Commutator Groups and Algebras*

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Let $H$ and $K$ be connected, Lie subgroups of a Lie group $G$. The group $[H, K]$, generated by all commutators $hkh^{-1}k^{-1}(heH, keK)$ is arcwise connected. Therefore, by a theorem of Yamabe, $[H, K]$ is a Lie subgroup. If $\mathfrak{h}$, $\mathfrak{k}$ denote the Lie algebras of $H$ and $K$, respectively, then the Lie algebra of $[H, K]$ is the smallest algebra containing $[\mathfrak{h}, \mathfrak{k}]$, which is invariant under $ad\mathfrak{h}$ and $ad\mathfrak{k}$. An immediate consequence is that if $H$ and $K$ are complex Lie subgroups, then $[H, K]$ is also complex.

Key words: Adjoint representation; commutator; Lie algebra; Lie group.

Let $G$ be a real Lie group, and let $H$ and $K$ be connected, Lie subgroups, with Lie algebras $\mathfrak{h}$ and $\mathfrak{k}$. The group $[H, K]$ generated by commutators $hkh^{-1}k^{-1}(heH, keK)$ is arcwise connected. This implies, by a theorem of Yamabe [1], that $[H, K]$ is a Lie subgroup. The question we raise in this note is: What is the Lie algebra of $[H, K]$? We shall prove that this is the smallest algebra which contains $[\mathfrak{h}, \mathfrak{k}]$ and is invariant under $ad\mathfrak{h}$ and $ad\mathfrak{k}$ (i.e., it is the smallest ideal in the algebra generated by $\mathfrak{h}$ and $\mathfrak{k}$ which contains $[\mathfrak{h}, \mathfrak{k}]$.) An immediate consequence is that if $H$ and $K$ are complex groups, then $[H, K]$ is also complex. Of course, one obtains as a special case the known fact that if $H$ and $K$ are normal, then $[\mathfrak{h}, \mathfrak{k}]$ is the Lie algebra of $[H, K]$. More generally, the Lie algebra of $[H, K]$ is the smallest algebra $\mathfrak{m}$ containing $[\mathfrak{h}, \mathfrak{k}]$, if and only if $\mathfrak{m}$ is invariant under $ad\mathfrak{h}$ and $ad\mathfrak{k}$.

We shall first give a list of notation and terminology. In the following, $G$ is a real Lie group, with Lie algebra $\mathfrak{g}$.

1. If $g \in G$ and $X$ is a tangent vector or vector field on $G$, then $gX$ and $Xg$ denote the left and right translation of $X$. (i.e., if $l_g$, $r_g$ are the left and right translations of $G$, then $gX = dl_g(X)$ and $Xg = dr_g(X)$.)

2. We denote by $ad$ and $Ad$ the adjoint representations of $\mathfrak{g}$ and $G$, respectively. Thus $ad(X)Y = [X, Y]$ and $Ad(g)Y = gYg^{-1}$, $\theta(g)$ denotes the inner automorphism $x \mapsto gxg^{-1}$. Note that $Ad(\exp X) = \exp (adX)$ and $\theta(g)\exp X = \exp [Ad(g)X]$.

3. If $X$ is a tangent vector at some point of $G$, $X$ denotes the corresponding left-invariant vector field. Of course, $\mathfrak{g}$ is the algebra of left-invariant vector fields on $G$.

4. If $X$ is a vector field, $x \in G$, then $X_x$ is the value of $X$ at $g$. If $\mathfrak{h} \subset \mathfrak{g}$ is a subalgebra, $\mathfrak{h}_g = \{X_g | X \in \mathfrak{h}\}$.

5. If $\mathfrak{h}$ and $\mathfrak{k}$ are subalgebras of $\mathfrak{g}$, $[\mathfrak{h}, \mathfrak{k}]$ denotes the linear span of the commutators $[[X, Y] | X \in \mathfrak{h}, Y \in \mathfrak{k}]$.

6. If $\mathfrak{m}$ is a subspace of $\mathfrak{g}$, $\mathfrak{m}$ denotes the subalgebra generated by $\mathfrak{m}$.

7. If $\mathfrak{m}$ is a subspace, $\mathfrak{h}$ a subalgebra, with $\mathfrak{m} \subset \mathfrak{h}$, then $\mathfrak{Z}(\mathfrak{m}, \mathfrak{h})$ denotes the smallest ideal in $\mathfrak{h}$ which contains $\mathfrak{m}$. Thus $\mathfrak{Z}(\mathfrak{m}, \mathfrak{h})$ is the linear span of $\mathfrak{m}$ and elements of the form $(adX_1)(adX_2) \ldots (adX_r)Y$, where $X_1 \in \mathfrak{h}$, $Y \in \mathfrak{m}$. We write $\mathfrak{Z}[\mathfrak{h}, \mathfrak{k}]$ for $\mathfrak{Z}([\mathfrak{h}, \mathfrak{k}], \mathfrak{g})$.

8. If $g : I \rightarrow G$ is a differentiable curve (where $I$ is an interval), we write $\frac{dg}{dt}_{t_0}$ for the tangent vector $dg|_{t_0}(\frac{d}{dt})$. We shall write $0((t-t_0)^k)$ for a function $X(t)$ (with values in $\mathfrak{g}$) if there exist numbers $M > 0$, $\epsilon > 0$ so that $||X(t)|| < M|t-t_0|^k$ when $|t-t_0| < \epsilon$. (Here $||X||$ is some norm in $\mathfrak{g}$.)

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247
For example, if \( g(t) \) is a differentiable curve in \( G \), with \( g(t_0) = 1 \) and \( \frac{dg}{dt} \big|_{t_0} = X \), then (for \( t \) near \( t_0 \)) \( \log \left[ g(t) \right] = (t-t_0)X + O((t-t_0)^2) \), or equivalently, \( g(t) = \exp \left[ (t-t_0)X + O((t-t_0)^2) \right] \).

The Campbell-Baker-Hausdorff formula implies:

\[
\exp (tX) \exp (tY) = \exp \left( t(X + Y) + \frac{t^2}{2} [X, Y] + O(t^3) \right).
\]

**Lemma 1:** Let \( H \) and \( K \) be connected Lie subgroups of \( G \), with Lie algebras \( \mathfrak{h} \) and \( \mathfrak{t} \).

(a) If \( h \in H \) and \( Y \in \mathfrak{t} \), then \( (Ad(h) - 1)Y \in \mathfrak{N}[\mathfrak{h}, \mathfrak{t}] \).

(b) If \( k \in K \) and \( X \in \mathfrak{h} \), then \( (Ad(k) - 1)X \in \mathfrak{N}[\mathfrak{h}, \mathfrak{t}] \).

**Proof:** We shall prove (a). Note that \( (Ad(h_1)h_2) - 1)Y = Ad(h_1) (Ad(h_2) - 1)Y + (Ad(h_1) - 1)Y \).

Thus, if the statement is true for \( h_1 \) and \( h_2 \), it is true for \( h_1h_2 \). Therefore it suffices to prove it for a set of generators. We shall show that it is true for \( h = \exp(X), X \in \mathfrak{h} \).

\[
Ad(\exp X) - 1 = \exp (adX) - 1 = \sum_{n=1}^{\infty} \frac{(adX)^n}{n!},
\]

while \( (adX)^n Y \in \mathfrak{N}[\mathfrak{h}, \mathfrak{t}] \).

**Lemma 2:** Let \( H \) and \( K \) be connected Lie subgroups of \( G \), with Lie algebras \( \mathfrak{h} \) and \( \mathfrak{t} \). Let \( h(t) \) (respectively \( k(t) \)) be a differentiable curve in \( H \) (respectively \( K \)) with \( h(0) = 1 \) (respectively \( k(0) = 1 \)).

Let \( g(t) = h(Vt)k(Vt)h(Vt) - k(Vt) \), then for each \( t_0 \geq 0 \) at which \( g(t) \) is defined,

\[
\frac{dg}{dt} \big|_{t_0} = \mathfrak{N}[\mathfrak{h}, \mathfrak{t}]_{g(t_0)}.
\]

**Proof:** For \( t_0 = 0 \),

\[
\begin{align*}
\frac{dg}{dt} \big|_{t_0} = \frac{d}{dt} g(t) \big|_{t_0} = X + Y,
\end{align*}
\]

where \( X = \frac{dh}{dt} \big|_{t_0} \) and \( Y = \frac{dk}{dt} \big|_{t_0} \). We must find out what happens when \( t_0 > 0 \).

Let \( u = Vt \), \( u_0 = \sqrt{t} \), \( h_0 = h(u_0) \), \( k_0 = k(u_0) \).

\[
g_0 = g(t_0) = h_0k_0h_0^{-1}k_0^{-1},
\]

\[
X = \frac{dh(u)}{du} \bigg|_{u_0}, \quad Y = \frac{dk(u)}{du} \bigg|_{u_0} \quad \text{and} \quad Z = \frac{dg(t)}{dt} \bigg|_{t_0}
\]

The translated curve \( g_0^{-1}g(t) \) has tangent vector \( g_0^{-1}Z \) at 1. We shall use properties of \( \exp \) to compute this tangent vector.

Noting that

\[
\frac{d}{du} (h_0^{-1}h(u)) \bigg|_{u_0} = h_0^{-1}X,
\]

and

\[
\frac{d}{du} (h_0h(u)^{-1}) \bigg|_{u_0} = - \frac{d}{du} (h(u)h_0^{-1}) \bigg|_{u_0} = -Xh_0^{-1}.
\]

We obtain

\[
\begin{align*}
(1) \quad & h_0^{-1}h(u) = \exp \left\{ (u-u_0)X + 0((u-u_0)^2) \right\}, \\
(2) \quad & h_0h(u)^{-1} = \exp \left\{ -(u-u_0)Ad(h_0)X + 0((u-u_0)^2) \right\}.
\end{align*}
\]

Similarly,

\[
\begin{align*}
(3) \quad & k(u)k_0^{-1} = \exp \left\{ (u-u_0)Ad(k_0)Y + 0((u-u_0)^2) \right\}, \\
(4) \quad & k(u)^{-1}k_0 = \exp \left\{ -(u-u_0)Y + 0((u-u_0)^2) \right\}.
\end{align*}
\]

Next, we see that

\[
g_0^{-1}g(t) = k_0h_0k_0^{-1}h_0^{-1}h(u)k(u)h(u)^{-1}k(u)^{-1}
\]

\[
= \theta(k_0) \left\{ \left[ \theta(h_0^{-1}) (h_0^{-1}h(u)k(u)k_0^{-1}) \right] [h_0h(u)^{-1}k(u)^{-1}k_0] \right\}.
\]

248
Now by using equations (1)–(4),
\[(5) \exp(sU + 0(s^2)) \exp(sV + 0(s^2)) = \exp\{s(U + V) + 0(s^2)\}\]
and
\[(6) \theta(g) \exp(U) = \exp[Ad(g)U],\]
a straightforward computation shows that \(g^{-1}g(t) = \exp[W(t)]\), where
\[
W(t) = (u - u_0) \{Ad(k_0h_0) (Ad(k_0^{-1}) - 1) \hat{X} + Ad(k_0) (Ad(h_0) - 1) \hat{Y}\} + 0((u - u_0)^2).
\]

It follows that the tangent vector \(g_0^{-1}Z\) is the value of the vector field
\[
\left.\frac{dW}{dt}\right|_{t_0} = \frac{1}{\sqrt{t_0}} \{Ad(k_0h_0) (Ad(k_0^{-1}) - 1) \hat{X} + Ad(k_0) (Ad(h_0) - 1) \hat{Y}\}
\]
at the unit element. Thus, by Lemma 1, \(g_0^{-1}Z \in \mathfrak{g}[\mathfrak{h}, \mathfrak{f}]_1\) and \(Z \in \mathfrak{g}[\mathfrak{h}, \mathfrak{f}]_{g_0}^g.\)

**LEMMA 3:** \([H, K]\) is normalized by \(H\) and \(K\).

**PROOF:** Let \(h \in H\) and \(h \in K\). Then
\[
h_1(hkh^{-1}k^{-1})h_1^{-1} = (h_1hkh^{-1}h^{-1}k^{-1}) (hk^{-1}h^{-1}).
\]
This shows that \([H, K]\) is normalized by \(H\). Since \([H, K] = [K, H]\), the same is true for \(K\).

**THEOREM:** Let \(G\) be a real Lie group, and let \(H\) and \(K\) be connected, Lie subgroups with Lie algebras \(\mathfrak{h}\) and \(\mathfrak{f}\). Then \([H, K]\) is a Lie subgroup with Lie algebra \(\mathfrak{g}[\mathfrak{h}, \mathfrak{f}]\).

**PROOF:** We have already remarked that Yamabe’s theorem implies that \([H, K]\) is a Lie subgroup. Let \(\mathfrak{m}\) denote its Lie algebra. We first show that \(\mathfrak{g}[\mathfrak{h}, \mathfrak{f}] \subseteq \mathfrak{m}\). If \(X \in \mathfrak{h}, Y \in \mathfrak{f}\), then the curve
\[
g(t) = \exp(\sqrt{t}X) \exp(\sqrt{t}Y) \exp(-\sqrt{t}X) \exp(-\sqrt{t}Y)
\]
lies in \([H, K]\). Its tangent vector at 1 is \([X, Y]_1\). This shows that \([\mathfrak{h}, \mathfrak{f}] \subseteq \mathfrak{m}\). Lemma 3 implies that \(\mathfrak{m}\) is invariant under \(ad(\mathfrak{h})\) and \(ad(\mathfrak{f})\). Therefore \(\mathfrak{g}[\mathfrak{h}, \mathfrak{f}] \subseteq \mathfrak{m}\).

Now we show that \(\mathfrak{m} \subseteq \mathfrak{g}[\mathfrak{h}, \mathfrak{f}]\). Equivalently, we show that \([H, K] \subseteq L\), where \(L\) is the connected Lie subgroup whose Lie algebra is \(\mathfrak{g}[\mathfrak{h}, \mathfrak{f}]\). Let \(h \in H, k \in K\), and let \(h(t)\) and \(k(t)\) be differentiable curves in \(H\) and \(K\), respectively, so that \(h(0) = 1 = k(0), h(1) = h\) and \(k(1) = k\).

Let
\[
g(t) = h(\sqrt{t})k(\sqrt{t})h(\sqrt{t})^{-1}k(\sqrt{t})^{-1}.
\]

Lemma 2 says that
\[
\left.\frac{dg}{dt}\right|_{t_0} \in \mathfrak{g}[\mathfrak{h}, \mathfrak{f}]_{g(t_0)} (0 \leq t_0 \leq 1).
\]
Therefore, the curve \(g(t)\) lies in the maximal connected integral manifold (through 1) of the differential system \(\mathfrak{g}[\mathfrak{h}, \mathfrak{f}]\). In other words, the curve \(g(t)\) lies in \(L\). Thus, the commutators \(hkh^{-1}k^{-1}eL\), and consequently \([H, K] \subseteq L\).

**COROLLARY 1:** Let \(G\) be a complex Lie group, and let \(H\) and \(K\) be connected (complex) Lie subgroups. Then \([H, K]\) is a (complex) Lie subgroup.

**PROOF:** Considering \(G\) with its real structure, we see that \([H, K]\) is a real Lie subgroup whose Lie algebra is \(\mathfrak{g}[\mathfrak{h}, \mathfrak{f}]\) (where \(\mathfrak{h}\) and \(\mathfrak{f}\) are the Lie algebras of \(H\) and \(K\)). Since \(\mathfrak{h}\) and \(\mathfrak{f}\) are complex, so is \(\mathfrak{g}[\mathfrak{h}, \mathfrak{f}]\) and \([H, K]\).

**COROLLARY 2:** The Lie algebra of \([H, K]\) is \(\mathfrak{A}[\mathfrak{h}, \mathfrak{f}]\), if and only if \(\mathfrak{A}[\mathfrak{h}, \mathfrak{f}]\) is invariant under \(ad(\mathfrak{h})\) and \(ad(\mathfrak{f})\).

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