Means and the Minimization of Errors*

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Let 0 < a < b. How should a number p be chosen so that the maximum 'relative error' obtained, by replacing a number x varying in the closed interval [a, b], by p, is a minimum? For a large number of 'relative errors,' p must be chosen as the geometric mean of a and b.

Key Words: Arithmetic mean, geometric mean, harmonic mean, means, relative error.

Let 0 < a < b. If x is a number contained in the closed interval [a, b], there are various measures of the "error" committed in replacing x by an approximation p. For example the "absolute error" |p - x|, or the "relative error" \[
\frac{|p-x|}{x}
\] In [1], Huntington suggests more general relative errors of the form \[
\frac{|p-x|}{\phi(p, x)}
\] where \(\phi(p, x)\) is a mean of p and x (that is for all p and x, \(\phi(p, x)\) lies between (not necessarily strictly) p and x). We shall consider errors of this type subject to a few other conditions. If \( E(p, x) = \frac{|p-x|}{\phi(p, x)} \) is given, for each p in \([a, b]\), let \( \lambda(p) = \max_x E(p, x) \) and let \( \mu \) in \([a, b]\) satisfy \( \lambda(\mu) = \min_p \lambda(p) \). The conditions we impose on \( \phi \) besides being a mean are that \( \lambda(p) \) exist and that \( \mu \) exist and be unique. If \( \phi^* \) denotes the 'transpose' of \( \phi(\phi^*(p, x) = \phi(x, p)) \), we will also require that \( \phi^* \) satisfy the same conditions as \( \phi \). In [2], Pólya showed that for \( \phi(p, x) = x, \mu = \frac{2ab}{a+b} \). In this note we compute \( \mu \) for a variety of other means. As a final condition imposed on \( \phi \) and hence on \( E \), we demand that if \( p' \) is strictly between \( p \) and \( x \) then \( E(p', x) \) be strictly smaller than \( E(p, x) \) and that \( E(x, x) = 0 \). A set of sufficient conditions on \( \phi \) to ensure all these requirements are

1. \( \phi \) is a mean.
2. \( \phi \) is continuous on the boundary of the square, \([a, b] \times [a, b]\).
3. \( \phi(t, u) \) and \( \phi(u, t) \) as functions of \( u \) are monotone in each of the intervals \([a, t]\) and \([t, b]\), for each \( t \in [a, b] \).

Under the conditions assumed in paragraph 1 preceding (A),

\[
\lambda(p) = \max (E(p, a), E(p, b)).
\]

and \( \mu \) can be characterized as the unique zero in \([a, b]\) of the equation

\[
E(\mu, a) = E(\mu, b)
\]

or

\[
(\mu - a)\phi(\mu, b) + (\mu - b)\phi(\mu, a) = 0
\]

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1 Figures in brackets indicate the literature references at the end of this paper.
For many choices of \( \phi \) the solution of (3) is completely routine and we merely record the results for some of them. By \( A(u, v) \), \( G(u, v) \), and \( H(u, v) \) we mean the arithmetic mean, geometric mean, and harmonic mean of \( u \) and \( v \), respectively.

<table>
<thead>
<tr>
<th>( \phi(p, x) )</th>
<th>( \mu(a, b) )</th>
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<tbody>
<tr>
<td>( H(a, b) )</td>
<td>( A(a, b) )</td>
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<td>( D(a, b) )</td>
<td>( G(a, b) )</td>
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<td>( A(u, v) )</td>
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We discuss another example in some detail. Let

\[
\phi_\tau(p, x) = A(p, x) + \frac{\tau}{2}(p - x). \tag{4}
\]

For \(-1 \leq \tau \leq 1\), \( \phi_\tau \) is admissible for our discussion. If \( \mu_\tau(a, b) \) is the corresponding value of \( \mu \), (3) becomes

\[
Q_\tau(\mu_\tau) = 0 \tag{5}
\]

where

\[
Q_\tau(u) = (\tau + 1)u^2 - \tau(a + b)u + (\tau - 1)ab, \tag{6}
\]

or equivalently

\[
Q_\tau(u) = \tau(u - a)(u - b) + u^2 - ab. \tag{7}
\]

From (7), \( Q_\tau(a) = a(a - b) < 0 \) and \( Q_\tau(b) = b(b - a) > 0 \). Since \( Q_\tau \) is of degree at most 2, there is a unique zero in the interval \([a, b]\). For \(-1 \leq \tau \leq 1\), this would follow from the general theory, but the argument just presented does not depend on \( \phi_\tau \) being a mean. For all real \( \tau \) we define \( \mu_\tau(a, b) \) as the unique zero of (5) in the interval \([a, b]\).

From (7) it follows that

\[
Q_{\tau+\delta}(u) - Q_\tau(u) = \delta(u - a)(u - b). \tag{8}
\]

Setting \( u = \mu_\tau \), for \( \delta > 0 \), we obtain \( Q_{\tau+\delta}(\mu_\tau) < 0 \). Since \( Q_{\tau+\delta}(b) > 0 \), we obtain \( a < \mu_\tau < \mu_{\tau+\delta} < b \). Hence as a function of \( \tau \), \( \mu_\tau \) is strictly increasing. As \( \tau \to +\infty \), one of the zeros of (5) approaches \( b \) and the other becomes unbounded. By the monotonicity we then have \( \mu_\tau(a, b) \to b \) as \( \tau \to +\infty \). Similarly \( \mu_\tau(a, b) \to a \) as \( \tau \to -\infty \). This motivates the definitions \( \mu_\infty(a, b) = +b \) and \( \mu_{-\infty}(a, b) = a \).

We may uniquely extend \( \mu_\tau \) for all pairs of positive numbers by defining \( \mu_\tau(x, y) = x \) and insisting that \( \mu_\tau(x, y) = \mu_\tau(y, x) \). With these extensions \( \mu \) is continuous in all variables \(-\infty \leq \tau \leq \infty\), \( 0 < x < \infty \), \( 0 < y < \infty \).

It is interesting to compare this one parameter family of means with the standard means \( M_\tau(a, b) \) \cite{3}. For \( \tau = -\infty, -1, 0, 1, \infty \); \( \mu_\tau = M_\tau \), each is homogeneous and symmetric. For \( \tau = 2 \), \( \mu_2(a, b) \leq M_2(a, b) \) for all \((a, b)\). We conjecture that for each \( \tau \), \( \mu_\tau \) and \( M_\tau \) are comparable.

From (6) or (7) we find that if \( x \neq 0 \)

\[
x^2Q_{-\tau}\left(\frac{ab}{x}\right) = -abQ_\tau(x). \tag{8}
\]
Also

\[ a \leq x \leq b \rightarrow a \leq \frac{ab}{x} \leq b. \]  \hspace{1cm} (9)

Hence

\[ \mu_\tau(a, b) = \frac{ab}{\mu_\tau(a, b)}. \]  \hspace{1cm} (10)

For \( \tau = 1 \), this is the elementary property

\[ G(A(a, b), H(a, b)) = G(a, b). \]  \hspace{1cm} (11)

From (4) it follows that \( \phi_\tau = \phi^*_\tau \). This suggests the following generalization of (10). We recall that a mean \( \phi \) is homogeneous if for positive \( x \), \( y \), and \( k \),

\[ \phi(kx, ky) = k\phi(x, y). \]

**THEOREM:** Let \( \phi \) be a homogeneous mean satisfying the conditions (A). Let \( \phi^* \) be the transposed mean. If \( \mu \) and \( \mu^* \) are the corresponding means of \( a \) and \( b \), then

\[ \mu\mu^* = ab. \]

**PROOF:** \( \mu \) is characterized by

(i) \( a < \mu < b \)

(ii) \( (\mu - a)\phi(\mu, b) + (\mu - b)\phi(\mu, a) = 0. \)

\( \mu^* \) is characterized by

(iii) \( a < \mu^* < b \)

(iv) \( (\mu^* - a)\phi^*(\mu^*, b) + (\mu^* - b)\phi^*(\mu^*, a) = 0. \)

Let

\[ P(x) = (x - a)\phi^*(x, b) + (x - b)\phi^*(x, a) \]

\[ P\left(\frac{ab}{\mu}\right) = \left(\frac{ab}{\mu} - a\right)\phi(\frac{ab}{\mu}, b) + \left(\frac{ab}{\mu} - b\right)\phi\left(\frac{ab}{\mu}, a\right). \]

Since \( \phi \) is homogeneous, \( \phi^*\left(\frac{ab}{\mu}, b\right) = \phi\left(b, \frac{ab}{\mu}\right) = \frac{b}{\mu}\phi(\mu, a) \) and \( \phi^*\left(\frac{ab}{\mu}, a\right) = \frac{a}{\mu}\phi(\mu, b) \).

Hence

\[ P\left(\frac{ab}{\mu}\right) = \frac{b}{\mu}\left(\frac{ab}{\mu} - a\right)\phi(\mu, a) + \frac{a}{\mu}\left(\frac{ab}{\mu} - b\right)\phi(\mu, b) \]

\[ = -\frac{1}{\mu^2}ab \{(\mu - b)\phi(\mu, a) + (\mu - a)\phi(\mu, b)\} = 0. \]
Since $a < \mu < b$, $a < \frac{ab}{\mu} < b$.

Hence $\mu^* = \frac{ab}{\mu}$ or $\mu \mu^* = ab$.

**Corollary:** If in addition to the hypotheses of the theorem, $\phi$ is symmetric, ($\phi = \phi^*$), then $\mu = G(a, b)$.

The corollary explains the frequent occurrence of $G(a, b)$ in table 1. Symmetry without homogeneity is not sufficient for $\mu(a, b) = G(a, b)$. For an example let $\phi(p, x) = \frac{p^2 + x^2}{p^2 + x^2 + 1} \max (p, x) + \frac{1}{p^2 + x^2 + 1} \min (p, x)$. By direct substitution in (3) it can be shown that $G(a, b)$ is not $\mu(a, b)$.

**References**


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