The Coefficients of the Powers of a Polynomial

Morris Newman

Institute for Basic Standards, National Bureau of Standards, Washington, D.C. 20234

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It is shown that if \( f(z) \) is a polynomial with no zeroes inside the unit circle and if \( r \) is any positive number, then the coefficients of \( f'(z) \) tend to zero like \( n^{-1-r} \), and this is best possible.

Key Words: Binomial coefficients, bounds, polynomials.

The general question which prompted this note is the following: Suppose that \( f(z) \) is analytic and zero-free inside the unit circle, and normalized so that \( f(0) = 1 \). Let \( r \) be real and positive and choose (for definiteness) that determination of \( f'(z) \) for which \( f'(0) = 1 \). Then what can be said about the order of magnitude of the coefficients of \( f'(z) \)? In particular, when will they converge to zero? In such a general setting no simple answer is to be expected and examples illustrating almost any sort of behavior may be given; but the case treated here, namely that when \( f(z) \) is a polynomial, is reasonably simple. In fact we shall prove the following theorem:

**Theorem 1.** Let \( f(z) \) be a polynomial of degree \( p \) which is zero-free inside the unit circle and such that \( f(0) = 1 \). Let \( r \) be positive and suppose that

\[
f'(z) = 1 + \sum_{n=1}^{\infty} a_r(n)z^n
\]

Then

\[
a_r(n) = O(n^{-1-r}), \quad n \to \infty,
\]

and consequently

\[
\lim_{n \to \infty} a_r(n) = 0.
\]

The statement is false for negative \( r \). For example if \( f(z) = (1 - z)^p \) and \( r = -2/p \) then \( a_r(n) = n + 1 \).

We require the following lemmas.

**Lemma 1.** Let \( r \) be arbitrary. Then

\[
\lim_{n \to \infty} (-1)^n \binom{r}{n} n^{1+r} = \frac{1}{\Gamma(-r)},
\]

and so

\[
\binom{r}{n} = O(n^{-1-r}), \quad n \to \infty.
\]

Formula (2) is just Euler's definition of the \( \Gamma \)-function.

Notice that if \( r \) is a positive integer then \( \binom{r}{n} \) vanishes for all sufficiently large \( n \), which agrees with the fact that \( \Gamma(s) \) has poles at \( s = 0, -1, -2, \ldots \).

**Lemma 2.** Let \( r \) be positive. For \( p = 1, 2, 3, \ldots \) define \( t_p(n) \) by

\[
\sum_{n=0}^{\infty} t_p(n)z^n = \left(1 + \sum_{n=1}^{\infty} n^{-1-r}z^n\right)^p.
\]

Then

\[
t_p(n) = O(n^{-1-r}).
\]

**Proof.** Differentiating (3) and comparing coefficients of corresponding powers of \( z \) in the result we obtain the recurrence formula

\[
nt_p(n) = p \sum_{k=1}^{n} k^{-r}t_{p-1}(n-k).
\]
The proof is by induction on $p$. For $p=1$, $t_p(n) = n^{-1-r}$, $n \geq 1$ and so the lemma is certainly true then. Assume the truth of the lemma for $p-1$, $p \geq 2$. Write (4) as

$$\sum_{1 \leq k \leq \frac{n-1}{2}} k^{-r}t_{p-1}(n-k) + p \sum_{\frac{n-1}{2} < k \leq n-1} k^{-r}t_{p-1}(n-k) + pn^{-r},$$

Then

$$\sum_{1 \leq k \leq \frac{n-1}{2}} k^{-r}t_{p-1}(n-k) = 0 \left\{ \sum_{1 \leq k \leq \frac{n-1}{2}} k^{-r}(n-k)^{-1-r} \right\} = 0 \left\{ n^{-r} \sum_{1 \leq k \leq \frac{n-1}{2}} k^{-r} \right\} = 0(n^{-r}),$$

(using the crude estimate $k^r \geq 1$), and

$$\sum_{\frac{n-1}{2} < k \leq n-1} k^{-r}t_{p-1}(n-k) = 0 \left\{ \sum_{\frac{n-1}{2} < k \leq n-1} k^{-r}(n-k)^{-1-r} \right\} = 0 \left\{ n^{-r} \sum_{\frac{n-1}{2} < k \leq n-1} (n-k)^{-1-r} \right\} = 0(n^{-r}),$$

since the series $\sum_{k=1}^\infty k^{-1-r}$ converges. Thus

$$nt_p(n) = 0(n^{-r}),$$
$$t_p(n) = 0(n^{-1-r}),$$

and the lemma is proved for $p$. This concludes the proof of the lemma.

We turn now to the proof of theorem 1. We may write

$$f(z) = (1 + \alpha_1 z)(1 + \alpha_2 z) \ldots (1 + \alpha_p z),$$

where, because of the assumptions made about $f(z)$, the numbers $\alpha_i$ are all of modulus not exceeding 1.

We have

$$f'(z) = 1 + \sum_{n=1}^\infty a_r(n)z^n = \prod_{i=1}^p (1 + \alpha_i z)^r$$

$$= \prod_{i=1}^p \left( \sum_{n=0} \left( \alpha_i z \right)^n \right)^r$$

$$\leq K \left( 1 + \sum_{n=0} \left( \frac{z}{n^{1+r}} \right)^r \right)$$

$$= K \sum_{n=0} z^n,$$

where $K$ is some suitable positive constant (lemma 1). Hence

$$|a_r(n)| \leq Kt_p(n),$$

and lemma 2 implies the truth of the theorem for all positive $r$.

It is clear that the critical case is when $f(z)$ has all its roots on the unit circle, since otherwise much stronger inequalities for the coefficients $a_r(n)$ will hold. In fact, we can prove

**Theorem 2.** Let $g(z)$ be a polynomial of degree $p$ such that $g(0) = 1$, and let $T$ be the distance from the origin to the nearest zero of $g(z)$. Let $r$ be positive,

$$g'(z) = 1 + \sum_{n=1}^\infty b_r(n)z^n.$$

Then

$$b_r(n) = 0(n^{-1-r} T^{-n}), \quad n \to \infty.$$  

**Proof.** It is only necessary to apply theorem 1 to the polynomial $g(Tz) = f(z)$ which now has no zeroes inside the unit circle and satisfies $f(0) = 1$.  

1. We write $\sum a_z z^r \leq \sum b_z z^r$ to mean that $b_z > 0$ and $|a_z| \leq b_z$.  

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