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Principal Submatrices VII: Further Results Concerning Matrices With Equal Principal Minors *

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This paper characterizes real symmetric matrices A such that all $t\times t$ principal minors are equal and all $t\times t$ nonprincipal minors are of fixed sign, for two consecutive values of t less than rank A. It also characterizes matrices A (over an arbitrary field) in which all $t\times t$ principal minors are equal and all nonprincipal $t\times t$ minors are equal, for one fixed value of t less than rank A.

Key Words: Matrix; principal submatrices; rank; symmetric matrix.

In the paper "Principal Submatrices V," [5],[1] a classification was found for symmetric matrices A for which all $t\times t$ principal minors of A are equal, for *three* consecutive values of t less than the rank of A. It is the purpose of this paper to present a similar theorem classifying the real symmetric matrices in which the condition on the principal minors is weakened to requiring that all $t\times t$ principal minors of A be equal, for *two* consecuitive values of t less than the rank of A, and in which a sign condition is imposed on the nonprincipal $t\times t$ minors for these two consecutive values of t. This result is presented in Theorem 1. In this paper we also classify all square matrices A (over an arbitrary field and not necessarily symmetric) in which the condition on the principal minors of A is weakened to requiring that all $t\times t$ principal minors of A be equal for *one* value of t less than the rank of A, and for this value of t the condition on the nonprincipal $t\times t$ minors of A is strengthened to requiring that they all be equal. This result is presented in Theorem 4.

THEOREM 1. *Let* r *be a fixed integer and let* A *be an* $n\times n$ *symmetric matrix over the real number field, such that:*

(i) *all* $r\times r$ *principal minors of* A *are equal;*

(ii) *all* $(r+1)\times(r+1)$ *principal minors of* A *are equal;*

(iii) *all nonprincipal* $r\times r$ *minors of* A *which do not vanish have a common sign;*

(iv) *all nonprincipal* $(r+1)\times(r+1)$ *minors of* A *which do not vanish have a common sign;*

(v) *rank* $A\geq r+2$.

Then A *is scalar:* $A=aI_n$.

PROOF. The proof follows the pattern of the analogous theorem (Theorem 13) in [5].

First case. Let $r=1$. Let $A=(a_{ij})$. The equality of the 1×1 principal minors forces all a_{ii} to be equal, say $a_{ii}=a$. The equality of all 2×2 principal minors forces all a_{ij}^2 to be equal, for $i\neq j$. Thus $a_{ij}=\pm b$, for $i\neq j$. Because all nonvanishing nonprincipal 1×1 minors have a common sign, we see that (choosing b properly) all $a_{ij}=b$ for $i\neq j$. We wish to show that $b=0$. The nonprincipal 2×2 minor det $A[1, 2|1, 3]=b(a-b)$ and the nonprincipal 2×2 minor det $A[2, 3|1, 2]=-b(a-b)$. The sign condition on these nonprincipal 2×2 minors now shows that $b=0$ or $a-b=0$. If $a-b=0$ then $A=bJ_n$, where J_n is the $n\times n$ matrix in which each entry is one. Since J_n has rank one, the possibility $a=b$ contradicts hypothesis (v). Therefore $b=0$ and hence $A=aI_n$ as claimed.

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[1] Figures in brackets indicate the literature references at the end of this paper.

Before continuing, we explain a certain (known) device. Let $A' = DAD^{-1}$, where D is a diagonal matrix in which the (i, i) entry is $(-1)^i$ for $i = 1, 2, \ldots, n$. Applying a well-known formula [7] to A', we have

$$C_s(\text{adj } A') = \det A^{s-1} C^{n-s}(A'). \tag{1}$$

Here, as in [5], adj A' is the adjugate of A', $C_s(X)$ denotes the sth compound of X, and $C^{n-s}(X)$ denotes the corresponding supplementary compound. If $i_1, \ldots, i_s$ and $j_1, \ldots, j_s$ are two strictly increasing sequences of integers, the entry

$$(-1)^{i_1+\cdots+i_s+j_1+\cdots+j_s} \det A'(i_1, \ldots, i_s | j_1, \ldots, j_s)$$

of $C^{n-s}(A')$ equals

$$\det A[i_{s+1}, \ldots, i_n | j_{s+1}, \ldots, j_n].$$

Here $i_{s+1}, \ldots, i_n$ and $j_{s+1}, \ldots, j_n$ are the increasing sequences in $1, 2, \ldots, n$ complementary to $i_1, \ldots, i_s$ and $j_1, \ldots, j_s$, respectively. (Note: round brackets $A(|)$ indicate rows and columns deleted, and square brackets $A[|]$ indicate rows and columns retained.) It follows from these remarks and (1) that if A has all $(n-s) \times (n-s)$ principal minors equal, then adj A' has all $s \times s$ principal minors equal. It also follows that if all nonvanishing $(n-s) \times (n-s)$ nonprincipal minors of A have a common sign, then all nonvanishing nonprincipal $s \times s$ minors of adj A' have a common sign.

Second case: $r = n-2$. This is the largest value of r permitted and implies that A is nonsingular. Taking $s = 1$ and $s = 2$ in the discussion of the previous paragraph, we see that adj A' is scalar, and thus $(A')^{-1}$ is also scalar. But then A' is scalar, which implies that A is scalar. The proof is complete in case 2.

The general case: Let $1 < r < n-2$. We seek to prove that each $r \times r$ principal submatrix A_r of A is scalar. Since $r > 1$, it is quite easy to see from this fact that A is itself scalar.

If $\det A_r \neq 0$, we may pass a complete nested chain through A_r (see [5]), and so we secure $(r+1)$-square, $(r+2)$-square, and $(r+3)$-square principal submatrices A_{r+1}, A_{r+2}, A_{r+3} such that

$$A_r \subset A_{r+1} \subset A_{r+2} \subset A_{r+3} \tag{2}$$

with at least one of A_{r+2} or A_{r+3} invertible. If $\det A_r = 0$, then every r-square principal minor of A is singular, since these principal minors are all equal, and hence (as all $(r+1)$-square principal minors are equal) all $(r+1)$-square principal submatrices of A are nonsingular. Let A_{r+1} be an $(r+1)$-square principal submatrix containing A_r. Passing a complete nested chain through A_{r+1}, we obtain the nested chain (2), with at least one of A_{r+2} or A_{r+3} invertible.

If $\det A_{r+2} \neq 0$, we may apply case 2 to A_{r+2} and so conclude that A_{r+2} is scalar. Therefore A_r is also scalar.

If $\det A_{r+2} = 0$, then $\det A_{r+3} \neq 0$ and we let $B = \text{adj } A'_{r+3}$. Applying the identity (1) to A'_{r+3}, we see that B has all 2×2 principal minors equal, all nonvanishing 2×2 nonprincipal minors of common sign, all 3×3 principal minors equal, and all nonvanishing nonprincipal 3×3 minors of common sign. If we can show that B is scalar, it will follow that A_{r+3} is scalar, and hence that A_r is scalar. To prove that B is scalar, it will suffice to prove that each 2×2 principal submatrix B_2 of B is scalar. Since B is at least 5×5, we may embed B_2 in a nested chain

$$B_2 \subset B_3 \subset B_4 \subset B_5 \tag{3}$$

with at least one of B_4 or B_5 invertible. If $\det B_4 \neq 0$, an application of case 2 to B_4 shows that B_2 is scalar. Therefore we may suppose every 4×4 principal submatrix of B_5 containing B_2 is singular. If every 4×4 principal submatrix of B_5 is singular, then B_5 satisfies the hypotheses of Theorem 13 of [5] and hence

$$B_5 = aI_5 + bDJ_5D^{-1},$$

where $D = \operatorname{diag}(\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4, \epsilon_5)$. Here $\epsilon_1, \ldots, \epsilon_5$ are each ± 1, and $a \neq 0$ as B_5 is nonsingular. We wish to show that $b = 0$. Since for $1 \leq i < j < k \leq 5$, we have det $B_5[i, j|i, k] = \epsilon_j\epsilon_k ba$ and det $B_5[i, k|j, k] = \epsilon_i\epsilon_j ab$, the sign condition on the nonprincipal 2×2 minors of B_5 yields (for $b \neq 0$) $\epsilon_1 = \epsilon_2 = \epsilon_3 = \epsilon_4 = \epsilon_5$. Thus, we have $B_5 = aI_5 + bJ_5$. But now det $B_5[1, 2|1, 3] = ba$ and det $B[2, 3|1, 2] = -ba$. The sign condition on the nonprincipal 2×2 minors of B_5 thus shows that $b = 0$. Hence B_5 is scalar, and therefore B_2 is scalar.

Remaining in our discussion of B_2 and B_5 is the case in which there is a $B_4 \not\supset B_2$ which is nonsingular. Suppose $B_2 = B_5[i, j|i, j]$ $(i < j)$. Then either B_4 overlaps B_2 in the (i, i) position or in the (j, j) position. Our B_4 satisfies the hypotheses of case 2. Hence B_4 is scalar, say $B_4 = aI_4$ with $a \neq 0$. Suppose B_4 and B_2 overlap in the (j, j) position. (The case in which B_4 and B_2 overlap in the (i, i) position can be obtained from this case by reversing the order of the rows and columns in B_5.) Let the (i, i) entry of B_5 be $a + b$. The equality of the principal 2×2 minors of B_5 shows that the nondiagonal entries in column i of B are $x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_5$ with $x_1^2 = \ldots = x_{i-1}^2 = x_{i+1}^2 = \ldots = x_5^2 = ab$. If $i \geq 3$, det $B_5[i, j|1, j] = ax_1 = -$det $B_5[2, i|1, 2]$. If $i = 2$, det $B_5[1, 5|1, 2] = ax_5 = -$det $B_5[3, 5|2, 3]$. If $i = 1$, det $B_5[2, 3|1, 2] = -ax_3 = -$det $B_5[3, 4|1, 4]$. Therefore we must have one of $x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_5$ zero, and hence $b = 0$. Thus all of $x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_5, b$ are zero and hence B_5 is scalar and therefore B_2 is scalar.

The proof is now finished.

As immediate consequences of Theorem 1 we have Theorems 2 and 3.

THEOREM 2. *If a symmetric completely nonnegative matrix* A *of rank* $\geq \mathrm{r}+2$ *has all* $\mathrm{r} \times \mathrm{r}$ *principal minors equal, and all* $(\mathrm{r}+1) \times (\mathrm{r}+1)$ *principal minors equal, then* A *is scalar.*

THEOREM 3. *A symmetric oscillation matrix cannot have all* $\mathrm{t} \times \mathrm{t}$ *principal minors equal for two consecutive values of* t.

It is not difficult to construct nontrivial examples of 3×3 completely positive matrices having all $t \times t$ principal minors equal, for one value of t. Therefore the hypothesis in Theorem 1 on the principal minors can be weakened only at the price of greatly strengthening the hypothesis concerning the nonprincipal minors.

THEOREM 4. *Let* A *be an* $\mathrm{n} \times \mathrm{n}$ *matrix with elements in a field* $\mathfrak{F}$. *Let* r *be a fixed integer,* $1 \leq \mathrm{r} < \mathrm{n}$. *Suppose:*

(i) *all* $\mathrm{r} \times \mathrm{r}$ *principal minors of* A *are equal;*

(ii) *all* $\mathrm{r} \times \mathrm{r}$ *nonprincipal minors of* A *are equal;*

(iii) *rank* $\mathrm{A} \geq \mathrm{r}+1$ *if* $\mathrm{r} \neq 1$.

Then:

(a) *if* $\mathrm{r} = 1$, $\mathrm{A} = \mathrm{aI_n} + \mathrm{bJ_n}$, *where* $\mathrm{a, b} \in \mathfrak{F}$;

(b) *if* $\mathrm{r} = \mathrm{n}-1$, $\mathrm{A} = \mathrm{D(aI_n + bJ_n)D^{-1}}$, *where* $\mathrm{a, b} \in \mathfrak{F}$, *and* $\mathrm{D} = \operatorname{diag}(-1, 1, -1, 1, \ldots, (-1)^{\mathrm{i}}, \ldots, (-1)^{\mathrm{n}})$;

(c) *if* $1 < \mathrm{r} < \mathrm{n}-1$, $\mathrm{A} = \mathrm{aI_n}$ *is scalar.*

PROOF. First case: $r = 1$. This is trivial.

Second case: $r = n-1$. Here adj DAD^{-1} satisfies the conditions of case 1. Therefore adj $DAD^{-1} = \alpha I_n + \beta J_n$. Thus $(DAD^{-1})^{-1} = \alpha' I_n + \beta' J_n$, and hence $DAD^{-1} = aI_n + bJ_n$. This yields the result in case 2.

General case: The assumptions (i) and (ii) of Theorem 4 are equivalent to

$$C_r(A) = aI_{\binom{n}{r}} + bJ_{\binom{n}{r}}, \text{ where } a, b \in \mathfrak{F}. \tag{4}$$

Since rank $C_r(A) = \binom{\text{rank } A}{r} > 1$ (because rank $A > r$), we see that $a \neq 0$. Let A_{r+1} be any principal $(r+1)$-square submatrix of A. Then $C_r(A_{r+1})$ is a principal submatrix of $C_r(A)$, so that (4) yields

$$C_r(A_{r+1}) = aI_{r+1} + bJ_{r+1}. \tag{5}$$

Let the eigenvalues of A_{r+1} be $\lambda_1, \ldots, \lambda_{r+1}$. These lie, of course, in an extension field of $\mathfrak{F}$. Then, as the eigenvalues of $aI_{r+1}+bJ_{r+1}$ are $a+(r+1)b$ (once), and a (r times), we may choose our notation for the eigenvalues of A_{r+1} so that

$$\lambda_1 \ldots \lambda_r = a+(r+1)b,\ \lambda_2\lambda_3 \ldots \lambda_{r+1} = a = \lambda_1\lambda_3 \ldots \lambda_{r+1}. \tag{6}$$

From (6) we see that all of $\lambda_1, \ldots, \lambda_{r+1}$ are nonzero and hence A_{r+1} is nonsingular.

Since A_{r+1} now satisfies the hypotheses of case 2, we see that A_{r+1} has the form

$$A_{r+1} = D(a_{r+1}I_{r+1}+b_{r+1}J_{r+1})D^{-1}, \tag{7}$$

where a_{r+1} and b_{r+1} are in $\mathfrak{F}$ and $D=\mathrm{diag}\,(-1, 1, -1, \ldots, (-1)^{r+1})$. From the formula (7) for A_{r+1}, it follows that the (i,j) element of A_{r+1} $(i<j<r+1)$ is the negative of the $(i,j+1)$ element of A_{r+1}. Applying this result to any A_{r+1} containing rows and columns i, j, $j+1$ of A, where $1 \leq i<j<n$, we see that the (i,j) element of A is the negative of the $(i,j+1)$ element of A, for $1 \leq i<j<n$. (An A_{r+1} exists containing rows and columns $i, j, j+1$ because $r+1 \geq 3$.) Next notice that

$$b = \det A[1, 2, \ldots, r|1, 2, \ldots, r-2, r+1, r+2], \tag{8}$$

since all nonprincipal r-square minors of A equal b. The last column of the minor in (8) is now known to be the negative of the second last column of this minor (8). Therefore $b=0$.

Hence each $C_r(A_{r+1}) = aI_{r+1}$. From the argument in case 2, it now follows that each A_{r+1} is scalar. Hence A is scalar. The proof is complete.

We remark that this proof can be shortened if $\mathfrak{F}$ is the real or complex number field.

We also remark that Theorem 4 is closely related to the question of the solvability of the matrix equation $C_r(X)=B$, where B is given. Recent results relating to this problem have been found by M. Marcus, M. Newman, A. Yaqub, H. Schwertdtfeger, W. Utz, and less recent results are to be found in papers by C. Ko, H. C. Lee, C. Yen, A. W. Wallace, D. E. Rutherford, A. C. Aitken, J. Williamson.

References

[1] Thompson, R. C., Principal submatrices of normal and Hermitian matrices, Illinois J. Math. **10,** 296–308 (1966).

[2] Thompson, R. C., and McEnteggert, P., Principal submatrices II: The upper and lower quadratic inequalities, J. Linear Algebra and Appl. **1,** 211–243 (1968).

[3] Thompson, R. C., Principal submatrices III: Linear inequalities, J. Res. NBS 72B (Math. Sci.) No. 1, 7–22 (1968).

[4] Thompson, R. C., Principal submatrices IV: On the independence of the eigenvalues of different principal submatrices, submitted to J. Linear Algebra and Appl.

[5] Thompson, R. C., Principal submatrices V: Some results concerning principal submatrices of arbitrary matrices, J. Res. NBS 72B (Math. Sci.) No. 2 (1968).

[6] Thompson, R. C., Principal submatrices VI: Cases of equality in certain linear inequalities submitted to J. Linear Algebra and Appl.

[7] Wedderburn, J. H. M., Lectures on matrices, Amer. Math. Soc. Colloq. Publ. **17,** 63–67 (New York, 1934).

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