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Inductance and Characteristic Impedance of a Strip-Transmission Line

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A general method is developed for determining the inductance and characteristic impedance of uniform transmission lines. A non-uniform current distribution is allowed in the transverse plane. The system is represented by a matrix equation which can be programmed for computer solution. The correct inductance and impedance are obtained as the result of a simple limiting process. The method is applied to one particular geometry, a four-tape stripline system. Results are given for the inductance, resistance, and current distribution as functions of frequency and resistivity for a particular geometry. A method for extending the results to strip lines with proportional dimensions is developed. An accuracy of one part in 10^5 was found to be feasible for the determination of the inductance per unit length.

Key Words: Characteristic impedance, inductance, and strip-transmission lines.

1. Introduction

The determination of the inductance, capacitance, or characteristic impedance of coaxial systems has been the subject of numerous reports. The system considered here is the special case of a strip-transmission line consisting of four vanishingly thin parallel tapes.

The approach used in this paper is to determine the current distribution in a transverse plane and therefore the transverse magnetic field configuration. Knowing the current distribution, the effective inductance per unit length is found directly. The characteristic impedance of a lossless line can then be determined from the inductance.

Almost all previous authors have used the approach of calculating the capacitance per unit length, C, after determining the configuration of the transverse electric field. The result is then used to calculate the characteristic impedance of the lossless line, using the relationship

$$Z_0 = 1/vC \tag{1}$$

where v is the velocity of propagation in the line. Several applications of this approach are listed in the references [1–4].[1] The results of Cohn [2] and Bates [3] give analytic results which are useful design tools.

The approach used in this paper, while nonanalytic, is quite general and without any geometric limitation in the transverse plane. The effect of finite conductor losses and frequency dependence can also be included. It is not necessary to assume any distribution or magnitude for the interdependent variables as this approach determines all necessary information simultaneously.

2. Consideration of Finite Losses

If the line has finite losses, the characteristic impedance and propagation constant, γ, can be calculated from

$$Z_0 = \sqrt{\frac{R + j\omega L}{G + j\omega C}} \tag{2a}$$

$$\gamma \equiv \alpha + j\beta = \sqrt{(R + j\omega L)(G + j\omega C)} \tag{2b}$$

where R and G are the series resistance and shunt conductance per unit length of the line and α and β are the attenuation and phase constant of the line. For a low-loss line, (2a) and (2b) can be closely approximated by

$$Z_0 = vL\left[1 + j\left(\frac{1}{2Q_c} - \frac{1}{2Q_L}\right)\right] \tag{3a}$$

$$\alpha = \frac{R}{2vL} + \frac{GvL}{2} \tag{3b}$$

$$\beta = \omega\sqrt{LC} = \frac{\omega}{v} \tag{3c}$$

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[1] Figures in brackets indicate the literature references at the end of this paper.

where $Q_L \equiv \omega L/R$ and $Q_c \equiv \omega C/G$. Terms involving Q^2 have been neglected. It is also assumed that

$$\frac{\sigma_c}{\omega\epsilon_c} \gg 1, \text{ and } \frac{R_s}{\eta} \ll 1$$

which must be satisfied for transmission line theory to be valid [5] where σ_c and ϵ_c are the conductivity and dielectric constant of the conductor, R_s is the skin-effect surface resistivity of the conductor, and η is the intrinsic impedance of the dielectric.

Both R and L are functions of frequency and resistivity as well as geometry for lines with finite loss. The inductance is therefore a function of resistance so that R must also be calculated to obtain accurate values of L. A computer method for calculating R and L as a function of frequency, resistivity, and geometry is developed in this paper. Values of v and G are readily obtained from properties of the dielectric material in the line. These values together with R and L can be used to obtain the attenuation constant from (3b). If the contribution of Q_L and Q_c in (3a) can be ignored, the characteristic impedance can be obtained from the simple relationship

$$Z_0 = vL.$$

which is analogous to (1).

3. D-C Inductance

Figure 1 shows the cross section of a four-tape stripline. The conductors marked 1 and 2 are to act as a single outer conductor while conductors 3 and 4 act as a single inner conductor in a go-and-return circuit. For conductors with finite resistance the current will be uniformly distributed throughout the outer and inner conductors at zero frequency. The total inductance of the system, $\mathscr{L}$, can be calculated from

$$\mathscr{L} = \frac{1}{4} \sum_{i=1}^{4} \sum_{j=1}^{4} M_{ij} \tag{4}$$

where M_{ij} is the mutual inductance between tape i and j, and M_{ii} is the self inductance of tape i. A recent report by Hoer and Love [6] gives exact equations for the inductances of rectangular conductors carrying a uniform current. Using these equations to calculate the M_{ij}, the exact d-c inductance of any rectangular coaxial system can be obtained.

4. A-C Resistance and Inductance

When an alternating voltage is applied to the system, the current is no longer uniformly distributed throughout the cross section of the conductors. The inductance as well as resistance then differs from the d-c values.

Consider a short length of a long uniform line, such as shown in figure 2. Writing V_2 in terms of V_1 gives

$$V_2 = V_1 \cosh \gamma\Delta l - I_1 Z_0 \sinh \gamma\Delta l. \tag{5}$$

Choose Δl such that $|\gamma\Delta l| \ll |$. Then (5) can be approximated by

$$V_2 = V_1 - I_1 Z_0 \gamma \Delta l$$

or

$$\frac{V_1 - V_2}{\Delta l} = I Z_0 \gamma. \tag{6}$$

Substituting (1) and (2) into (5) gives

$$\frac{\Delta V}{\Delta l} \equiv \frac{V_1 - V_2}{\Delta l} = I(R + j\omega L). \tag{7}$$

Our object now will be to calculate the total current, I, for some arbitrary value of $\Delta V/\Delta l$ and from this result obtain R and L from (7). With R and L known, the transmission line constants may be obtained from (3).

To determine the total current in the stripline, each tape is mathematically divided into a number of smaller, parallel sections as shown in figure 3. An equivalent circuit of the transmission line then looks

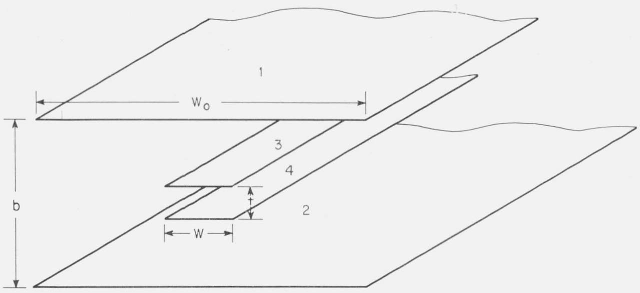

FIGURE 1. *Cross section of a four-tape stripline.*

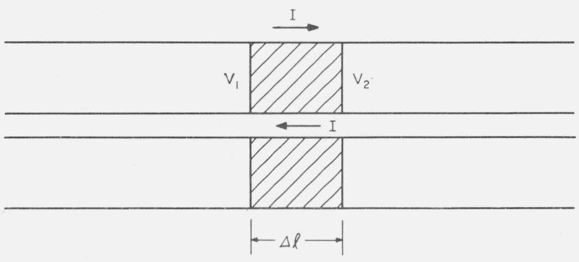

FIGURE 2. *Short length of a long uniform transmission line.*

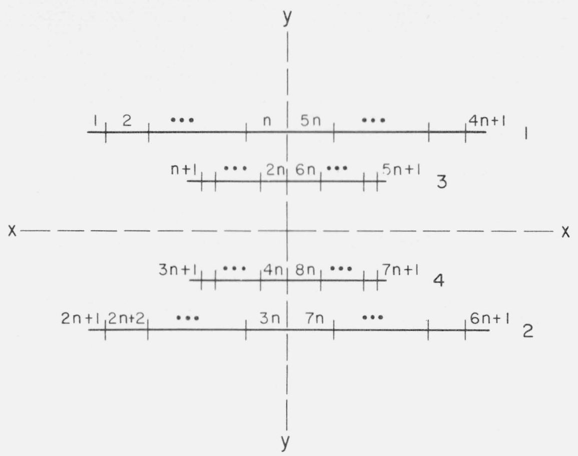

FIGURE 3. *Cross section of the stripline showing how the sections are labeled.*

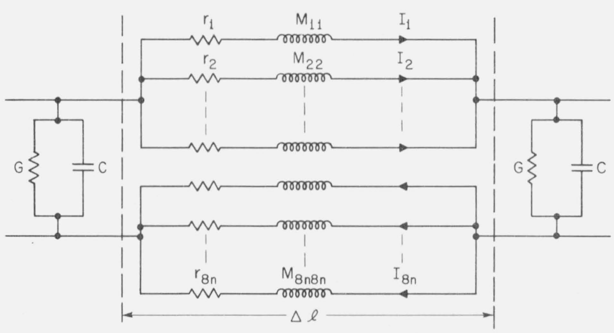

FIGURE 4. *Equivalent circuit for the stripline with the tapes divided into small parallel sections.*

like that in figure 4. The width of each section is chosen small enough so that the current density may be considered uniform throughout that section. The appendix contains dc inductance equations which can be used to calculate the self, and the mutual inductance per unit length of these sections. The resistance per unit length of each section will be the dc resistance.

Although the method of subdivision is almost completely arbitrary, one system will be explained, which, although strange in appearance, will simplify later expressions considerably. Each tape is divided into $2n$ sections labeled in such a way that the index increases as the y axis is approached from any edge as shown in figure 3. The widths of the sections need not be equal and indeed will not be in later calculations. Since the geometry has symmetry about the origin and both axes, this symmetry is retained in subdividing the tapes. Symmetry produces the following relations between the currents in the different sections. For the outer tapes,

$$I_j = I_{2n+j} = I_{4n+j} = I_{6n+j}, \; j = 1, 2, \ldots, n.$$

For the inner tapes,

$$I_{n+j} = I_{3n+j} = I_{5n+j} = I_{7n+j}, \; j = 1, 2, \ldots, n.$$

The total voltage drop per unit length along the length of any one section may be written,

$$V_k = r_k I_k + j\omega \sum_{l=1}^{8n} M_{kl} I_l. \tag{8}$$

For the purposes of this report the d-c resistance per unit length, r_k, of each section will be used in the form of resistivity/area. There are $8n$ equations which may be written in matrix form as

$$\begin{pmatrix} V_1 \\ \cdot \\ \cdot \\ \cdot \\ V_{8n} \end{pmatrix} = \left[\begin{pmatrix} r_1 & & 0 \\ & \ddots & \\ 0 & & r_{8n} \end{pmatrix} + j\omega \begin{pmatrix} M_{1,1} & \cdots & M_{1,8n} \\ \vdots & & \vdots \\ M_{8n,1} & \cdots & M_{8n,8n} \end{pmatrix} \right] \begin{pmatrix} I_1 \\ \cdot \\ \cdot \\ \cdot \\ I_{8n} \end{pmatrix} \tag{9}$$

As a result of the current symmetry conditions, only $2n$ of the equations are independent. The four groups $k = 1$ to $2n$, $2n+1$ to $4n$, $4n+1$ to $6n$, and $6n+1$ to $8n$ are all equivalent.

Choosing the first group, with $k = 1$ to $2n$ as the $2n$ independent equations to be solved, rewrite (9) as

$$\begin{pmatrix} V_1 \\ \cdot \\ \cdot \\ \cdot \\ V_{2n} \end{pmatrix} = \left[\begin{pmatrix} r_1 & & 0 \\ & \ddots & \\ 0 & & r_{2n} \end{pmatrix} + j\omega \begin{pmatrix} \mathcal{M}_{1,1} & \cdots & \mathcal{M}_{1,2n} \\ \vdots & & \vdots \\ \mathcal{M}_{2n,1} & \cdots & \mathcal{M}_{2n,2n} \end{pmatrix} \right] \begin{pmatrix} I_1 \\ \\ \\ I_{2n} \end{pmatrix} \tag{10}$$

where

$$\mathcal{M}_{ij} = \sum_{p=0}^{3} M_{i,2pn+j}, \qquad i, j = 1, 2, \ldots 2n.$$

Equation (10) can now be written in the form

$$V = [R + j\omega\mathcal{M}]I. \tag{11}$$

Since V_k and I_k from (8) are complex, let

$$V_k = e_k + jf_k, \qquad I_k = a_k - jb_k. \tag{12}$$

In terms of column matrices,

$$V = E + jF, \qquad I = A - jB. \tag{13}$$

Substituting (13) into (11) produces two real matrix equations:

$$E = RA + \omega \mathscr{M} B, \tag{14}$$

$$F = -RB + \omega \mathscr{M} A. \tag{15}$$

Solving for A and B produces

$$A = R^{-1}[(1) + (\omega \mathscr{M} R^{-1})^2]^{-1}[E + (\omega \mathscr{M} R^{-1})F] \tag{16}$$

$$B = R^{-1}[(1) + (\omega \mathscr{M} R^{-1})^2]^{-1}[(\omega \mathscr{M} R^{-1})E - F]. \tag{17}$$

In these equations for A and B, E and F have not yet been determined. To simplify their determination the following matrix definitions are used:

$$\psi = R^{-1}[(1) + (\omega \mathscr{M} R^{-1})^2]^{-1}$$

$$\phi = \omega \mathscr{M} R^{-1}$$

$$\tau = (1_1, \ . \ . \ . \ , 1_n, -1_{n+1}, \ . \ . \ . \ , -1_{2n})$$

$$\theta = (0_1, \ . \ . \ . \ , 0_n, 1_{n+1}, \ . \ . \ . \ , 1_{2n}) \tag{18}$$

Since the total current in the outer conductor must equal the total current in the inner conductor, we have

$$\sum_{k=1}^{n} a_k - \sum_{k=n+1}^{2n} a_k \equiv 0$$

$$\sum_{k=1}^{n} b_k - \sum_{k=n+1}^{2n} b_k \equiv 0$$

or in matrix notation, using (16) and the definitions in (18)

$$\tau A = \tau \psi [E + \phi F] \equiv 0 \tag{19}$$

$$\tau B = \tau \psi [\phi E - F] \equiv 0. \tag{20}$$

The scalar voltage drops in the inner sections and outer sections obey the following conditions:

$$\left.\begin{array}{l} f_1 = f_k \\ e_1 = e_k \end{array}\right\} k = 1, 2, \ . \ . \ . \ , n$$

$$\left.\begin{array}{l} f_{n+1} = f_l \\ e_{n+1} = e_l \end{array}\right\} l = n+1, n+2, \ . \ . \ . \ 2n$$

$$e_1 + e_{n+1} + j(f_1 + f_{n+1}) = \Delta v / \Delta l.$$

Since the value of $\Delta v/\Delta l$ is arbitrary, it is set equal to $1 + j0$. This condition applied to the above results produces

$$e_1 + e_{n+1} = 1$$

$$f_1 + f_{n+1} = 0.$$

The column matrices E and F can now be written in terms of known matrices and one scalar unknown each:

$$\begin{aligned} E &= \theta^T + e_1 \tau^T \\ F &= f_1 \tau^T \end{aligned} \tag{21}$$

where the superscript T indicates the transposed matrix. Substituting (21) into (19) and (20) and solving for e_1 and f_1 produces

$$e_1 = -\frac{(\tau\psi\phi\tau^T)(\tau\psi\phi\theta^T) + (\tau\psi\theta^T)(\tau\psi\tau^T)}{(\tau\psi\tau^T)^2 + (\tau\psi\phi\tau^T)^2} \tag{22}$$

and

$$f_1 = \frac{(\tau\psi\phi\theta^T)(\tau\psi\tau^T) - (\tau\psi\theta^T)(\tau\psi\phi\tau^T)}{(\tau\psi\tau^T)^2 + (\tau\psi\phi\tau^T)^2} \tag{23}$$

Every term in parenthesis on the right sides of these equations is a calculable scalar. The result obtained can be substituted into (16) and (17) and the current in each section can be determined. With the currents known, the approximate inductance and resistance per unit length can be calculated from (7), which in terms of the components of the current in each section becomes

$$L_{\mathrm{eff}} = \frac{\dfrac{1}{4\omega} \displaystyle\sum_{k=1}^{n} b_k}{\left(\displaystyle\sum_{k=1}^{n} a_k\right)^2 + \left(\displaystyle\sum_{k=1}^{n} b_k\right)^2} \tag{24}$$

and

$$R_{\mathrm{eff}} = \frac{\dfrac{1}{4} \displaystyle\sum_{k=1}^{n} a_k}{\left(\displaystyle\sum_{k=1}^{n} a_k\right)^2 + \left(\displaystyle\sum_{k=1}^{n} b_k\right)^2}. \tag{25}$$

5. Limiting Value of L_n and R_n

The correct value of inductance and resistance per unit length is obtained in the limit as n becomes infinite. A good estimate of the correct L and R can be obtained from several approximations calculated for different values of n. The approximate values of L and R are plotted against n as shown in figure 5 and 6. The asymptotes to these curves are the values as n becomes infinite.

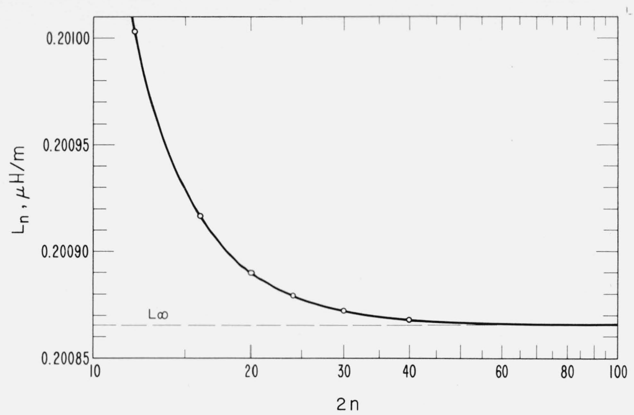

FIGURE 5. *Approximate inductance versus number of subdivisions, showing limiting value of inductance for infinite* n.

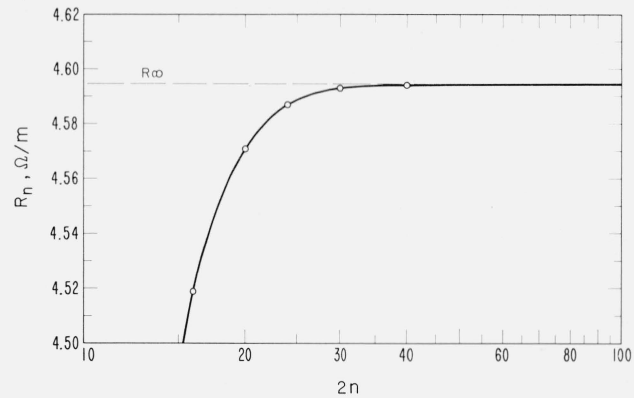

FIGURE 6. *Approximate resistance versus number of subdivision, showing limiting value of resistance for infinite* n.

The curve of L versus n can be approximated by an equation of the form

$$L_n = L\infty + a_n - \mathscr{S} \tag{26}$$

where L_n is the approximate inductance obtained from (24) for a given n, $L\infty$ is the inductance as n becomes infinite, and a and $\mathscr{S}$ are constants. Equation (26) can be solved for $L\infty$ if L_n is calculated for four different n's chosen such that

$$\frac{n_1}{n_2} = \frac{n_3}{n_4}. \tag{27}$$

Then from (26)

$$L\infty = L_{n_4} - \frac{(L_{n_3} - L_{n_4})(L_{n_2} - L_{n_4})}{(L_{n_1} - L_{n_2}) - (L_{n_3} - L_{n_4})}. \tag{28}$$

A plot of $(L_n - L\infty)$ versus n for these 4 values of L_n will be a straight line on log−log paper only if (26) is a valid representation of the curve L_n versus n. Thus, four points can be used to verify (26) as well as to calculate $L\infty$. Once it has been shown that (26) is a valid representation of the curve, $L\infty$ can be calculated from three values of n chosen such that

$$\frac{n_1}{n_2} = \frac{n_2}{n_3}. \tag{29}$$

Then (26) gives

$$L\infty = L_{n_3} - \frac{(L_{n_2} - L_{n_3})^2}{(L_{n_1} - L_{n_2}) - (L_{n_2} - L_{n_3})}. \tag{30}$$

The value of $L\infty$ for the curve shown in figure 5 was calculated from three sets of values of n. For values of n equal to 4, 6, and 9, (30) gave a value of $L\infty = 0.201352$ μH/m. For values of n equal to 9, 12, and 16, (30) gave a value of $L\infty = 0.201344$. For values of n equal 12, 15, 16, and 20, (28) gave a value of $L\infty = 0.201343$. The maximum difference is 5 parts in 10^5. The latter value of $L\infty$ was used to obtain the curve of $(L_n - L\infty)$ versus n shown in figure 7. The time to run these sets of n on a large high speed computer was 10, 35, and 75 seconds respectively.

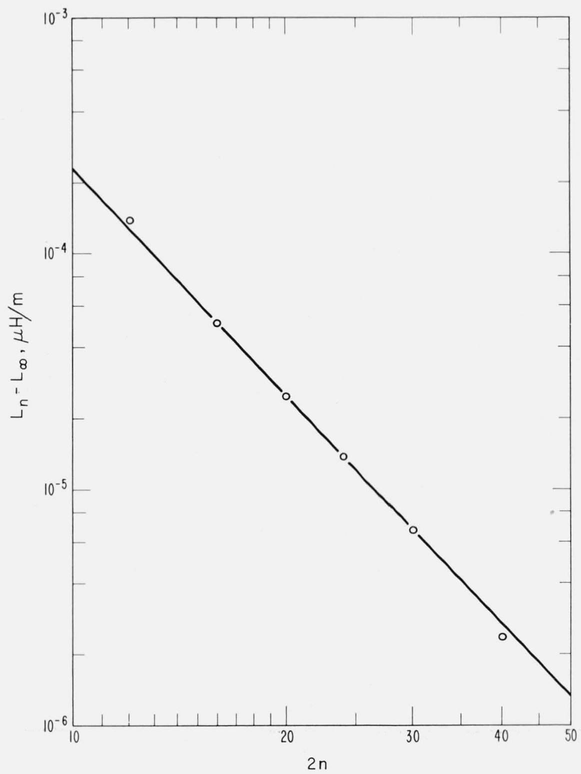

FIGURE 7. *Difference between approximate and limiting value of inductance versus number of subdivisions.*

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No simple approximation was found for the curve of R versus n.

6. Subdividing the Tapes

The exponent, $\mathscr{S}$, in (26) is an indication of the rate of convergence of L_n to its limiting value, $L\infty$. Subdividing the tapes into segments of equal width gave a $\mathscr{S}$ of approximately 1. The variation of the current density across the width of the inner and outer tapes is shown in figure 8. This curve suggests that the width of the segments of the inner tapes be made smaller near the edges where the curve is steepest. One such method of subdivision is to let the width, w_{kn+j} of the $kn+j$ segment be given by,

$$w_{kn+j}=\frac{C}{(n+2-j)^{\eta}},\quad \begin{matrix} k=1,3,5,7. \\ j=1,2,\ldots n \end{matrix} \tag{31}$$

where C is determined by setting the sum of the widths equal to $W/2$. Thus

$$\sum_{j=1}^{n}\frac{C}{(n+2-j)^{\eta}}=\frac{W}{2}$$

or

$$C=\frac{W}{2\sum_{j=1}^{n}\frac{1}{(n+2-j)^{\eta}}}. \tag{32}$$

The constant, η, was chosen such that the rate of convergence was near maximum. With the outer tape divided into segments of equal width and the inner tape divided into segments having widths calculated from (31), an η of 3 was sufficient to yield a $\mathscr{S}$ of approximately 3.

7. Typical Results

If the line has losses, the R and L will be a function of the actual dimensions instead of just the ratios t/b, W/b and W_0/b. The R and L of a stripline having dimensions $b=W=0.5$ cm and $t=0.025$ cm, was calculated as a function of frequency f, resistivity ρ, and width of outer conductor W_0. The thickness, T, of all tapes was assumed to be equal and small enough to permit calculation of the M_{ij} in (10) using inductance equations for zero thickness tapes.[2] Machine storage limitations prevented including the variation of current density with thickness. This limits the strict application of these results to tapes whose thickness is of the order of a skin depth. For the geometry considered here the errors introduced even at high frequencies should be negligible.

[2] See appendix.

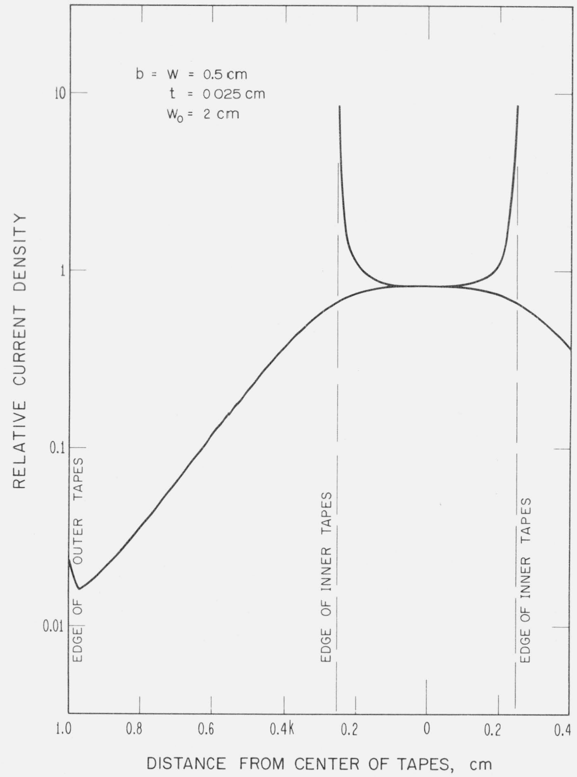

FIGURE 8. *Variation of current density in the inner and outer tapes.*

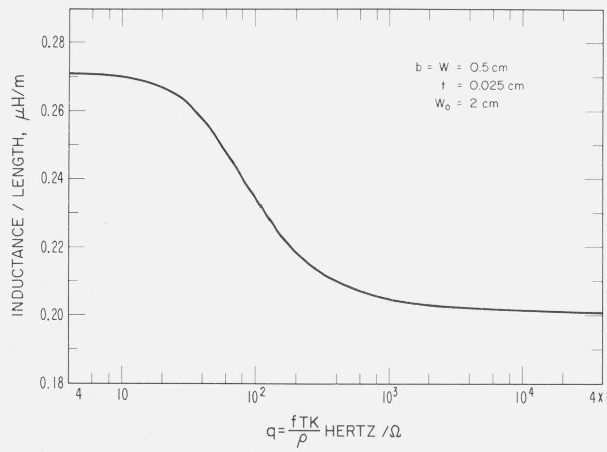

FIGURE 9. *Variation of inductance per unit length with* q.

The variation of inductance per unit length with frequency and resistivity is shown in figure 9. The reason for using the variable $q=fTk/\rho$ is that one curve of L versus q is sufficient for all f, T, k, and ρ, provided that T meets the requirements just discussed. The scale factor k is a dimensionless proportionality constant for striplines having proportionate dimensions and shape. If all dimensions, except the thickness, are multiplied by a scale factor, k, the matrix M in (16) and (17) remains unchanged. The matrix R^{-1} in (16) and (17) is a function of k and may be written

$$R^{-1}=\begin{pmatrix} \frac{Tkw_1}{\rho} & & & 0 \\ & \cdot & & \\ & & \cdot & \\ 0 & & & \frac{Tkw_{2n}}{\rho} \end{pmatrix}=\frac{Tk}{\rho}\mathscr{W} \qquad (33)$$

where

$$\mathscr{W}=\begin{pmatrix} w_1 & & & 0 \\ & \cdot & & \\ & & \cdot & \\ 0 & & & w_{2n} \end{pmatrix}.$$

Equations (16) and (17) can now be written,

$$A=\frac{Tk}{\rho}\mathscr{W}\left[(1)+\left(\frac{\omega Tk}{\rho}\mathscr{M}\mathscr{W}\right)^2\right]^{-1} \cdot\left[E+\left(\frac{\omega Tk}{\rho}\mathscr{M}\mathscr{W}\right)F\right] \qquad (34)$$

$$B=\frac{Tk}{\rho}\mathscr{W}\left[(1)+\left(\frac{\omega Tk}{\rho}\mathscr{M}\mathscr{W}\right)^2\right]^{-1} \cdot\left[\left(\frac{\omega Tk}{\rho}\mathscr{M}\mathscr{W}\right)E-F\right]. \qquad (35)$$

Since the matrices $\mathscr{M}$ and $\mathscr{W}$ are independent of f, T, k, and ρ, one curve of L versus the variable $q=fTk/\rho$ will be sufficient for all f, T, k, and ρ. The same result is true for R/R_{dc} versus q. For the strip line under consideration, with $T=10^{-4}$ cm, $\rho=2\times10^{-6}$ Ω cm, and $k=1$, a q of 5×10^4 hertz/Ω corresponds to a frequency of 10^3 hertz.

A curve of R/R_{dc} for the stripline versus q is shown in figure 10. An explanation of the shape of the curve can be had from an examination of the current density curves such as shown in figures 8 and 11. The current density in the outer tapes remains uniform up to about $q=5$. From $q=5$ to about $q=5000$ the current distribution is changing. Above a q of about 5000 the current distribution in the outer tapes no longer changes. The resistance of the outer tape would therefore remain constant up to a q of 5, increase until a q of 5000 is reached and then remain constant again. The current density in the inner tapes remains uniform up to a q of 50. The current then begins to crowd to

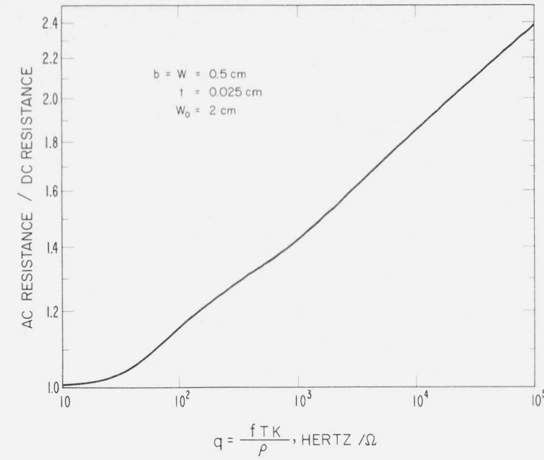

FIGURE 10. *Variation of resistance with* q.

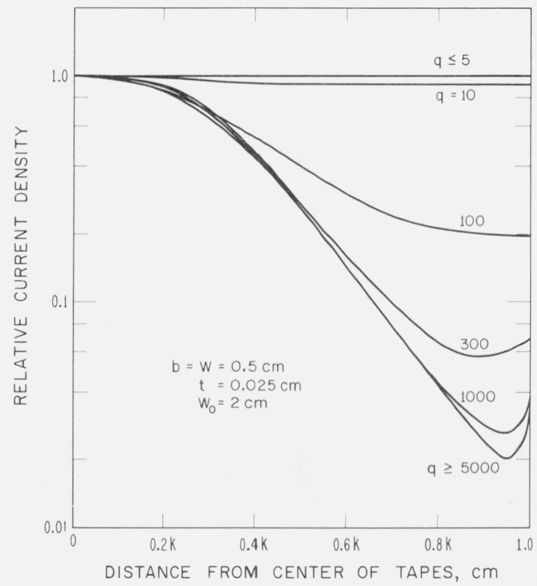

FIGURE 11. *Normalized current density in outer tapes as a function of* q.

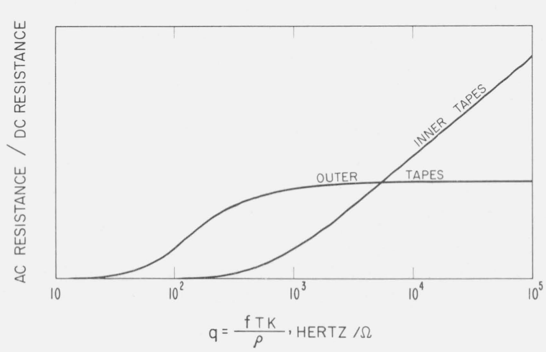

FIGURE 12. *Variation of Resistance with* q *for the outer and inner tapes.*

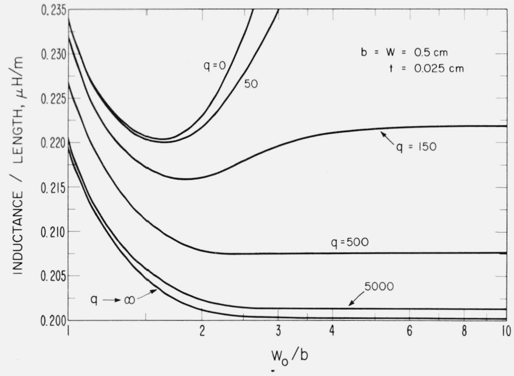

FIGURE 13. *Variation of inductance per unit length with width of outer tapes.*

the edges of the inner tapes. The current continues to move to the edges as long as the value of q increases, never reaching a constant distribution as the current in the outer tape does. The curves of the change in resistance of the outer and the inner tapes with q would look like those shown in figure 12. The sum of these two curves gives a curve like that shown in figure 10.

The variation of inductance per unit length with the width of the outer tapes is shown in figure 13 for different values of q, for the dimensions given above.

8. Summary

The inductance and resistance per unit length of strip transmission line are functions of the current distribution in the conductors. At frequencies where the current distribution throughout the conductors is uniform, the inductance and resistance can be calculated exactly. For frequencies where the current distribution is not uniform, the conductors are divided into sections. From these sections the approximate current distribution of the conductors is calculated. Approximate values of the inductance and resistance per unit length are then calculated from the approximate current distribution. Using high speed computers, it is not difficult to divide the conductors into sufficiently small sections to calculate the inductance to an accuracy of one part in 10^5 or better. The accuracy on resistance is considerably less, depending on frequency. Other methods of subdividing the tapes are being considered to improve the convergence of R_n to its limiting value, R_∞.

The variation of the inductance per unit length with frequency, resistivity, and width of the outer conductors is given for a particular geometry. Tables and graphs of the inductance and resistance per unit length for other geometries are being prepared for future publication.

The technique of calculating R and L as given in this paper for a stripline can be applied to other lines having more complicated cross sections provided that the conductors can be divided into sections whose d-c self and mutual inductances can be calculated. This technique is used in a forthcoming paper by Brooke and Cruz [7] to calculate the L and hence Z_0 of a lossless, rectangular line with a rotating center conductor.

The authors gratefully acknowledge the assistance and helpful criticism of C. M. Allred, and W. E. Little.

9. Appendix

The mutual inductance per unit length between two long, thin, parallel tapes such as shown in figure 14 can be obtained from equation 8 of reference [6] and is

$$\mathscr{M}_t=\frac{0.2}{ad}\left[\frac{P^2-x^2}{4}\ln(P^2+x^2)-xP\,\mathrm{Tan}^{-1}\frac{x}{P}\right]\begin{matrix}E-a,\ E+d\\(x)\\E+d-a,\ E\end{matrix}$$

$$+0.2\left[\ln 2l+\frac{1}{2}\right],\ \mu\mathrm{H/m} \qquad \text{(A1)}$$

where the limits, which have been retained for compactness, are substituted as follows:

$$[f(x)]\begin{matrix}s_1s_3\\(x)\\s_2s_4\end{matrix}\equiv\sum_{i=1}^{4}(-1)^{i+1}f(s_i).$$

The origin coincides with the left edge of one tape. If the left edge of the second tape is in any quadrant other than the first as shown, one or both of the values of E and P will be negative. The self-inductance per unit length of a long thin tape is

$$L_t=0.2\ln\frac{1}{a}+0.2\left[\ln 2l+\frac{1}{2}\right],\ \mu\mathrm{H/m}. \qquad \text{(A2)}$$

In both of these expressions it is assumed that the current density is uniform throughout the conductors.

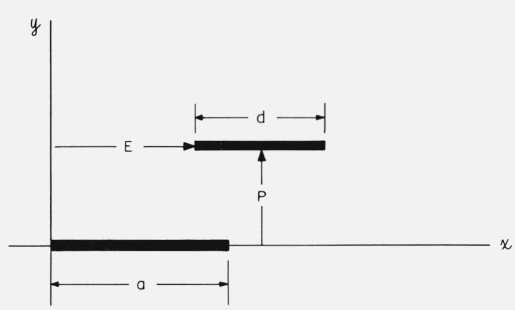

FIGURE 14. *Cross sectional view of two long parallel thin tapes.*

Note that the term

$$0.2\left[\ln 2l+\frac{1}{2}\right] \tag{A3}$$

appears in both M_t and L_t as the only term involving length. However, if M_t and L_t are substituted into (8), the terms involving length will exactly cancel. Substituting (A1) and (A2) into (8) gives

$$V_k = r_k I_k + j\omega \sum_{l=1}^{8n} M'_{kl} I_l + j\omega \left[0.2\left(\ln 2l+\frac{1}{2}\right)\right] \sum_{l=1}^{8n} I_l \tag{A4}$$

where the M'_{kl} are the expressions in (A1) and (A2) with the (A3) term excluded. But the last term is zero because

$$\sum_{l=1}^{8n} I_l \equiv 0.$$

That is, the total current in the outer conductors is equal and opposite to the total current in the inner conductors. Therefore the M_k in (8) may be calculated from (A1) and (A2) with the (A3) term excluded.

10. References

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