Bounds for the Solutions of Second-Order Linear Difference Equations

F. W. J. Olver

Institute for Basic Standards, National Bureau of Standards, Washington, D.C. 20234

(August 22, 1967)

Simple bounds are established for the solutions of second-order homogeneous linear difference equations in ranges in which the solutions are exponential in character. The results are applied to a recent algorithm for the computation of subdominant solutions of second-order linear difference equations, homogeneous or otherwise. Strict and extremely realistic bounds are obtained for the truncation error associated with the algorithm in a number of examples, including Anger-Weber functions, Struve functions, and the solution of a differential equation in Chebyshev series.

Key Words: Chebyshev series, difference equations, error bounds, Miller algorithm, recurrence relations, special functions.

1. Introduction

Consider the difference equation

\[ p_{r+1} - b p_r + p_{r-1} = 0, \]  
(1.01)

in which \( b \) is a real constant exceeding 2. The solution may be expressed as

\[ p_r = \frac{p_1 - \lambda_2 p_0}{\lambda_1 - \lambda_2} \lambda_1^r + \frac{\lambda_1 p_0 - p_1}{\lambda_1 - \lambda_2} \lambda_2^r, \]

where \( p_0 \) and \( p_1 \) are at our disposal, and

\[ \lambda_1 = \frac{1}{2} b + \sqrt{\frac{1}{4} b^2 - 1}, \quad \lambda_2 = \frac{1}{2} b - \sqrt{\frac{1}{4} b^2 - 1}. \]

Eventually \( |p_r| \) grows in proportion to \( \lambda_1^r \) as \( r \to \infty \), unless \( p_1 = \lambda_2 p_0 \). The larger the value of \( b \), the faster is the rate of growth.

Now consider the equation

\[ p_{r+1} - b p_r + p_{r-1} = 0, \]  
(1.02)

in which \( b_1, b_2, b_3, \ldots \) is a given sequence of real numbers, not necessarily monotonic, each of which exceeds the \( b \) of eq (1.01). If the same values of \( p_0 \) and \( p_1 \) are used for the two difference equations, then it is reasonable to expect—and not difficult to prove—that the solution of (1.02) generally increases at a faster rate than that of (1.01). Similarly, if, for every \( r, b_r < B \), where \( B \) is another constant, then we expect the solution of (1.02) to grow at a slower rate than the solution of

\[ p_{r+1} - B p_r + p_{r-1} = 0 \]  
(1.03)

having the same initial values.

The first purpose of the present paper is to generalize these intuitive results and present them as precise theorems. We shall allow the coefficient of \( p_{r-1} \) in (1.02) to take values other than unity, and we shall not insist that the solutions of difference equations under comparison agree at both \( r = 0 \) and \( r = 1 \). Some related results for the tails of continued fractions have been given by Blanch [1] and Merkes [2].

The second purpose of this paper is to apply these theorems to a recent algorithm for computing subdominant solutions of homogeneous or inhomogeneous second-order difference equations [3], with a view to obtaining bounds for the truncation error.

2. Comparison Theorem

Theorem 1. Let

\[ q_{r+1} = \beta_r q_r - \alpha_r q_{r-1}, \]  
(2.01)
\[ Q_{r+1} = B_r Q_r - A_r Q_{r-1}, \]  
(2.02)

where \( \{\alpha_r\}, \{\beta_r\}, \{A_r\}, \{B_r\}, r = 1, 2, \ldots \), are sequences of numbers such that

\[ \frac{B_r}{A_r} < \frac{1}{2} \quad \text{for all } r \]
If 
\[ Q_1 - q_1 \geq Q_0 - q_0 \geq 0, \text{ and } q_1 \geq \max(q_0, 0), \] (2.04)
then \( q_r \) and \( Q_r \) are nondecreasing functions of \( r \), and
\[ q_r \leq Q_r, \ r = 0, 1, \ldots \]

To establish this result, we begin with the identity
\[ q_{r+1} - q_r = (\beta_r - \alpha_r - 1) q_r + \alpha_r (q_r - q_{r-1}), \] (2.05)
and observe that both \( \beta_r - \alpha_r - 1 \) and \( \alpha_r \) are nonnegative. From (2.04) we see that \( q_1 \leq Q_1 \). Hence from (2.05), with \( r = 1 \), it follows that \( q_2 - q_1 \geq 0 \). Continuing this argument by induction, we see that \( q_{r+1} \geq q_r, \ r = 0, 1, \ldots \). Similarly from the identity
\[ Q_{r+1} - Q_r = (B_r - A_r - 1) Q_r + A_r (Q_r - Q_{r-1}), \] (2.06)
we deduce that \( \{Q_r\} \) is a nondecreasing sequence.

Next, subtraction of (2.05) and (2.06) yields
\[
(Q_{r+1} - Q_r) - (q_{r+1} - q_r) = (\beta_r - \alpha_r - 1) (q_r - q_{r-1}) + \alpha_r (Q_r - q_{r-1}) - (A_r - \alpha_r) (Q_r - Q_{r-1}).
\]

Assume that \( r \geq 1 \) and \( Q_r - q_r \geq Q_{r-1} - q_{r-1} \geq 0 \) as is certainly the case when \( r = 1 \) in consequence of (2.04). Then using (2.03) and the facts that \( Q_r \geq 0 \) and \( Q_r - Q_{r-1} \geq 0 \), we deduce that \( (Q_{r+1} - Q_r) - (q_{r+1} - q_r) \geq 0 \). Hence by induction \( Q_r \geq q_r \). This completes the proof.

3. Bounds for the Solutions When the \( a_r \) Are Nonnegative

**Theorem 2.** Let
\[ p_{r+1} = b p_r - a p_{r-1}, \] (3.01)
where \( \{a_r\} \) and \( \{b_r\}, \ r = 1, 2, \ldots \), are sequences of numbers such that
\[ b_r \geq 1 + a_r \geq 1, \] (3.02)
and suppose that
\[ p_0 > 0, \quad \kappa = p_1 / p_0 \geq 1. \] (3.03)

Then \( p_r \) is a nondecreasing function of \( r \), and
\[ p_0 \{\min(\kappa, \Lambda)\}^r \leq p_r \leq p_0 \{\max(\kappa, \Lambda)\}^r \] (r = 0, 1, \ldots). (3.04)

Here \( \Lambda, \Lambda \) are the largest roots of the equations
\[ \lambda^2 - b \lambda + a = 0, \quad \Lambda^2 - B \Lambda + A = 0, \] (3.05)
respectively, and
\[ a = \inf_{r \geq 1} a_r, \quad b = a + \inf_{r \geq 1} (b_r - a_r), \] (3.06)
\[ A = \sup_{r \geq 1} a_r, \quad B = A + \sup_{r \geq 1} (b_r - a_r). \] (3.07)

Before entering into the proof of this theorem, we observe that the numbers \( b \) and \( a \) always exist and are subject to the inequalities
\[ b \geq 1 + a \geq 1. \] (3.08)

Moreover,
\[ \lambda = \frac{1}{2} b + \sqrt{\frac{1}{4} b^2 - a} \geq \frac{1}{2} a + \frac{1}{2} + \frac{1}{2} a - \frac{1}{2} \geq \max(a, 1). \] (3.09)

On the other hand, either or both of the sequences \( \{a_r\} \) and \( \{b_r - a_r\} \) may be unbounded, causing either \( B \) or both \( B \) and \( A \) to be infinite; in this event \( \Lambda \) is undefined and the second of the inequalities (3.04) is inapplicable. Except in these cases, we have \( a \).

That \( p_r \) is a nondecreasing function of \( r \) is an immediate consequence of (3.02), (3.03), and Theorem 1. We shall establish the inequalities (3.04) in turn by further use of this theorem.

Suppose first that \( \kappa > \Lambda \). Define \( q_r = p_0 \lambda^r \). Then \( q_0 = p_0, q_1 = p_0 \lambda \) and, in consequence of (3.05),
\[ q_{r+1} = b q_r - a q_{r-1} \] (r \geq 1). (3.11)

We apply Theorem 1, with (3.11) filling the role of (2.01), and (3.01) that of (2.02). Conditions (2.03) are satisfied in consequence of the definitions of \( a \) and \( b \), and (2.04) is satisfied since \( \kappa \geq \Lambda \). Therefore \( p_r \geq q_r = p_0 \lambda^r \), in agreement with the first of (3.04).

Now consider the case when \( \kappa < \Lambda \). Define \( q_r = p_0 \lambda^r \). Then \( q_0 = p_0, q_1 = p_1, \) and
\[ q_{r+1} = b q_r - a q_{r-1} \] (r \geq 1), (3.12)
where \( \tilde{a} = \min(a, \kappa) \) and \( \tilde{b} = \kappa + \tilde{a} \kappa^{-1} \). The desired result \( p_r \geq q_r \) will follow if Theorem 1 can be applied to eqs (3.12) and (3.01). Conditions (2.04) are clearly satisfied; the remaining conditions are
\[ a_r \geq \tilde{a} \geq 0, \quad b_r - a_r \geq \tilde{b} - \tilde{a} \geq 1. \]

The first two inequalities follow from the definitions of \( a \) and \( \tilde{a} \). And since
\[ \tilde{b} - \tilde{a} - 1 = (\kappa - 1) (1 - \tilde{a} \kappa^{-1}) \geq 0, \quad b - a \leq b_r - a_r, \]

The last of (3.10) may be verified by expressing \( 2x - a + (b^2 - 4b + 3) x^{-1} \), where \( c = \inf(b_r - a_r) \geq 1 \), showing that \( \alpha \lambda / b \) and \( \alpha \lambda / c \) are both positive.
it will suffice to show that

\[ b - a \geq \tilde{b} - \tilde{a}. \quad (3.13) \]

If \( \tilde{a} = \kappa \), then \( \tilde{b} = \kappa + 1 \) and (3.13) follows immediately from (3.08). If, on the other hand, \( \tilde{a} = a < \kappa \), then

\[ (b - a) - (\tilde{b} - \tilde{a}) = b - \kappa - a\kappa^{-1} \]

\[ = \lambda + \frac{a}{\lambda} - \left( \frac{\kappa + a}{\kappa} \right) \]

\[ = (\lambda - \kappa) \left( 1 - \frac{a}{\lambda \kappa} \right) > 0, \]

since \( a < \kappa \) and \( 1 \leq \kappa < \lambda \). This completes the proof of the left-hand inequality in (3.04).

To establish the right-hand inequality in (3.04), suppose first that \( \kappa \leq \Lambda \). Defining \( Q_r = p_0 \Lambda^r \), we have \( Q_0 = p_0, Q_1 = p_0 \Lambda \), and

\[ Q_{r+1} = BQ_r - A Q_{r-1} \quad (r \geq 1). \quad (3.14) \]

It is readily verified that Theorem 1 is applicable to eqs (3.01) and (3.14). Hence \( p_r \leq p_0 \Lambda^r \), in agreement with (3.04).

Now consider the case when \( \kappa > \Lambda \). Define \( Q_r = p_0 \kappa^r \), so that \( Q_0 = p_0, Q_1 = p_1 \), and

\[ Q_{r+1} = BQ_r - A Q_{r-1} \quad (r \geq 1), \quad (3.15) \]

where \( \tilde{B} = \kappa + A \kappa^{-1} \). The desired result \( p_r \leq Q_r \) will follow immediately if Theorem 1 is applicable to (3.01) and (3.15). Conditions (2.04) are obviously satisfied. Conditions (2.03) demand that

\[ A \geq a_r \geq 0, \quad \tilde{B} - A \geq b_r - a_r \geq 1. \]

In consequence of (3.02) and (3.07), these inequalities will follow if we can show that \( \tilde{B} \geq B \). Now

\[ \tilde{B} - B = \kappa + \frac{A}{\kappa} - \left( \Lambda + \frac{A}{\Lambda} \right) = (\kappa - \Lambda) \left( 1 - \frac{A}{\Lambda \kappa} \right). \]

This is nonnegative because \( \kappa > \Lambda \), \( \kappa \geq 1 \), and \( \Lambda \geq A \); compare (3.10). The proof of Theorem 2 is now complete.

4. Bounds for the Solutions When the \( a_r \) Are Nonpositive

If the quantities \( a_r \) are negative then Theorem 2 is inapplicable. The analysis of this case is somewhat easier, however, and Theorem 1 is not needed. For convenience, we replace \( a_r \) by \(-a_r\).

**Theorem 3. Let**

\[ p_{r+1} = b_r p_r + a_r p_{r-1}, \quad (4.01) \]

where

\[ a_r \geq 0, \quad b_r \geq 0, \quad p_0 > 0, \quad \kappa = p_1/p_0 \geq 1. \quad (4.02) \]

Then

\[ p_0 \{ \min (\kappa, \lambda) \}^r \leq p_r \leq p_0 \{ \max (\kappa, \lambda) \}^r \]

\[ (r = 0, 1, \ldots), \quad (4.03) \]

where \( \lambda, \Lambda \) are the largest roots of the equations

\[ \lambda^2 - b \lambda - a = 0, \quad \Lambda^2 - B \Lambda - A = 0, \quad (4.04) \]

respectively, and

\[ a = \inf_{r \geq 1} a_r, \quad b = \inf_{r \geq 1} b_r, \quad (4.05) \]

\[ A = \sup_{r \geq 1} a_r, \quad B = \sup_{r \geq 1} b_r. \quad (4.06) \]

To prove the left-hand inequality in (4.03) when \( \kappa \geq \Lambda \), assume that \( p_r \geq p_0 \lambda^r \) and \( p_{r-1} \geq p_0 \lambda^{r-1} \), as is the case when \( r = 1 \). Then from (4.01) we have

\[ p_{r+1} \geq b p_0 \lambda^r + a p_0 \lambda^{r-1} = p_0 \lambda^{r+1}, \]

as required.

Now consider the case when \( \kappa < \Lambda \). The roots of the quadratic equation \( x^2 - bx - a = 0 \) are \( \lambda \) and \(-a/\lambda\). Since \(-a/\lambda < \kappa < \lambda\), it follows that

\[ \kappa^2 - b \kappa - a < 0. \quad (4.07) \]

Assume that \( p_r \geq p_0 \kappa^r \) and \( p_{r-1} \geq p_0 \kappa^{r-1} \) as is the case when \( r = 1 \). Then from (4.01), (4.05), and (4.07), we derive

\[ p_{r+1} \geq b p_0 \kappa^r + a p_0 \kappa^{r-1} > p_0 \kappa^{r+1}. \]

The right-hand inequality in (4.03) may be established in a similar way.

5. Anger-Weber Functions

In the remaining part of this paper we show how the preceding theorems may be applied to the truncation errors associated with the algorithm of [3].

Consider first the computation of the Anger-Weber function \( E_r(z) \) given in [3], section 6, Example 1. The truncation error of the approximation \( y_{14}^{14} \) is given by

\[ [3], \text{section 5}. \]
where

\[ E_{14} = \sum_{s=14}^{\infty} \frac{e_s}{p_sp_{s+1}}, \]

(5.02)

and

\[ p_r \text{ satisfies (3.01) above with } a_r = 1, \ b_r = 2r, \ p_0 = 0, \]

\[ p_1 = 1, \text{ and} \]

\[ e_r = e_{r-1} - d_r p_r, \quad d_r = -\frac{2}{\sqrt{\pi}}r(1 - (-1)^r), \]

(5.03)

with \( e_0 = E_0(1) \).

We apply Theorem 2 of section 3 to the sequence \( \{p_{r+1}\} \), using the numerical entries given in [3], table 1. Working to five significant figures (for the sake of illustration), we have

\[ \kappa = p_{15}/p_{14} = 1.1127 \times 10^{15}/(3.9793 \times 10^{15}) = 27.962, \]

\[ a = 1, \quad b = 1 + \inf_{r \geq 15} (2r - 1) = 30. \]

Hence

\[ \lambda^2 - 30\lambda + 1 = 0, \quad \lambda = 29.967 > \kappa. \]

Therefore

\[ p_r \geq p_{14} \kappa^{r-14} \quad (r \geq 14). \]

(5.04)

Next, from (5.03) we have

\[ e_s = e_{14} + \frac{(4/\pi)(p_{15} + p_{17} + \ldots + p_s)}{p_{s+1}} \quad (s > 14), \]

(5.05)

where \( s' = s \) if \( s \) is odd, and \( s' = s - 1 \) if \( s \) is even. Since \( p_s \) is a nondecreasing function of \( s \) (Theorem 2), we have

\[ e_s \leq e_{14} + \frac{(4/\pi)(s-14)p_s}{p_{s+1}} \quad (s \geq 14). \]

(5.06)

Application of this inequality to (5.02) gives

\[ E_{14} \leq e_{14} \sum_{s=14}^{\infty} \frac{1}{p_{s+1}} + \frac{4}{\pi} \left( \frac{1}{p_{16}} + \frac{2}{p_{17}} + \frac{3}{p_{18}} + \ldots \right). \]

(5.07)

Substituting (5.04) on the right-hand side of (5.07), summing the resulting series and using numerical entries given in [3], table 1, we obtain

\[ E_{14} \leq \frac{e_{14}}{p_{14}^2 (\kappa - 1)} + \frac{4}{\pi p_{14} (\kappa - 1)^2} = 8.824 \times 10^{-17}. \]

(5.08)

The actual value of \( E_{14} \) was estimated in [3] to be

\[ 8.248 \times 10^{-17} \]

by direct numerical summation of the terms in the expansion (5.02). Our strict bound (5.08) exceeds this estimate by about 7 percent, which is very satisfactory in view of the somewhat crude step leading from (5.05) to (5.06).

Strict bounds for the errors \( \epsilon_{14}^{(14)}(r \geq 14) \) may be computed by substituting the numerical value (5.08) for \( E_{14} \) in (5.01) and using the values of \( p_r \) given in [3], table 1. Obviously they, too, will overestimate the actual truncation errors by about 7 percent.

6. Struve Functions

As a second example, consider the Struve function \( H_{\alpha}(0.1) \) computed in [3], section 6, Example 2. The relevant formulas in this case are

\[ e_r = e_{15} = \frac{(0.05)r}{\sqrt{\pi} \Gamma \left( r + \frac{3}{2} \right)} p_r, \]

(6.01)

and

\[ p_{r+1} = 20rp_r - p_{r-1}, \quad e_r = e_{r-1} - \frac{(0.05)^r}{\sqrt{\pi} \Gamma \left( r + \frac{3}{2} \right)} p_r. \]

(6.02)

Applying Theorem 2 to the sequence \( \{p_{r+1}\} \) using the numerical entries in [3], table 2, we have

\[ \kappa = p_{16}/p_{15} = 299.99643 \ldots, \]

\[ a = 1, \quad b = 1 + \inf_{r \geq 16} (20r - 1) = 320, \]

\[ \lambda = 160 + \sqrt{25599} \geq \kappa. \]

Hence

\[ p_r \geq p_{15} \kappa^{r-15} \quad (r \geq 15). \]

(6.03)

Inspection of table 2 of [3] suggests that

\[ |e_r| \leq e_{15} \quad (r \geq 15). \]

(6.04)

Deferring the proof of this inequality for the moment and substituting (6.03) and (6.04) in the second of (6.01), we derive

\[ |E_{15}| \leq \frac{e_{15}}{p_{15}^2 (\kappa - 1)} = 0.23277 \times 966 \ldots \times 10^{-62}. \]

(6.05)

The actual value of \( E_{15} \) may be estimated by computation of \( p_{17}, p_{18}, \ldots \) and direct summation of the expansion in (6.01). This gives \( 0.23277 \times 943 \ldots \times 10^{-62} \). Therefore strict bounds for \( \epsilon_{15}^{(15)}(r \leq 15) \) computed from (6.05) and the first of (6.01) will exceed
the estimated errors by less than one part in \(10^6\), a striking example of the power of Theorem 2.

It remains to establish (6.04). Write

\[
u_r = (0.05)p_r \left(\sqrt{\pi r} \left(r + \frac{3}{2}\right)\right),
\]
so that

\[e_r = e_{15} - u_{16} - a - \ldots - u_r \quad (r \geq 16).
\]

The \(p_r\) and \(u_r\) are all positive, in consequence of (6.03). From the first of (6.02) we see that

\[p_{r+1} < 20rp_r.
\]

Accordingly, we derive

\[u_{r+1} < 2ru_r/(2r+3),
\]
and thence

\[
\sum_{r=16}^{\infty} u_r < u_{15} \left(\frac{30}{33} + \frac{30}{33} \frac{32}{35} + \frac{30}{33} \frac{32}{35} \frac{34}{37} + \ldots\right) = 30u_{15},
\]
the last relation being a consequence of Gauss’ formula for the hypergeometric function \(F(15, 1; 16; \cdot; 1)\); see [4], eq (15.1.20). Since

\[30u_{15} = 0.01418 \ 23625 \ldots
\]
the stated result now follows.

7. Clenshaw’s Numerical Method for Ordinary Differential Equations

As a final illustration, we consider a recurrence relation arising in the solution of a differential equation by the method of Clenshaw; compare Example 4 of [3], section 11. Suppose first that we are interested in the solution of the equation

\[(2r-1)f_{r-1} - 12rf_r + (2r+1)f_{r+1} = 0 \quad (7.01)
\]
satisfying the conditions \(f_0 = 1\) and \(f_r \to 0\) as \(r \to \infty\). Approximations to \(f_r\) are provided by the function \(f_r^{(7)}\) given to six decimal places in table 4 of [3] and to ten decimal places in table 5 of the same reference. Let us seek bounds for the magnitude of the truncation error

\[\phi_r^{(7)} = f_r - f_r^{(7)}. \quad (7.02)
\]
As in the other examples, we have

\[\phi_r^{(7)} = E_7p_r, \quad E_7 = \sum_{s=7}^{\infty} e_s / p_sp_{s+1}. \quad (7.03)
\]

In the present case \(p_r\) satisfies the difference eq (7.01), and \(e_r = 1/(2r+1)\). Numerical values of \(p_r\) are given

in table 4 of [3] for \(r=0, 1, \ldots, 8\). Applying Theorem 2 to the sequence \(\{p_r\}\), we have

\[\kappa = ps/p_7 = 5.4389, \quad (7.04)
\]
correct to five significant figures, and

\[a = \inf_{r \geq 8} \frac{2r-1}{2r+1} = \frac{15}{17} \quad b = \frac{15}{17} + \inf_{r \geq 8} \frac{10r+1}{2r+1} = \frac{96}{17}.
\]

Accordingly,

\[17\lambda^2 - 96\lambda + 15 = 0, \quad \lambda = 5.4862 > \kappa.
\]
Hence

\[p_r \geq p_7\kappa^{r-7} \quad (r \geq 7). \quad (7.05)
\]
Substitution of the last inequality in the second of (7.03) yields

\[E_7 \leq \frac{1}{p_7^2} \left(\frac{1}{15\kappa} + \frac{1}{17\kappa^3} + \frac{1}{19\kappa^5} + \ldots\right)
\]
\[< \frac{1}{15p_7^2 (\kappa - \kappa^{-1})} = 0.67713 \times 10^{-10}. \quad (7.06)
\]
The corresponding bounds for the errors \(\phi_r^{(7)}\) are given in the second column of table A below, in units of the tenth decimal place. They exceed only slightly the estimates for the errors obtained by subtracting the entries in the columns headed \(f_r^{(12)}\) and \(f_r^{(7)}\) in table 5 of [3], and given in the third column of table A.

<table>
<thead>
<tr>
<th>Table A</th>
</tr>
</thead>
<tbody>
<tr>
<td>(r)</td>
</tr>
<tr>
<td>---</td>
</tr>
<tr>
<td>0</td>
</tr>
<tr>
<td>1</td>
</tr>
<tr>
<td>2</td>
</tr>
<tr>
<td>3</td>
</tr>
<tr>
<td>4</td>
</tr>
<tr>
<td>5</td>
</tr>
<tr>
<td>6</td>
</tr>
<tr>
<td>7</td>
</tr>
</tbody>
</table>

Secondly, suppose that we are interested in the solution of (7.01), \(y_r\), say, which satisfies the condition

\[y_0 = y_1 = y_2 = y_3 + \ldots = 1. \quad (7.07)
\]
(This is the form in which the example was originally proposed in [3].) Then \(y_r\) is related to \(f_r\), defined above, by

\[y_r = f_r/F, \quad F = \frac{1}{2} f_0 + f_1 + f_2 + \ldots. \quad (7.08)
\]
We seek bounds for the errors

\[e_r^{(7)} = y_r - y_r^{(7)} \quad (r = 0, 1, \ldots, 7). \quad (7.09)
\]
\[ y^{(7)}_r = f^{(7)}_r / F_7, \quad F_7 = \frac{1}{2} f_0^{(7)} + f_1^{(7)} + f_2^{(7)} + \ldots + f_6^{(7)}. \]  
\[ \sigma_7 \leq 13403 \times 10^{-10}; \]

Using (7.02) and (7.08), we find that
\[ \epsilon^{(7)}_r = \epsilon^{(7-1)}_r - \epsilon^{(7-2)}_r, \]  
where
\[ \epsilon^{(7-1)}_r = \frac{\phi^{(7)}_r}{F_7 + \sigma_7}, \quad \epsilon^{(7-2)}_r = \frac{\sigma_7 f^{(7)}_r}{F_7 (F_7 + \sigma_7)}. \]
and
\[ \sigma_7 = \frac{1}{2} \phi^{(7)}_0 + \phi^{(7)}_1 + \phi^{(7)}_2 + \ldots + \phi^{(7)}_7 + f_8 + f_9 + f_{10} + \ldots. \]  

In order to assess an upper bound for \( \sigma_7 \), we need an upper bound for
\[ f_r = p_r \sum_{s=r}^{\infty} \frac{1}{(2s+1)p_s p_{s+1}} \]  
when \( r \geq 8 \); compare [3], eq (5.03). To construct this, we make a second application of Theorem 2 to the sequence \( \{ p_{r+7} \} \). We have
\[ A = \sup_{r \geq 8} \frac{2r-1}{2r+1} = 1, \quad B = 1 + \sup_{r \geq 8} \frac{10r+1}{2r+1} = 6. \]
Accordingly
\[ \Lambda^2 - 6\Lambda + 1 = 0, \quad \Lambda = 3 + \sqrt{8} = 5.8284 > \kappa, \]
where \( \kappa \) is defined by (7.04). Hence
\[ p_r \leq p_7 \Lambda^{-r-7} \quad (r \geq 7). \]
Substituting this result and (7.05) in (7.14), we find that
\[ f_r \leq \frac{\Lambda^{-r} \sum_{s=r}^{\infty} \frac{1}{(2s+1)\kappa^{2s-13}}}{p_7 (2r+1)(\kappa - \kappa^{-1})\kappa^{2r-14} (r \geq 7)}, \]
and hence
\[ \sum_{r=8}^{\infty} f_r < \frac{\Lambda}{17p_7(\kappa - \kappa^{-1})(\kappa^2 - \Lambda)} = 2007 \times 10^{-10}, \]
correct to ten decimal places. Adding this value to \( 10^{-10} \) times the sum of the entries in the column of table A headed 0.67713\( p_r \), we derive

\[ \epsilon^{(7-1)}_r \leq \frac{\phi^{(7)}_r}{F_7} \leq 0.11303 \times 10^{-9}, \quad \epsilon^{(7-2)}_r \leq \frac{\sigma_7 f^{(7)}_r}{F_7^2} \leq 3735 \times 10^{-9}, \]
both \( \epsilon^{(7-1)}_r \) and \( \epsilon^{(7-2)}_r \) obviously being nonnegative. The values of \( 0.11303p_r \) and \( -3735f^{(7)}_r \) are given in the fourth and fifth columns of table A. Comparison with the estimated values of \( 10^9 \epsilon^{(7)}_r \), extracted from table 5 of [3] and given in the final column, is again fully satisfactory.

### 8. Summary

In the first part of this paper (secs. 1–4) simple lower and upper bounds are established for the solutions of second-order homogeneous linear difference equations, in ranges in which the solutions are exponential in character.

In the second part (secs. 5–7), the results are applied to an algorithm for the computation of subdominant solutions of second-order difference equations which was introduced recently by the writer. It is shown by means of examples how to compute strict and extremely realistic bounds for the truncation error associated with the algorithm. The bounds depend only on simple properties of the coefficients in the difference equation, and are independent of asymptotic theories of the solutions.

The writer is grateful to C. W. Clenshaw and G. F. Miller for valuable criticisms of an earlier form of this paper.

### 9. References