Polar Factorization of a Matrix*

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(April 24, 1967)

It is known that if $A$ is a bounded linear operator with closed range on a Hilbert space then $A$ can be factored as $A = UH$, with $U$ a partial isometry and $H$ nonnegative and self adjoint. For the finite-dimensional case a strictly matrix-theoretic derivation is given based on the concept of a generalized inverse. Certain properties of the factors are given as well as conditions under which $H$ or both $U$ and $H$ are uniquely determined by $A$. A pivotal item in the derivation is the representation of a square partial isometry as the product of a unitary matrix and an orthogonal projection. This representation is new, of some interest in itself and greatly simplifies the derivations.

Key Words: Generalized inverse, matrix, partial isometry.

1. Introduction

It is well known but not well discussed in the matrix literature\(^1\) that a square matrix, $A$, can be factored as $A = UH$ where $U$ is a partial isometry and $H$ is positive semidefinite. The primary purpose of this paper is to give a fairly direct demonstration of this factorization, and the main result is thus not new (see footnote 1). The demonstration, as well as the deduction of certain properties of the factors, is based on a characterization of partial isometries which is new and of some interest per se. While not strictly necessary it is possible and illuminating to cast part of the development in terms of generalized inverses of singular matrices.

2. Notation and Preliminaries

In what follows all matrices are considered to have complex entries. We denote by $\rho(A)$, $R(A)$, $N(A)$ and $A^*$ rank, range, null space and conjugate transpose, respectively, of any given matrix. When $A$ is nonsingular, $A^{-1}$ denotes the inverse. For generalized inverses a special terminology is used. This terminology, previously introduced and related to others \([5, 6]\)\(^2\) is as follows: For a given matrix $A$ denote by $C_1(A)$ the set of all matrices $B$ such that $ABA = A$. Then $C_3(A)$ is defined as the set of all matrices $B$ such that $BeC_1(A)$ and $AeC_1(B)$; $C_4(A)$ is the set of all matrices $B$ such that $BeC_4(A)$ and $AB$ is hermitian; finally $C_4(A)$ is the set of all matrices $B$ such that $BeC_4(A)$ and $BA$ is hermitian. We note that the set $C_4(A)$ contains a single uniquely determined matrix which is the Moore-Penrose generalized inverse \([7]\). We call a matrix $BeC_1(A)$ a $C_1$-inverse of $A$. The relation between a $C_1$-inverse, as here defined, and the “weak generalized inverse” of Goldman and Zelen \([3]\) has been noted elsewhere \([5]\). Repeated use will be made of the following fact: If $BeC_1(A)$ then $\rho(B) \geq \rho(A) = \rho(AB) = \rho(BA)$, with strict equality if and only if $BeC_1(A)$ \([5, 9]\).

We call a matrix $A$ a partial isometry if there exists a subspace, $S$, such that $x^*A^*Ax = x^*x$, when $x \in S$, and $Ax = 0$, when $x \in S^\perp$, where $S^\perp$ is the orthogonal complement of $S$. This definition is equivalent to the requirement that $A^*A$ be an orthogonal projection \([2, 3, p. 150]\).

3. The Polar Factorization

We begin with the following two lemmas

**Lemma 1.** The square matrix, $A$, is a partial isometry if and only if $A = QE$, where $Q$ is an isometry and $E$ is an orthogonal projection.

**Proof.** If $A = QE$, with $Q^*Q = I$ and $E = E^2 = E^*$, we have $A^*A = E$ and $A$ is a partial isometry. Let $A = QH$ be the usual polar factorization of $A$, where $Q$ is unitary and $H$ is positive semidefinite. If $A$ is a partial isometry then $A^*A = H^2$ is hermitian and idempotent. If so then $H$, the positive semidefinite square root of $H^2$, is also hermitian idempotent.

**Remark.** It is an obvious consequence of Lemma 1 that $A$ is a partial isometry if and only if $A = FQ$ where $F$ is an orthogonal projection and $Q$ is unitary. For, from $A = QE$ we have $A = QE(Q^*Q)$, and we identify $F$ with the orthogonal projection $QEQ^*$. Conversely $A = FQ = QF^*FQ = QE$.

**Lemma 2.** Let $A$ be normal and $BeC_1(A)$. Then if $E = AB$ is normal, $E$ is uniquely determined by $A$, and $EA = AE = A$.

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*An invited paper.
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\(^1\) The factorization, with condition for both factors to be unique, cf. Theorem 2 to follow, is given as a problem in [4, p. 171]. The factorization is well-known as a result for bounded operators with closed range on a Hilbert space [1, 8].
\(^2\) Figures in brackets indicate the literature references at the end of this paper.
Proof. From \( ABA = EA \) it follows that \( Ax = \lambda x \)
implies \( Ex = x \), provided \( \lambda \neq 0 \). Let \( \rho(A) = r \). Then there are
linearly independent \( x_i \) such that \( Ex_i = x_i \), \( 1 \leq i \leq r \), and since \( \rho(E) = \rho(A) \) we have \( R(E) = R(A) \).
Since \( E \) is a normal projection, it is an orthogonal projection and thus
uniquely determined by its range and hence by \( A \). Further, since \( E \) and \( A \) are normal, \( EA = A \)
shows that \( N(E) = N(A) \). Hence \( E \) and \( A \) have a complete
set of eigenvectors in common and must commute.

Theorem 1. Let \( A \) be any n-square matrix. Then there
exists a partial isometry \( U \) and a positive semidefinite matrix \( H \), such that

\[
\begin{align*}
(i) & \quad A = UH, \\
(ii) & \quad U^*A = H, \\
(iii) & \quad N(U) = N(H) = N(A), \\
(iv) & \quad U \text{ maps all of } n\text{-space onto } R(A) \\
(v) & \quad UH = HU \text{ if and only if } A \text{ is normal, and in this case } U \text{ is normal.}
\end{align*}
\]

Proof. Let \( A = QH \) be the usual polar factorization of \( A \), where \( Q \) is unitary and \( H \) is positive semidefinite. Let \( P \) be any \( C_1 \)-inverse of \( H \) such that \( E = HP \)
is an orthogonal projection. Then \( HPH = EH = H \)
and we have \( A = QH = QEH = UH \), where \( U = QE \) is,
by Lemma 1, a partial isometry. Thus (i) is proved. Now \( U^*U = E \)
and hence, from (i), \( U^*A = EH = H \)
which is (ii). It is clear that \( N(U) = N(E) \) and that \( N(A) = N(H) \); and since \( \rho(E) = \rho(H) \), \( EH = H \) shows that \( N(E) = N(H) \). Thus \( N(U) = N(E) = N(H) = N(A) \),
which gives (iii). Given \( A \), the projection \( E = HP \) is,
by Lemma 2, uniquely determined. If \( P \) is chosen to be non-
singular, as is plainly possible (see after (2) below), then \( A = UH = UEP^{-1} = UP^{-1} \),
and (iv) is evident. Suppose \( A \) to be normal. Then \( A = QH = HP \)
and from this and \( EH = HE = H \), which we have from 
Lemma 2 (but which in this case is obvious from \( EH = H \) since \( E \) and \( H \) are hermitian), it follows that \( EA = AE = A \). But then \( AE = A = HQE = HU = UH \). Conversely, suppose \( UH = HU \). We have at once that \( AE = EA = A \), which shows that \( N(A^*) = N(E) = N(A) \).
Given this, and \( N(H) = N(E) \), we have from \( A^* = H \)
that \( QyE \in N(E) \) whenever \( y \in N(E) \). We can now assert that \( HQy = HQy = 0 \),
when \( y \in N(E) \). Finally, \( HU = HQE = UH = QH \) implies that \( HQx = QHE = HQ = EQH \).
We have proved that \( HQ = EQH \) and hence that \( A \) is normal. Given this, from \( A = QH = QEH = EQH = EQH \),
we have \( QEx = EQx \) when \( x \in R(E) \), and we have seen that \( QyE \in N(H) \) when \( y \in N(E) \). Thus \( QE = EQ \) and \( U \) is normal.

Theorem 2. Let \( A = UH \), where \( U \) is a partial
isometry and \( H \) is positive semidefinite. Consider the
conditions: (i) \( U^*A = H \), (ii) \( \rho(U) = \rho(H) \), (iii) \( N(U) = N(H) \).
Then, if (i) holds, \( H \) is uniquely determined; (iii) holds
if and only if (i) and (ii) hold, and in that case both \( U \)
and \( H \) are uniquely determined.

Proof. By Lemma 1, we may replace \( U \) by \( QE \) with
\( Q \) unitary and \( E \) an orthogonal projection. Then, if (i)
holds, \( U^*A = EH = H \). This being so, we have \( A = UH = QEH = QH \), and \( HE = A^*A \). Thus \( H \) is the unique
positive semidefinite square root of \( A^*A \). We next
show that (iii) is equivalent to (i) and (ii) together.
Let (i) and (ii) hold. Then, with \( U = QE \), (i) gives \( U^*A = EH = H \), which with (ii) implies (iii). Let (iii) hold.

We obviously then have (iii). Further, with \( U = QE \),
(iii) states \( N(E) = N(H) \). Let \( x_1, x_2, \ldots, x_r \) be
any orthonormal basis of \( N(E) = N(H) \). Then from \( E = I - \sum x_i x_i^* \), we have \( HE = HE = H \). This being the case, \( U^*A = EH = H \) which is (i). Now let \( A = U_1H_1 = U_2H_2 \)
be any two factorizations of \( A \) and assume (iii). Since \( (iii) \) implies (i), \( H_1 = H_2 = H \) and we have \( U_1H_1 = U_2H_2 \)
which implies \( U_1x = U_2x \), when \( x \in R(E) \).
But (iii) now also requires \( N(U_1) = N(U_2) = N(H) \) and hence \( U_1y = U_2y \) for \( y \in N(H) \). Thus \( U_1 = U_2 \).

If \( H \) is hermitian, then \( H = T \operatorname{diag}(\Lambda, 0)T^* \),
where \( T \) is unitary, \( \Lambda \) is real, diagonal and nonsingular.
In the following discussion let this unitary similarity via \( T \)
be denoted by \( H \sim \operatorname{diag}(\Lambda, 0) \). Then for arbitrary \( K \), \( L \), and \( D \) of appropriate sizes and shapes any \( P \)
such that \( P \sim \begin{bmatrix} \Lambda^{-1} & K \\ L & D \end{bmatrix} \)
is a \( C_1 \)-inverse of \( H \). For, from
\[\begin{cases} E = HP \sim \begin{bmatrix} I & \Lambda K \\ 0 & 0 \end{bmatrix} \end{cases}\]
we have that \( HP \) is idempotent and has the rank of \( H \)
and this is known [5] to be necessary and sufficient for \( P \mathcal{E} (H) \).
Now \( E \), in (2), is hermitian if and only if \( K = 0 \). Thus given \( K = 0 \), any \( P \) as in (1) will serve in the
proof of (i), (ii), (iii) and (v) of Theorem 1, and any \( P \)
as in (1) with \( D \) nonsingular will serve in the proof
of (iv) of Theorem 1. Now we could, in the proof of
Theorem 1 except for (iv), forthwith have taken \( P \mathcal{E} \mathcal{E} (H) \)
or \( P \mathcal{E} \mathcal{E} (H) \), for in both cases \( E = HP \) is hermitian.
For the proof of (iv), we could have then noted that for \( P \mathcal{E} \mathcal{E} (H), P + E_0 \) is nonsingular when \( E_0 \) is the
principal idempotent matrix of \( H \) (and of \( P \)) associated with the
zero root, and \( H(P + E_0) = HP = E \). Of course
the Theorem 1 could be proved, without reference to
generalized inverses, by simply producing \( P \) as in
in (1) with \( K = 0 \), noting that \( E \) as in (2) is then hermitian
idempotent, and that, subject to \( K = 0 \), \( E \) is invariant
under choices of \( P \). The pivotal idea of the proof is the
observation that given \( A = QH \), we have
(i) of Theorem 1 at once, in view of Lemma 1, if we
can produce an orthogonal projection, \( E \), such that \( EH = H \).
This possibility is suggested by considering
generalized inverses and that it is indeed possible is
perceived at once by considering the Moore-Penrose
generalized inverse, but as we have seen, other “inverses”
will serve as well.

In the proof of (iv) of Theorem 1 and in the above
discussion we have encountered an observation which
may be set out as a corollary.

Corollary. If \( A \) is any square matrix, there exist matrices \( P \) such that \( AP \) is a partial isometry. Further there
exist such matrices \( P \) which are normal, in particular
positive definite.
PROOF. As we have seen any $P$ as in (1) with $K = 0$ has the required property. Any $P$ as in (1) with $K = 0$, $L = 0$, $D$ normal and nonsingular is normal and nonsingular and has the required property. In particular, if $K = 0$, $L = 0$, and $D$ is positive definite, we have a positive definite $P$ from (1).

From Theorem 1, the corollary and the usual polar factorization $A = QH$, we have the following statement: If $A$ is any square matrix, there exists an isometry $Q$ and a partial isometry $U$ such that $Q^*A = U^*A = H$, where $H$ is positive semidefinite. If $A$ is nonsingular there exists a positive definite matrix, $C$, such that $AC = Q$ is an isometry, but there always exists a positive definite $P$ such that $AP = U$ is a partial isometry.

4. References


