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Partially Isometric Matrices*

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The complex, not necessarily square matrix A is called a partial isometry if the vectors x and Ax have the same Euclidean norm whenever x is in the orthogonal complement of the null space of A. The main results of the paper give necessary and sufficient conditions for a matrix to be a partial isometry, for a partial isometry to be normal and for the product of two partial isometries to be a partial isometry. A factorization for an arbitrary matrix involving partial isometries is given. The concept of a generalized inverse is used in establishing the primary results.

Key Words: Generalized inverse, matrix, partial isometry.

1. Introduction

It is the purpose of this paper to study certain properties and characterizations of partial isometries. In most cases the results are new but in some cases (Lemma 2, Corollary 2 and part of Theorem 5) we give generalizations of results due to Erdelyi [3].[1] No attempt is made to give a complete review or a unified treatment. Emphasis is placed upon those properties and characterizations for which analogous or parallel theorems exist with respect to other "partial properties." In a later paper [8] we define matrices which are partially involutory, partially circular or partially orthogonal and devote some discussion to the comparative anatomy of these four kinds of "partial" matrices (partial isometries, partial involutions, matrices which are partially circular and matrices which are partially orthogonal).

2. Preliminaries and Notation

All matrices considered have complex entries. For any matrix V we denote by $\rho(V)$, $N(V)$, $R(V)$ and V^* the rank, null space, range and conjugate-transpose, respectively of V. For a subspace S we denote by $S^\perp$ the orthogonal complement of S. For any vector x we define the Euclidean norm in the usual way by $\|x\|^2=x^*x$. For generalized inverses we adopt the terminology previously used [5, 6, 7]: a matrix B is called a C_i-inverse of the matrix A if and only if B satisfies the first i of the relations (1) $A=ABA$, (2) $B=BAB$, (3) $AB=(AB)^*$, (4) $BA=(BA)^*$. The C_4-inverse is the Moore-Penrose inverse and is unique. If B is a C_i-inverse of A we sometimes write $B\epsilon C_i(A)$.

3. Partial Isometries

The primary definition of a partial isometry is taken to be: A is a partial isometry if and only if $\|Ax\|=\|x\|$ whenever $x\epsilon N(A)^\perp$. In what follows many of the results deal with equivalent characterizations of a partial isometry.

LEMMA 1. *Of the following statements any one implies each of the others: (i) A is a partial isometry, (ii) A*A is idempotent, (iii) AA* is idempotent, (iv) A=AA*A, (v) A*=A*AA*.*

PROOF. It is no restriction to assume that A has n columns and $\rho(A)=r$. Define $E=A^*A$ and let x_i, $1\leq i\leq r$ be orthonormal vectors such that $Ex_i=\lambda_i x_i$, $\lambda_i\neq 0$. The x_i form a basis for $N(A)^\perp$. If E is idempotent, $\lambda_i=1$, $x_i^*Ex_i=x_i^*x_i$ and A is a partial isometry. Conversely, let A be a partial isometry. Then,

$$x_i^*Ex_i=\lambda_i(x_i^*x_i)=x_i^*x_i,$$

which requires $\lambda_i=1$, and E is idempotent. Thus (i)$\leftrightarrow$(ii). By a known theorem [5] the matrix B is a C_1-inverse of A if and only if BA is idempotent and $\rho(BA)=\rho(A)$ and if and only if AB is idempotent and $\rho(AB)=\rho(A)$. By this theorem, (ii) is equivalent to (iii) and to (iv), and clearly (iv) is equivalent to (v).

Lemma 1 is well known [3, 4] and can be summarized by the statement: A is a partial isometry if and only if A^* is the C_4-inverse of A. We present Lemma 1 for ready reference and to show clearly that $A^*\epsilon C_1(A)$ characterizes a partial isometry and that, since $\rho(A)=\rho(A^*)$ and A^*A and AA^* are hermitian, $A^*\epsilon C_1(A)$ implies $A^*\epsilon C_4(A)$. Later [8], we study matrices such that $T(A)\epsilon C_1(A)$, where $T(A)$ is a matrix valued function of A, and draw analogies between such matrices and partial isometries ($T(A)=A^*$).

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[1] Figures in brackets indicate the literature references at the end of this paper.

The following lemma generalizes a theorem of Erdelyi [3] which states that if A is a partial isometry then every matrix unitarily similar to A is a partial isometry.

Lemma 2. *The matrix* A *is a partial isometry if and only if* T_1AT_2 *is a partial isometry, where* T_1 *and* T_2 *are unitary.*

Proof: Let $P = T_1AT_2$, where T_1 and T_2 are any unitary matrices such that the product P is defined. Then $PP^* = T_1AA^*T_1^*$. If A is a partial isometry then, by Lemma 1, AA^* is idempotent, hence PP^* is idempotent and, by Lemma 1, P is a partial isometry. By this very result, if P is a partial isometry then

$$A = T_1^*PT_2^*$$

is a partial isometry.

Theorem 1. *Let* M *be the* $m \times n$ *matrix with ones in the first* r *diagonal positions and zero entries elsewhere. Then an* $m \times n$ *matrix,* A, *of rank* r *is a partial isometry if and only if* $A = UMV$ *where* U *and* V *are unitary.*

Proof: Let $A = UMV$. Since M^*M is idempotent, M is, by Lemma 1, a partial isometry and, by Lemma 2, A is a partial isometry. Let A be $m \times n$ and $\rho(A) = r$. Then by a known theorem [2] we can write $A = U\Gamma V$. Here U and V are unitary, the first r diagonal elements of Γ are the nonzero roots of A^*A and all other entries of Γ are zero. If A is a partial isometry then, by Lemma 1, A^*A is idempotent, the first r diagonal elements of Γ are unity and $\Gamma = M$.

The following corollary of Theorem 1 has been given direct proof elsewhere [6].

Corollary 1. *The square matrix* A *is a partial isometry if and only if* $A = QE$ *and if and only if* $A = FQ$ *where* Q *is unitary,* E *and* F *are orthogonal projections.*

Proof: If the partial isometry $A = UMV$ is square, we can write

$$A = (UV)(V^*MV) = QE = (UMU^*)(UV) = FQ,$$

where $Q = UV$ is unitary, $E = V^*MV$ and $F = UMU^*$ are orthogonal projections. Conversely if $A = QE$ then $A^*A = E$ and, by Lemma 1, A is a partial isometry. Similarly if $A = FQ$, then $AA^* = F$ implies A is a partial isometry.

Theorem 2. *If* A *is an* $m \times n$ *partial isometry of rank* r, *then* $A = U_rV$, *where* U_r *is an* $m \times n$ *matrix the first* r *columns of which are an orthonormal set of vectors while the remaining columns are zero vectors, and* V *is unitary. Conversely every matrix of the form* U_rV *with* U_r *and* V *as described is a partial isometry.*

Proof: If A is a partial isometry of rank r we have from Theorem 1,

$$A = (UM)V = [u_1, u_2, \ldots, u_r, 0, \ldots, 0]V,$$

where u_i, $1 \leq i \leq r$, are the first r columns of U. Conversely, given orthonormal vectors u_i, $1 \leq i \leq r$, in m-space, the $m \times n$ matrix

$$U_r = [u_1, u_2, \ldots, u_r, 0, \ldots, 0]$$

is by Lemma 1, a partial isometry, since $U_r^*U_r$ is idempotent. By Lemma 2, U_rV is a partial isometry if V is unitary.

Corollary 2. *A square matrix of rank* r *is a partial isometry if and only if it is unitarily similar to a matrix the first* r *columns of which are an orthonormal set of vectors and the remaining columns of which are zero vectors.*

Proof: By Theorem 2, the square partial isometry A of rank r can be written $A = U_rV$ and we have $VAV^* = VU_r = [q_1, q_2, \ldots, q_r, 0, \ldots, 0]$ where $q_i = Vu_i$, $1 \leq i \leq r$. Conversely, given orthonormal vectors q_i, $1 \leq i \leq r$, the square matrix

$$[q_1, q_2, \ldots, q_r, 0, \ldots, 0]$$

is a partial isometry and by Lemma 2 so is any matrix unitarily similar to it.

Remarks: There are evidently "row versions" of Theorem 2 and Corollary 2. Further, in Theorem 2 and Corollary 2, the non-zero columns of U_r can be placed in any r positions by replacing V by PP^*V where P is a permutation matrix. Corollary 2 was given as a theorem by Erdelyi [3]. A more direct proof is: By Corollary 1, $A = QE$. Let $T^*ET = \mathrm{diag}(I_r, 0)$, T unitary. Then from $T^*AT = (T^*QT)\ \mathrm{diag}\ (I_r, 0)$ the 'only if' statement is read. The 'if' statement is proved as above.

The following theorem may be viewed as a generalization of the theorem [1, 4, 6], which we obtain as a corollary below, that a square matrix A can be factored as $A = PH$ where P is a partial isometry and H is positive semidefinite.

Theorem 3. *Every matrix of rank* r *can be factored as* $A = P_1\Gamma V = U\Gamma P_2 = P_1\Gamma P_2$ *where* U *and* V *are unitary,* P_1 *and* P_2 *are partial isometries,* Γ *has positive entries on the first* r *diagonal positions and zero entries elsewhere.*

Proof: We factor A as $A = U\Gamma V$ where U, V and Γ are as in the proof of theorem 1. Let L be the C_4-inverse of Γ. Then $\Gamma L\Gamma = \Gamma$ and $E_1 = \Gamma L$ and $E_2 = L\Gamma$ are orthogonal projections. Further

$$\Gamma = E_1\Gamma = \Gamma E_2 = E_1\Gamma E_2.$$

Thus we have

$$A = (UE_1)\Gamma V = U\Gamma(E_2V) = (UE_1)\Gamma(E_2V).$$

But by Corollary 1, $P_1 = UE_1$ and $P_2 = E_2V$ are partial isometries.

Corollary 3. *The square matrix* A *can be factored as* $A = PH$ *where* P *is a partial isometry and* H *is positive semidefinite.*

Proof: From Theorem 3, if $A = P_1\Gamma V$ is square we have $A = (P_1V)(V^*\Gamma V)$ where $H = V^*\Gamma V$ is positive semidefinite and, by Lemma 2, $P = P_1V$ is a partial isometry.

Theorem 4: *Let* A *be a partial isometry of rank* r. *Then* A *is unitarily similar to diag* (C, 0), *where* C *is* r*-square and nonsingular, if and only if* A *is normal. In that case* C *is unitary.*

PROOF: Let $A = P^*$ diag $(C, 0)P$, where P is unitary. Then, by Lemma 1, $AA^* = P^*$ diag $(CC^*, 0)P$ is idempotent and hence CC^* is idempotent. But then, since C is nonsingular, C is unitary and A is normal. Conversely, let A be a normal partial isometry of rank r. Then there is a unitary P such that $A = P^*$ diag $(C, 0)P$, where C is r-square of rank r. Then as above C is unitary.

COROLLARY 4: *Let* A *be a partial isometry of rank* r. *Then* A *is an* EPr *matrix (i.e.,* N(A) = N(A*)) *if and only if* A *is normal.*

PROOF: By a known theorem [9], A is an EPr matrix if and only if A is unitarily similar to diag $(C, 0)$, where C is r-square and nonsingular. But, according to Theorem 4, this is the case for a partial isometry if and only if A is normal.

REMARK: Corollary 4 can be obtained in a different way and without reference to Theorem 4. It is known [11] that a matrix A commutes with its C_4-inverse if and only if A is an EPr matrix. If A is a partial isometry, then by Lemma 1 and the remarks following that lemma we have $A^* \epsilon C_4(A)$. Thus if A is an EPr partial isometry, A and A^* commute and A is normal. Conversely every (complex) normal matrix is an EPr matrix.

The next lemma is a special case of a generalization [8] of the statement [10] that an isometry is an involution if and only if it is hermitian. We prove it here from a different point of view as a step toward Lemma 4 which is repeatedly used in the proof of Theorem 5.

LEMMA 3. *A partial isometry is a projection if and only if it is an orthogonal projection.*

PROOF: By a known theorem [4], if E is a projection then $\|Ex\| \leq \|x\|$, for all x, if and only if $E = E^*$. With this in mind we observe from Theorem 1 that if A is a partial isometry then $\|Ax\| = \|UMVx\| = \|MVx\| \leq \|x\|$, since M^*M is an orthogonal projection, and the lemma follows.

LEMMA 4: *Let* E *and* F *be orthogonal projections. Then* C = FE *is a partial isometry if and only if* C *is an orthogonal projection.*

PROOF: Let C be a partial isometry, then

$$C^*C = EFE = EC \text{ and } C = CC^*C = CEC = C^2,$$

where the first equality follows from Lemma 1 and the last from $CE = C$. But now C is both a partial isometry and a projection and by Lemma 3, is an orthogonal projection. Conversely, by Lemma 1, every orthogonal projection is a partial isometry.

THEOREM 5: *Let* W *and* V *be partial isometries and* P = WV. *Then* P *is a partial isometry if and only if* W*W *and* VV* *commute. If either* W *and* VV* *or* V *and* W*W *commute then* P *is a partial isometry.*

PROOF: If both W and V are square, the proof of the first statement in the theorem is quite direct and we give this special case first: By Corollary 1, write $W = QE$ and $V = FU$, E and F orthogonal projections, Q and U unitary. Then $P = QEFU$. If P is a partial isometry then, by Lemma 2, EF is a partial isometry and is thus, by Lemma 4 an orthogonal projection. Thus $EF = FE$, but $E = W^*W$ and $F = VV^*$. Conversely, if $EF = FE$ then EF is a partial isometry and so, by Lemma 2, is $P = QEFU$. Now let W and V be arbitrary except that WV is defined, and still define $E = W^*W$ and $F = VV^*$. Then $PP^*P = WFEV$. If $EF = FE$, we have $PP^*P = (WE)(FV) = WV = P$, since by Lemma 1, $WE = W$ and $FV = V$, and then, by Lemma 1, P is a partial isometry. Now suppose P is a partial isometry. Then by Lemma 1, $PP^*P = P$, or $WFEV = WV$, from which it follows (left multiplication by W^* and right multiplication by V^*E) that $EFEFE = EFE = (EFE)(EFE)$. Thus EFE is a projection. But $EFE = (FE)^*(FE)$ and, by Lemma 1, FE is a partial isometry. By Lemma 4, FE is an orthogonal projection and hence $EF = FE$. As for the second assertion of the theorem, suppose W and F commute. It then follows that E and F commute and by what we have proved so far $WF = FW$ is sufficient for P to be a partial isometry. Similarly if V and E commute it follows that E and F commute and $VE = EV$ is sufficient for P to be a partial isometry.

The last statement of Theorem 5 is a theorem of Erdelyi [3]. It assumes that either W or V is square. The above proof was based on the first statement in the theorem and implications

$$W(VV^*) = (VV^*)W \rightarrow (W^*W)(VV^*) = (VV^*)(W^*W)$$

and

$$V(W^*W) = (W^*W)V \rightarrow (W^*W)(VV^*) = (VV^*)(W^*W).$$

(Note that neither reverse implication is necessarily true.) A greatly shortened direct proof of Erdelyi's theorem is this: With W, V, E and F defined as in Theorem 5 and its proof, we observe that if W and F commute then F commutes with both E and WW^*. If so, then $PP^* = WFW^* = F(WW^*) = (WW^*)F$ shows PP^* to be idempotent and by Lemma 1, P is a partial isometry. In the same way if V and E commute then E commutes with both F and VV^* so that

$$P^*P = V^*EV = (V^*V)E = E(VV^*),$$

P^*P is idempotent, and P is a partial isometry.

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