Principal Ideals in Matrix Rings

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It is shown that every left ideal of the complete matrix ring of a given order over a principal ideal ring is principal, and a partial converse is proven.

Key words: Dedekind ring; matrix ring; non-Noetherian ring; principal ideal ring.

1. Introduction

Let $R$ be a ring with a unity 1, and let $n$ be a positive integer. It is well-known [3, p. 37] that every two-sided ideal of $R_n$ (the complete matrix ring of order $n$ over $R$) is necessarily of the form $M_n$, where $M$ is a two-sided ideal of $R$. Simple examples show that this result no longer holds for one-sided ideals. In this note we investigate the left ideals of $R_n$ in the case when $R$ is a principal ideal ring (an integral domain in which every ideal is principal). We shall prove

**Theorem 1:** If $R$ is a principal ideal ring, then every left ideal of $R_n$ is principal.

The proof of Theorem 1 depends upon the fact that if $A$ is any $p \times q$ matrix over $R$, then a unit matrix $U$ of $R_p$ exists such that the $p \times q$ matrix $UA$ is upper triangular [2, p. 32].

We also establish the following partial converse to Theorem 1:

**Theorem 2:** If $R$ is not Noetherian or if $R$ is a Dedekind ring but not a principal ideal ring, then $R_n$ contains a nonprincipal left ideal.

For general information on rings, see [3]. For information on Dedekind rings, see [1, p. 101].

2. Proofs

We denote the matrix of $R_n$ which has 1 in position $(i, j)$ and 0 elsewhere by $E_{ij}$. We first prove

**Lemma 1:** Suppose that every left ideal of $R$ has a finite $R$-basis. Then so has every left ideal of $R_n$.

**Proof:** Let $a$ be a left ideal of $R_n$. Let $a_k$, $2 \leq k \leq n$, be the subset of $a$ consisting of all matrices of $a$ whose first $k-1$ columns are 0; and set $a_1 = a$. Then, as is easily verified, $a_k$ is a left ideal of $R_n$ for $1 \leq k \leq n$.

Let $M_{ik}$, $1 \leq i \leq n$, be the set of elements of $R$ occurring in the $(i, k)$ position of all matrices of $a_k$, $1 \leq k \leq n$. Then $M_{ik}$ is a left ideal of $R$ (since $a_k$ is a left ideal of $R_n$) and so has a finite $R$-basis, say

$$m_{lk}, 1 \leq l \leq l_{ik}.$$

Hence we can find $l_{ik}$ matrices of $a_k$, say $A_{ik}$, such that the $(i, k)$th entry of $A_{ik}$ is $m_{lk}$. It follows that the $l_{ik}$ matrices

$$Bl_{ik} = E_{ik}A_{ik}, 1 \leq i, k \leq n, 1 \leq l \leq l_{ik},$$

also belong to $a_k$, have $m_{lk}$ as their $(i, k)$th entry, but have nonzero entries in the $i$th row only. These

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1 Figures in brackets indicate the literature at the end of this paper.
matrices constitute a finite $R$-basis for $a$. For suppose that $A$ is any element of $a$. We first find elements $r_i^1$ of $R$ such that

$$A - \sum_{i=1}^{n} \sum_{l=1}^{k} r_{i}^{1} B_{i l}^{1} = A_{2} \epsilon a_{2};$$

we then find elements $r_{i}^{2}$ of $R$ such that

$$A_{2} - \sum_{i=1}^{n} \sum_{l=1}^{k} r_{i}^{2} B_{i l}^{2} = A_{3} \epsilon a_{3};$$

and continuing in this manner, we determine elements $r_{i}^{k}$ of $R$ such that

$$A = \sum_{k=1}^{n} \sum_{i=1}^{k} \sum_{l=1}^{k} r_{i}^{k} B_{i l}^{k}.$$

This completes the proof.

We now prove Theorem 1. Let $a$ be a left ideal of $R_n$. By Lemma 1, $a$ possesses a finite $R$-basis, say $B_1, B_2, \ldots, B_t$. Let $B$ be the $nt \times t$ matrix

$$B = \begin{pmatrix}
B_1 \\
B_2 \\
\vdots \\
B_t
\end{pmatrix}$$

Let $U$ be a unit matrix of $R_{nt}$ such that $UB = T$ is upper triangular. Thus

$$UB = T = \begin{bmatrix}
H \\
0
\end{bmatrix},$$

where $H$ is an $n \times n$ upper triangular matrix, and the zero block 0 is $(nt - n) \times n$. We shall show that $a = R_n H$. For if we write $U = (U_{ij})$, where the matrices $U_{ij}$ are $n \times n$, then

so that $H \epsilon a$, implying that

$$\sum_{j=1}^{t} U_{i j} B_{j} = H,$$

so that $R_n H \subseteq a$.

If we then write $U^{-1} = V = (V_{ij})$, where the matrices $V_{ij}$ are $n \times n$, then $V$ belongs to $R_{nt}$ (since $U$ is a unit matrix of $R_{nt}$) and from $B = V T$ we find that

implying that

$$B_i = V_{i1} H, \quad 1 \leq i \leq t,$$

$$a \subseteq R_n H.$$

This completes the proof of Theorem 1.

To prove Theorem 2, we first observe that for any left ideal $M$ of $R$, the left ideal $M_n$ of $R_n$ can be principal only if $M$ has a set of $n$ or fewer generators. In particular, if $R$ is non-Noetherian, $M$ can be chosen to violate this condition.

We now assume that $R$ is a Dedekind ring and that any ideal in $R$ can be generated by at most $n$ elements. Let $S$ be a nonprincipal ideal in $R$, and let $\mathcal{S}$ be the subset of $R_n$ consisting of all matrices with first column entries in $S$ and all other entries arbitrary members of $R$. Clearly, $\mathcal{S}$ is a left ideal in $R_n$. We shall show that $\mathcal{S}$ is not principal.

Suppose the contrary. Let $X = (x_{ij})$ generate $\mathcal{S}$, so that $\mathcal{S} = R_n X$. Clearly the $x_{1l}$ generate $S$; $S = \{x_{11}, x_{21}, \ldots, x_{n1}\}$. We may assume that $x_{11}$ is not zero, since we may interchange the rows of $X$ by left multiplication by a permutation matrix. Let $d = \det X$. Since $\mathcal{S}$ contains nonsingular matrices (for example, $\text{diag}(x_{11}, 1, \ldots, 1)$) $X$ must be nonsingular and thus $d$ is a nonzero element of $S$. Let $Y = (y_{ij})$ be the adjoint of $X$, so that

$$XY = YX = dI.$$
Then $Y \in R_n$, and if $C$ is any matrix in $\mathcal{G}$, every element of $CY$ must be divisible by $d$. First choose $C = x_{ii} E_{11}, 1 \leq i \leq n$. We obtain

$$\{d\} \{x_{ii} y_{ij}\}, \quad 1 \leq i, j \leq n. \quad (1)$$

Next choose $C = E_{ij}, 2 \leq i \leq n$. We obtain

$$\{d\} \{y_{ij}\}, \quad 2 \leq i \leq n, \quad 1 \leq j \leq n. \quad (2)$$

Put $y = \{y_{11}, y_{12}, \ldots, y_{1n}\}$. Then (2) implies that $y \{d\}^{n-1} \{\det Y\}$; and since $\det Y = d^{n-1}, y = \{1\} = R$. Hence $\{y_{11}, y_{12}, \ldots, y_{1n}\} = \{1\} = R$. But now (1) and (2) imply that $\{d\} \{x_{ii}\}, 1 \leq i \leq n$. Write

$$x_{ii} = \beta_i d, \quad \beta_i \in R, \quad 1 \leq i \leq n. \quad (3)$$

Since $d \in S$ and the $x_{ii}$ are a basis for $S$, elements $r_i$ of $R$ exist such that

$$d = \sum_{i=1}^{n} r_i x_{ii}.$$ 

But now (3) implies that

$$\sum_{i=1}^{n} r_i \beta_i = 1,$$

and hence

$$S = \{x_{11}, x_{21}, \ldots, x_{nn}\} = \{\beta_1, \beta_2, \ldots, \beta_n\} \{d\} = \{d\}.$$ 

Thus $S$ is principal, a contradiction. This completes the proof of Theorem 2.

3. References


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