Symmetrizable Generalized Inverses of Symmetrizable Matrices*

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The matrix $A$ is said to be symmetrizable by $V$ when $V$ is positive definite and $AV$ is hermitian. Several lemmas regarding symmetrizability are given. For three classes of generalized inverses it is shown that if $A$ is symmetrizable by $V$ there exists a generalized inverse in each class which is symmetrizable by $V$. The Moore-Penrose inverse (or pseudo-inverse) of a matrix symmetrizable by $V$ is also symmetrizable by $V$ if and only if the matrix and the pseudo-inverse commute.

Key Words: Generalized inverse, symmetrizable matrix.

1. Introduction

We call a matrix $A$ symmetrizable if there exists a positive definite $V$ such that $AV$ is hermitian. In that case $A$ is said to be symmetrizable by $V$. Given an $A$ symmetrizable by $V$ we inquire for the existence of generalized inverses of $A$ which are symmetrizable by the same matrix $V$. For several classes of generalized inverses which have been previously discussed [3, 4, 7 and references therein] it is shown that such symmetrizable generalized inverses exist. In particular it is shown that the $C_2$-inverse [3, 4] (also called reflexive generalized inverse [7] or semi-inverse [1]) of a symmetrizable matrix which commutes with the matrix is symmetrizable by the same $V$. Finally it is shown that the Moore-Penrose inverse, $B$, of a matrix $A$ symmetrizable by $V$ is symmetrizable by $V$ if and only if $A$ and $B$ commute.

2. Preliminaries and Notation

All matrices are considered to have complex entries. For any matrix $M$, we denote by $M^*$ and $\rho(M)$ the conjugate transpose and rank of $M$ respectively. We write $Ae^{\mathcal{A}}(V)$ when and only when $V$ is positive definite and $AV$ is hermitian. If $Ae^{\mathcal{A}}(V)$, we say that $A$ is symmetrizable to the right by $V$. We consider only symmetrizability to the right. Since we show,Lemma 1, that $A$ is symmetrizable to the right by $V$ if and only if $A$ is symmetrizable to the left by $V^{-1}$, an analogous set of results could be derived regarding left symmetrizability. For a given matrix $A$ we define $G_i(A)$ to be the set of all matrices $B$ which satisfy the first $i$ of the relations (i) $ABA=A$, (ii) $BAB=B$, (iii) $AB=(AB)^*$ and (iv) $BA=(BA)^*$. A matrix $B\in G_i(A)$ is called a $G_i$-inverse of $A$. The correspondence between this terminology and others which are in use has been noted elsewhere [3, 4]. The set $G_4(A)$ contains a unique determined matrix which is the Moore-Penrose inverse [6] of $A$. If $B\in G_i(A)$, $i < 4$, then $B$ is not uniquely determined by $A$ unless other conditions are imposed. For example, if $B\in G_2(A)$ and commutes with $A$ then $B$ is uniquely determined by $A$ [4].

3. Symmetrizable Matrices

In this section we give several lemmas which are needed in the remainder of the paper.

**LEMMA 1.** Let $A$ be a given matrix, $V$ be positive definite, and $T$ be the positive definite square root of $V$. If any one of the matrices $S_1=AV$, $S_2=V^{-1}A$, and $S_3=V^{-1}AT$ is hermitian, then all are hermitian. There exists a positive definite $H$ such that $H^{-1}AH$ is hermitian, if and only if $A$ is similar to a real diagonal matrix.

**PROOF.** From $T^2=V$ and the definitions of the $S_i$ we have $S_1=VS_2V=TS_3T$ from which it follows at once that if any $S_i$ is hermitian then every $S_i$ is hermitian. If $H^{-1}AH$ is hermitian then $A$ is similar to a hermitian matrix and thus has real roots and is diagonalizable. Conversely, let $P^{-1}AP=A$ where $A$ is real and diagonal. If $P=HQ$ is the polar factorization of $P$, where $H$ is positive definite and $Q$ is unitary, then we have that $H^{-1}AH=Q\Lambda Q^*$ is hermitian.

For ready reference we have included the above simple proof of Lemma 1, but the content of the lemma is known: That $Ae^{\mathcal{A}}(V)$ is equivalent to $Ae^{\mathcal{A}}(V^{-1})$ has been shown [2]. Further if $S_1=S_2^*$,
then $A$ can be written as the product of two hermitian matrices one of which is positive definite. That this is possible if and only if $A$ is similar to a real diagonal matrix is a known theorem [9].

LEMMA 2. If $A \in \mathcal{S}(V)$ then $A^p \in \mathcal{S}(V)$ for every integer $p \neq 0$.

PROOF. $A \in \mathcal{S}(V)$, by Lemma 1, implies

$$T^{-1}AT = S = S^*,$$

where $T^2 = V$ and $T$ is positive definite. But then $T^{-1}AT = S^p$ is hermitian and Lemma 1 then gives $A^p \in \mathcal{S}(V)$.

Lemma 2 has been proved in a much more general context [8] and a slightly different proof has been given elsewhere [2].

LEMMA 3. Let $A \in \mathcal{S}(V)$ and $B \in \mathcal{S}(V)$. Then $AB \in \mathcal{S}(V)$ if and only if $AB = BA$.

PROOF. Let $C = AB$ and $T^{-1}CT = P$. Then

$$(T^{-1}AT)(T^{-1}BT) = P.$$

We choose $T$ to be the positive definite square root of $V$ and then, by Lemma 1, $P$ is the product of two hermitian matrices. Thus $P = P^*$ if and only if $A$ and $B$ commute. But, by Lemma 1, $P = P^*$ if and only if

$$C \in \mathcal{S}(V).$$

LEMMA 4. Let $A \in \mathcal{S}(V)$, $B \in \mathcal{S}(V)$ and define $C_1 = AB$, $C_2 = BA$. If $C_1$ is hermitian, then $C_1$ is similar to $C_2$. If $C_1$ and $C_2$ are hermitian, then $C_1 = C_2$.

PROOF. By Lemma 1 we may write $C_1$ as the product of two hermitian matrices: $C_1 = (AV)(V^{-1}B)$. If $C_1 = C_2^*$ we have $C_1 = V^{-1}BAV = V^{-1}C_2V$, and the first assertion is proved. If additionally $C_2 = C_2^*$, then $C_2 = V^{-1}C_2V = VC_2V^{-1} = C_1^*$ which implies $C_2 = V = C_2$. But then $C_1 = C_2$.

4. Symmetrizable Generalized Inverses

THEOREM 1. Let $A \in \mathcal{S}(V)$. Then there exist matrices $B_i \in \mathcal{S}(V)$, $i = 1, 2, 3$, such that $B_i \in \mathcal{C}_i(A)$.

PROOF. Let $AV = S = S^*$. Then there exists [7] an $H = H^*$ such that $HeC_1(S)$. Given this,

$$SHS = S = AV = AVHAV$$

shows that $B_1 = VHeC_1(A)$. Further $V^{-1}B_1 = H$ implies, by Lemma 1, that $B_1 \in \mathcal{S}(V)$. By a known theorem [3], if $B_2 = B_1AB_1$ then $B_2 \in \mathcal{C}_2(A)$. But

$$V^{-1}B_2 = (V^{-1}B_1)(AV)(V^{-1}B_1) = HSH$$

is hermitian and, by Lemma 1, we have $B_2 \in \mathcal{S}(V)$. Now let $K \in \mathcal{C}_3(S)$. Then [6], $K = K^*$. Further,

$$SKS = S = AV = AVKAV$$

and $SKS = K = KAV$ show that $B_3 = VKeC_3(A)$. Since $SK$ is hermitian and $SK = (AV)(V^{-1}B_3) = AB_3$, we have $B_3 \in \mathcal{C}_3(A)$. Finally $V^{-1}B_3 = K = K^*$ implies, by Lemma 1, that $B_3 \in \mathcal{S}(V)$.

THEOREM 2. Let $A \in \mathcal{S}(V)$. Then there exists a $B \in \mathcal{S}(V)$, uniquely determined by $A$, such that $AB = BA$. Further, $B \in \mathcal{S}(V)$.

PROOF. From Lemma 1, $A \in \mathcal{S}(V)$ implies that $A$ is diagonal and hence that $p(A) = p(A^2)$. Given this condition on the rank of $A$, it follows from a known theorem [4] that there exists a uniquely determined $BeC_2(A)$ which commutes with $A$; furthermore this $B$ is a polynomial $g(A)$ in $A$. From the construction [4] of $B = g(A)$, the coefficients of $g$ are real if the roots of $A$ are real, a condition insured by Lemma 1. This being the case, $A \in \mathcal{S}(V)$, implies, by Lemma 2, that $g(A) \in \mathcal{S}(V)$ and the theorem is proved.

THEOREM 3. Let $A \in \mathcal{S}(V)$ and $B \in \mathcal{S}(V)$ commute. Then $T^{-1}BT \in \mathcal{S}(V)$, where $T$ is the positive definite square root of $V$.

PROOF. By Theorem 2, $A \in \mathcal{S}(V)$ and given this we have, from Lemma 3, that the projection $C = AB = BA$ is such that $C \in \mathcal{S}(V)$. By Lemma 1, $T^{-1}CT$ is a hermitian projection. Since $BeC_2(A)$ is clearly equivalent to $T^{-1}BT \in \mathcal{S}(V)$, we have shown

$$(T^{-1}AT)(T^{-1}BT) = (T^{-1}BT)(T^{-1}AT) = T^{-1}CT$$

to be hermitian.

THEOREM 4. Let $A \in \mathcal{S}(V)$ and $B \in \mathcal{S}(V)$. Then $B \in \mathcal{S}(V)$ if and only if $AB = BA$.

PROOF. Let $B \in \mathcal{S}(V)$. Then from $BeC_4(A)$ we have that $AB$ and $BA$ are hermitian and it follows from Lemma 4 that $AB = BA$. Conversely let $AB = BA$. Then it follows from Theorem 2 that $B \in \mathcal{S}(V)$.

It is known [4, 5] that $BeC_4(A)$ commutes with $A$ if and only if $B$ is a polynomial in $A$, and that $BeC_4(A)$ is a polynomial in $A$ if and only if $A$ is an $EPr$ matrix [5]. We combine these results with Theorem 4 to obtain:

THEOREM 5. Let $A \in \mathcal{S}(V)$ and $BeC_4(A)$. Then the following conditions are equivalent.

(i) $B \in \mathcal{S}(V)$
(ii) $AB = BA$
(iii) $B$ is a polynomial in $A$
(iv) $A$ is an $EPr$ matrix.

5. References