobenhavn, 72–79 (1948).

Lehmer, D. H.,

(1) On a conjecture of Ramanujan, J. London Math. Soc. **11,** 114–118 (1936).
(2) On the Hardy-Ramanujan Series for the partition function, J. London Math. Soc. **12,** 171–176 (1937).
(3) An application of Schläfli's modular equation to a conjecture of Ramanujan, Bull. Amer. Math. Soc. **44,** 84–90 (1938).
(4) On the series for the partition function, Trans. Amer. Math. Soc. **43,** 271–295 (1938).
(5) On the remainders and convergence of the series for the partition function, Trans. Amer. Math. Soc. **46,** 362–373 (1939).
(6) Two non-existence theorems on partitions, Bull. Amer. Math. Soc. **52,** 538–544 (1946).
(7) On the partition of numbers into squares, Amer. Math. Monthly **55,** 476–481 (1948).
(8) A triangular number formula for the partition function, Scripta Math. **17,** 17–19 (1951).

Lehner, J.,

(1) A partition function connected with the modulus 5, Duke Math. J. **8,** 631–655 (1941).
(2) Ramanujan identities involving the partition function for the moduli 11^α, Amer. J. Math. **65,** 492–520 (1943).
(3) Proof of Ramanujan's partition congruence for the modulus 11^3, Proc. Amer. Math. Soc. **1,** 172–181 (1950).

Lindenbaum, A.,

(1) Sur les ensembles dans lequels toutes les equations d' une famille donnee ont un nombre de solutions fixé d' avance, Fundam. Math. Warszawa **20,** 1–29 (1933).

Livingood, J.,

(1) A partition function with the prime modulus $p > 3$, Amer. J. Math. **67,** 194–208 (1945).

Lorentz, G. G.,

(1) Multiplicity of representation of integers by sums of elements of two given sets, J. London Math. Soc. **28,** 464–467 (1953).

Luckey, P.,

(1) Auf wieviel verschiedene Arten Läßt sich ein Geldbetrag in Kleiner Munze Ohne und mit Benutzung des 4-pfg-Stucks auszahlen, Unterr. Bl. **39,** 50–52 (1933).

MacMahon, P. A.,

(1) Combinatory Analysis Vol. **1,** Cambridge Univ. Press 1915; Vol. **2** (1916), xix + 340 pp.
(2) Note on the parity of the number which enumerates the partitions of a number, Proc. Cambridge Phil. Soc. **20,** 281–283 (1920–21).
(3) Divisors of numbers and their continuations in the theory of partitions. Proc. London Math. Soc. (2) **19,** 75–113 (1921).
(4) The theory of modular partitions, Proc. Cambridge Phil Soc. **21,** 197–204 (1922–23).
(5) The partitions of infinity with some arithmetic and algebraic consequences, Proc. Cambridge Phil. Soc. **21,** 642–650 (1922–23).
(6) The parity of $p(n)$, the number of partitions of n, when $n \leq 1000$, J. London Math. Soc. **1,** 225–226 (1926).

Majumdar, K. N.,

(1) On the parity of the partition function $p(n)$. J. Indian Math. Soc. (N.S.) **13,** 23–24 (1949).

Makowski, A.,

(1) Partitions into unequal primes, Bull. Acad. Polon Sci. Sere. Sci. Math. Astr. Phys. **8,** 125–126 (1960).

Meinardus, G.,

(1) Über das partitionen-problem eines reellquad-ratischen Zahlen körpers, Math. Ann. **126,** 343–361 (1953).
(2) Über Partitionen mit Differenzenbedingungen, Math. Z. **61,** 289–302 (1954/1955).
(3) Partitionen und Teilerfunktionen, Arch. Math. **7,** 52–54 (1956).
(4) Zur additiven Zahlentheorie in Mehreren Dimensionen I. Math. Ann. **132,** 333–346 (1956).

Melzak, Z. A.,

(1) Partition functions and spiralling in plane random walk, Canad. Math. Bull. **6,** 231–237 (1963).

Mian, A. M. and Chowla, S.,

(1) The differential equations satisfied by certain functions, J. Ind. Math. Soc. (N.S.) **8,** 27–28 (1944).

Mordell, L. J.,

(1) An identity in combinatorial analysis. Proc. Glasgow Math. Assoc. **5,** 197–200 (1962).

Motzkin, T. S.,

(1) Ordered and cyclic partitions, Riveon Lematematika 1, 61–67 (1947).

Nanda, V. S.,

(1) Tables of solid partitions, Proc. Nat. Inst. Sci. India **19,** 313–314 (1953).
(2) Bose-Einstein condensation and the partition theory of numbers, Proc. Nat. Inst. Sci. India **19,** 681–690 (1953).
(3) Bipartite partitions, Proc. Cambridge Philos. Soc. **53,** 273–277 (1957).

Narayana, T. V. and Fulton, G. E.,

(1) A note on the compositions of an integer, Canad. Math. Bull. **1,** 169–173 (1958).

Newman, D. J.,

(1) The evaluation of the constant in the formula for the number of partitions of n, Amer. J. Math. **73,** 599–601 (1951).
(2) A simplified proof of the partition formula, Michigan Math. J. **9,** 283–287 (1962).

Newman, M.,

(1) Remarks on some modular identities. Trans. Amer. Math. Soc. **73,** 313–320 (1952).
(2) An identity for the coefficients of certain modular forms. J. London Math. Soc. **30,** 488–493 (1955).
(3) On the existence of identities for the coefficients of certain modular forms, J. London Math. Soc. **31,** 350–359 (1956).
(4) A table of the coefficients of the powers of η (τ), Nederl. Akad. Wetensch. Proc. Ser. A. 59= Indag. Math. **18,** 204–216 (1956).
(5) Some theorems about $p_r(n)$. Canad. J. Math. **9,** 68–70 (1957).
(6) Congruences for the coefficients of modular forms and some new congruences for the partition function, Canad. J. Math. **9,** 549–552 (1957).
(7) Further identities and congruences for the coefficients of modular forms, Canad. J. Math. **10,** 577–586 (1958).
(8) Congruences for the coefficients of modular forms and for the coefficients of $j(\tau)$, Proc. Amer. Math. Soc. **9,** 609–612 (1958).
(9) Construction and application of a class of modular functions II, Proc. London Math. Soc. (3) **9,** 373–387 (1959).
(10) Modular forms whose coefficients possess multiplicative properties, Ann. of Math. (2) **70,** 478–489 (1959).
(11) Weighted restricted partitions, Acta Arith. **5,** 371–380 (1959).
(12) Periodicity modulo m and divisibility properties of the partition function, Trans. Amer. Math. Soc. **97,** 225–236 (1960).
(13) Subgroups of the modular group and sums of squares, Amer. J. Math. **82,** 761–778 (1960).
(14) Congruences for the partition function to composite moduli, Illinois J. Math. **6,** 59–63 (1962).
(15) Modular forms whose coefficients possess multiplicative properties II, Ann. of Math. (2) **75,** 242–250 (1962).
(16) Note on the partition function, Amer. Math. Monthly **71,** 1022 (1964).

Nicol, C. A.,

(1) On restricted partitions and a generalization of the Euler ϕ number and the Moebius function, Proc. Nat. Acad. Sci. (U.S.A.) **39,** 963–968 (1953).

Nicol, C. A., and Vandiver, H. S.,

(1) A von Sterneck arithmetical function and restricted partitions with respect to a modulus, Proc. Nat. Acad. Sci. (U.S.A.) **40,** 825–835 (1954).
(2) On generating functions for restricted partitions of rational integers, Proc. Nat. Acad. Sci. (U.S.A.) **41,** 37–42 (1955).

Niven, I.,

(1) On a certain partition function, Amer. J. Math. **62,** 353–364 (1940).

Oderfeld, J., and Pleszezynska, E.,

(1) On some applications of partitions (Polish, Russian and English summaries) Zastos. Mat. **6,** 189–198 (1962).

Ostmann, Hans-Heinrich,

(1) Über eine Rekursionsformel in der Theorie der Partitionen, Math. Nachr. **13,** 157–160 (1955).

Pagni, P.,

(1) Studio Sulle partizioni numeriche I, II, III period, Mat. (4) **32,** 172–183, 199–211, 294–301 (1954).

Palama, G.,

(1) Saggio di una nuova trattazione delle multigrade, Boll. Un. Mat. Ital. (3) **3,** 263–278 (1948).
(2) Su di una questione relativa alla partizione di n. Boll. Un. Mat. Ital. (3) **13,** 558–563 (1958).

Parameswaran, S.,

(1) Partition functions whose logarithms are slowly oscillating. Trans. Amer. Math. Soc. **100,** 217–240 (1961).

Pennington, W. B.,

(1) On Mahler's partition problem, Ann. of Math. (2) **57,** 531–546 (1953).

Petersson, H.,

(1) Uber Modulfunktionen und partitionen-probleme, Abh. Deutsch Akad. Wiss Berlin Kl. Math. Allg. Nat. **1954** No. 2, p. 59.
(2) Uber die arithmetischen Eigenschaften eines systems multiplikativer modulfunktionen von Primzahlstufe. Acta Math. **95,** 57–110 (1956).
(3) Über Partitionen-probleme in Verbindung mit Potenzresten nach einem Primzahlmodul, Math. Z. **66,** 241–268 (1956).

Pleszczynska, E.,

(1) Recurrence formula for the number of limited partitions with different components, Zastos. Mat. **6,** 305–307 (1961/1962).

Putter, J.,

(1) On a modular equation connected with partitions, Riveon Lematematika **3**, 42–43 (1949).

Rademacher, H.,

(1) On the partition function, Proc. London Math. Soc. (2) **43**, 241–254 (1937).
(2) A convergent series for the partition function. Proc. Nat. Acad. Sci. (U.S.A.) **23**, 78–84 (1937).
(3) The Fourier coefficients of the modular invariant $J(\tau)$, Amer. J. Math. **60**, 501–512 (1938).
(4) Fourier expansions of modular forms and problems of partition. Bull. Amer. Math. Soc. **46**, 59–73 (1940).
(5) The Ramanujan identities under modular substitutions. Trans. Amer. Math. Soc. **51**, 609–636 (1942).
(6) On the expansion of the partition function in a series, Ann. of Math. (2) **44**, 416–422 (1943).
(7) Generalization of the reciprocity formula for Dedekind sums, Duke Math. J. **21**, 391–397 (1954).
(8) On the transformation of $\log \eta(\tau)$. J. Indian Math. Soc. (N.S.) **19**, 25–30 (1955).
(9) On the Selberg formula for $A_k(n)$, J. Indian Math. Soc. (N.S.) **21** (1957), 41–55 (1958).

Rademacher, H., and Whiteman, A.,

(1) Theorems on Dedekind sums, Amer. J. Math. **63**, 377–407 (1941).

Rademacher, H., and Zuckerman, H. S.,

(1) On the Fourier coefficients of certain modular forms of positive dimension, Ann. Math. Princeton (2) **39**, 433–462 (1938).
(2) A new proof of two of Ramanujan's identities, Ann. Math. Princeton (2) **40**, 473–489 (1939).

Raghavan, S.,

(1) Modular forms of degree n and representation by quadratic forms, Ann. of Math. (2) **70**, 446–477 (1959).
(2) Modular forms of degree n and representation by quadratic forms, Contributions to function theory (Internat. Colloq. Function Theory, Bombay 1960), pp. 181–183, Tata Institute of Fundamental Research, Bombay.

Ramanathan, K. G.,

(1) Identities and congruences of the Ramanujan type, Canadian J. Math. **2**, 168–178 (1950).

Ramanujan, S.,

(1) Congruence properties of partitions, Proc. London Math. Soc. (2) **18** (1920); Math. Z. **9**, 147–153 (1921).
(2) Collected papers (Cambridge Univ. Press 1927, xxxvi+355 pp).

Ricci, G.,

(1) Sulla partizione degli interi in addendi prime Col procedimanto del residue minimo, Boll. Un. Math. Ital. (3) **10**, 1–10 (1955).

Rieger, G. J.,

(1) Ein einfacher Beweis der Funktional-gleichungen der Thetanellwërte. Arch. Mat. **11**, 454–456 (1960).

Robertson, M. M.,

(1) Partitions of large multipartites, Amer. J. Math. **84**, 16–34 (1962).
(2) The evaluation of a definite integral which occurs in asymptotic partition theory, Proc. Roy. Soc. Edinburgh Sect. A. **65**, 283–309 (1960/61).
(3) Asymptotic formulae for the number of partitions of a multipartite number, Proc. Edinburgh Math. Soc. (2), **12**, 31–40 (1960/61).

Rogers, L. J.,

(1) On the expansion of some infinite products. Proc. London Math. Soc. **24**, 337–352 (1893).

Rogers, L. J., and Ramanujan, S.,

(1) Proof of certain identities in Combinatory analysis. Proc. Cambridge Phil. Soc. **19**, 211–216 (1916–19).

Roselle, D. P.,

(1) Restricted k-partite partitions, Math. Nachr. **32**, 139–148 (1966).

Roth, K. F., and Szekeres, G.,

(1) Some asymptotic formulae in the theory of partitions, Quart. J. Math. Oxford Ser. (2) **5**, 241–259 (1954).

Rudert, W. S., and Lill, H. G.,

(1) Uber partitionen und ein lineares diophantisches problem, Numer. Math. **8**, 407–411 (1966).

Rushforth, J. M.,

(1) Congruence properties of the partition function and associated functions, Proc. Cambridge Phil. Soc. **48**, 402–413 (1952).

Sandham, H. F.,

(1) Five series of partitions, J. London Math. Soc. **27**, 107–115 (1952).
(2) Two identities due to Ramanujan, Quart. J. Math. Oxford Ser. (2) **3**, 179–182 (1952).
(3) Two series of partitions, Amer. Math. Monthly **61**, 104–106 (1954).

Sapiro-Pjateckij,

(1) Über eine asymptotische Formel für die Anzahl der Abelschen Gruppen, deren ordnung n nicht ubersteigt. Math. Shornik n, Ser. **26**(68), 479–486 (1950).

Scheenfeld, L.,

(1) A transformation formula in the theory of partitions, Duke Math. J. **11**, 873–887 (1944).

Schrutka, L.,

(1) Zur additiven Zahlentheorie, Wien Ber. **126**, 1081–1163 (1917).

Schur, I.,

(1) Ein Beitrag zur additiven Zahlentheorie und zur Theorie der Kettenbruche, Berl. Ber. 302–321 (1917).
(2) Zur additiven Zahlentheorie—S. B. Akad. Berlin 488–495 (1926).

Scorza, G.,

(1) Osservazioni varie sulla teoria delle sostituzioni e sulle partizioni di numeri interi in numeri interi, Palermo Rend. **36**, 163–170 (1913).

Selberg, A.,

(1) Über einige arithmetische Identitaten, Avhl. Norske Vidensk Akad. Oslo I Nr. 8, 23 pp. (1936).

Shah, S. M.,

(1) An inequality for the arithmetical function $g(x)$. J. Indian Math. Soc. **3**, 316–318 (1939).

Shanks, D.,

(1) A short proof of an identity of Euler, Proc. Amer. Math. Soc. **2**, 747–749 (1951).

Simons, W. H.,

(1) Congruences involving the partition function $p(n)$, Bull. Amer. Math. Soc. **50**, 883–892 (1944).

Slater, L. J.,

(1) A new proof of Roger's transformations of infinite series, Proc. London Math. Soc. (2) **53**, 460–475 (1951).
(2) Further identities of the Rogers-Ramanujan type. Proc. London Math. Soc. (2) **54**, 147–167 (1952).

Stern, M. A.,

(1) Eine Bemerkung über divisorensummen, Acta Math. **6**, 327–328 (1885).

Sterneck, R.,

(1) Über die Anzahl der Zerle ungen einer ganzen Zahl in 6 Summanden-Arch. d. Math. u. Phys. (3) **3**, 195–216 (1902).
(2) (i) Ein Analogon Zur additiven Zahlentheorie I. Wien. Ber. **111**, 1567–1601 (1902); (ii) Wien. Ber. **113**, 326–340 (1904).
(3) Über ein Analogon zur additiven Zahlentheorie, D. M. V. **12**, 110–113 (1903).
(4) Geometrische Ableitung des Satzes Von De Morgan-Sylvester und seines Analogons für 4 Summanden, Palermo Rend. **32**, 88–94 (1911).
(5) Zum Euler-Legendreschen Satz der additiven Zahlentheorie. Wien. Ber. **126**, 1345–1354 (1917).

Stohr, A.,

(1) Gelöste ünd ungelöste Fragen über Basen der naturlichen Zahlenreihe I and II, J. Reine Angew. Math. **194**, 40–65, 111–140 (1955).

Straus, E. G.,

(1) On a problem in the theory of partitions. Proc. Amer. Math. Soc. **13**, 192–196 (1962).

Subba Rao, M. V.,

(1) Ramanujan's trigonometrical sum and relative partitions. J. Indian Math. Soc. (N.S.) **15,** 57–64 (1951).
(2) Some remarks on the partition function, Amer. Math. Monthly **73,** 851–854 (1966).

Subrahmanya Sastri, V. V.,

(1) Three series involving partitions. Math. Student **29,** 47–51 (1961).
(2) Some results involving partition functions. J. Indian Math. Soc. (N.S.) **26,** 97–113 (1962).

Sudler, Culbreth, Jr.,

(1) An estimate for a restricted partition function, Quart. J. Math. Oxford Ser. (2) **15,** 1–10 (1964).
(2) Two algebraic identities and the unboundedness of a restricted partition function. Proc. Amer. Math. Soc. **15,** 16–20 (1964).
(3) A direct proof of two theorems on two-line partitions, Proc. Amer. Math. Soc. **16,** 161–168 (1965).
(4) Two enumerative proofs of an identity of Jacobi, Proc. Edinburgh Math. Soc. (2) **15,** 67–71 (1966).

Sugai, I.,

(1) On unequal partitions of integers, Math. Mag. **33,** 129–138 (1959–60).

Szekeres, G.,

(1) An asymptotic formula in the theory of partitions, Quart. J. Math. Oxford Ser. (2) **2,** 85–108 (1951).
(2) Some asymptotic formulae in the theory of partitions II, Quart. J. Math. Oxford Ser. (2) **4,** 96–111 (1953).

Tanturri, A.,

(1) Della partizioni dei numeri. Ambi, terni, quaterne, ecinquine di data somma, Torini Atti **52,** 902–918 (1916–17).

Tietze, H.,

(1) Systeme Von Partitionen Und Gitterpunktfiguren,
i. S. B. Math. Abt. Bayer Akad. Wiss. 23–54 1940.
ii. *S. B. Math. Abt. Bayer Akad. Wiss.* 69–131 1940.
iii. *S. B. Math. Abt. Bayer Akad. Wiss.* 133–145 1940.
iv. *S. B. Math. Abt. Bayer Akad. Wiss.* 147–166 1940.
v. *S. B. Math. Abt. Bayer Akad. Wiss.* 1–37 1941.
vi. *S. B. Math. Abt. Bayer Akad. Wiss.* 39–55 1941.
vii. *S. B. Math. Abt. Bayer Akad. Wiss.* 165–170 1941.
viii. *S. B. Math. Abt. Bayer Akad. Wiss.* 171–186 1941.
ix. *S. B. Math. Abt. Bayer Akad. Wiss.* 187–191 1941.
(2) Über Tripel Konjugierter Partitionen, Abh. Math. Sem. Hansischen Univ. **14,** 273–284 (1941).
(3) Über die Glaischersche Verallgemeinerung eines Euler-Schen Satzes Über Partitionen. J. Reine Angew. Math. **191,** 64–68 (1953).
(4) Über einge-schränkte und uneigeschränkte Partitionen. Bayer Akad. Wiss Math. Nat. Kl S. B., 11–16 (1960).

Todd, J. A.,

(1) A theorem in arithmetic, Math. Gaz. London **21,** 423–424 (1937).
(2) A table of partitions, Proc. London Math. Soc. (2) **48,** 229–242 (1943).

Vahlen, Th.,

(1) Beitrage zu einer additiven Zahlentheorie, J. Reine Angew. Math. **112,** 1–36 (1893).
(2) Zur Partition der Zahlen, S. B. Akad. Berlin, 394–400 (1929).

Vaidya, A. M.,

(1) A formula for the partitions of n into four parts. J. Gujrat Univ. **5,** No. 1/2, 172–176 (1962).
(2) A congruence property of $C_k(n)$, Amer. Math. Monthly **72,** 38–40 (1965).

Venkata Narayana, T.,

(1) Sur les treillis formes par les partitions d'un entier et leurs applications a la theorie des probabilites, C. R. Acad. Sci. Paris **240,** 1188–1189 (1955).

Vinogradov, I. M.,

(1) Die Methode der trigonometrischen Summen in der Zahlentheorie, Akad. Nauk SSSR, Trudy Mat. Inst. Steklov **23,** 109 pp. (1947).

Wall, H. S.,

(1) A continued fraction related to some partition formulas of Euler, Amer. Math. Monthly **48,** 102–108 (1941).

Watson, G. N.,

(1) A new proof of the Rogers-Ramanujan identities. J. London Math. Soc. **4,** 4–9 (1929).
(2) Two tables of partitions. Proc. London Math. Soc. (2) **42,** 550–556 (1937).
(3) Ramanujan's Vermutung über Zerfallungszahlen, J. Reine Angew. Math. **179,** 97–128 (1938).

Whiteman, A. L.,

(1) A sum connected with the partition function, Bull. Amer. Math. Soc. **53,** 598–603 (1947).

Wintner, A.,

(1) On restricted partitions with a basis of uniqueness. Rev. Un. Mat. Argentina **13,** 99–105 (1948).

Wright, E. M.,

(1) Asymptotic partition formulae, (I) Plane partitions. Quart. J. Math. **2,** 177–189 (1931).
(2) Asymptotic partition formulae, (II) Weighted partitions, Proc. London Math. Soc. (2) **36,** 117–141 (1934).
(3) Asymptotic partition formulae, (III) Partitions into k-th powers, Acta Math. **63,** 143–191 (1934).
(4) Partitions of multi-partite numbers. Proc. Amer. Math. Soc. **7,** 880–890 (1956).
(5) The number of partitions of a large bi-partite number. Proc. London Math. Soc. (3) **7,** 150–160 (1957).
(6) Partitions of large bipartites, Amer. J. Math. **80,** 643–658 (1958).
(7) A definite integral in the asymptotic theory of partitions. Proc. London Math. Soc. (3) **8,** 312–320 (1958).
(8) Partitions into k parts, Math. Ann. **142,** 311–316 (1960/61).
(9) Partitions of multipartite numbers into a fixed number of parts. Proc. London Math. Soc. (3) **11,** 499–510 (1961).
(10) A simple proof of a known result in partitions. Amer. Math. Monthly **68,** 144–145 (1961).
(11) A closure estimate for a restricted partition function, Quart. J. Math. Oxford Ser. (2). **15,** 283–287 (1964).
(12) Direct proof of the basic theorem on multipartite partitions, Proc. Amer. Math. Soc. **15,** 469–472 (1964).
(13) Partition of multipartite numbers into k parts, J. Reine Angew. Math. **216,** 101–112 (1964).
(14) Proof of a conjecture of Sudler's. Quart. J. Math. Oxford Ser. (2), **15,** 11–15 (1964).
(15) An extension of a theorem of Gordon, Proc. Glasgow Math. Assocn. **7,** 39–41 (1965).
(16) Coefficients of a reciprocal generating function. Quart. J. Math. Oxford Ser. (2) **17,** 39–43 (1966).
(17) The generating function of solid partitions, Proc. Roy. Soc. Edinburgh, Sect.-A **67,** 185–195 (1965/67).
(18) Rotatable Partitions, J. London Math. Soc. **43,** 501–505 (1968).

Zeller, Chr.

(1) Zu Eulers Recursionsformel für die Divisorensummen. Acta Math. **4,** 415–416 (1884).

Ziaud Din, M.,

(1) On formulae in partitions and divisors of a number, derived from symmetric functions, Proc. Nat. Acad. Sci. India Sect. A **13,** 221–224 (1943).

Zubrzycki, S.,

(1) A recursive formula for the number of restricted partitions. (Polish, Russian and English summaries.) Zastos Mat. **6,** 231–234 (1962).

Zuckerman, H. S.,

(1) Identities analogous to Ramanujan's identities involving the partition function. Duke Math. J. **5,** 88–110 (1939).
(2) The computation of the smaller coefficients of $J(\tau)$, Bull. Amer. Math. Soc. **45,** 917–919 (1939).

(Paper 74B1–313)