Three Observations on Nonnegative Matrices*

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Some results on nonnegative matrices are proved, of which the following is representative: Let $A = (a_{ij})$ be a nonnegative row stochastic matrix. If $\lambda \neq 1$ is an eigenvalue of $A$, then

$$|\lambda| \leq \min \left(1 - \sum_j \min a_{ij}, \sum_j \max a_{ij} - 1\right).$$

1.

1. Introduction

In this note, the following results are proved about nonnegative matrices $A = (a_{ij})$ of order $n$.

**THEOREM A:** If $A$ is symmetric, and $c_i = \sum_j a_{ij}$ then

$$\sum_{i,j} (A_m)_{i,j} \leq \sum_i c_i^m, \quad m = 1, 2, \ldots.$$ 

This proves a conjecture of London [3], who has already proved Theorem A for small $m$ and all $n$, and small $n$ and all $m$.

**THEOREM B:** If $A$ is stochastic (i.e., $\sum_j a_{ij} = 1$), and $\lambda \neq 1$ is an eigenvalue of $A$, then

$$|\lambda| \leq \min \left(1 - \sum_j \min a_{ij}, \sum_j \max a_{ij} - 1\right).$$

Goldberg ([1], Theorem 1) has shown that $|\det A|$ is at most the reciprocal of the right-hand side of the above equation, and Theorem C is an effort to amplify Goldberg's theorem by characterizing the right-hand side in terms of elements of $A^{-1}$.

2. Proof of Theorem A

We denote by the cardinality of a set $S$ by $|S|$, and the set $\{1, \ldots, n\}$ by $N$. For any $S \subseteq N$, $u_S$ is the vector $(u_1, \ldots, u_n)$ with $u_j = 1$ if $j \in S$, $u_j = 0$ if $j \notin S$. A nonnegative matrix $A$ is substochastic if $u_S^T A u_S = 0$, and $A u_N \leq u_N$. A subpermutation matrix is a substochastic matrix in which every entry is 0 or 1. If $x$ and $y$ are nonnegative vectors, then $x \leq y$ means

$$\max_{i,j} |b_{ij}| = \max_{y: y \neq 0} \frac{1}{\sum_j y_j}; \quad \max_{i,j} |y_i| = \max_{y: y \neq 0} \frac{1}{\sum_j y_j}.$$

The following are known (see [3]).

(2.1) $\max_{|S| = k} (u_S, x) \leq \max_{|S| = k} (u_S, y)$ \quad $k = 1, \ldots, n.$

The following are known (see [3]).

(2.2) If $x$ and $y$ are nonnegative vectors, $x \leq y$ if and only if there exists a substochastic matrix $A$ such that $x = Ay$.

(2.3) The convex hull of the subpermutation matrices is the set of substochastic matrices.

If $a=(a_1, \ldots, a_n)$ and $b=(b_1, \ldots, b_n)$ are nonnegative vectors with

$$a_1 \geq \ldots \geq a_n \text{ and } b_1 \geq \ldots \geq b_n, \quad ab \text{ is the vector } (a_1b_1, \ldots, a_nb_n).$$
Also, we define \( \Xi_{n} \) to be the set of nonnegative matrices \( A \) such that
\[
A_{n} \preceq a \quad \text{and} \quad (a_{i} A')' \preceq a. \tag{2.5}
\]
Note that \( \Xi_{n} \) is the set of substochastic matrices.

**Lemma 2.1**: If \( a \) and \( b \) satisfy (2.4), \( X \in \Xi_{a}, Y \in \Xi_{b} \), then \( XY \in \Xi_{a+b} \).

**Proof**: By (2.1) and (2.5), we need only show, for \( S \subseteq N \), that
\[
(u_{S} X Y u_{S})' \preceq b_{S} a_{S}. \tag{2.6}
\]
and
\[
(u_{S} X Y u_{S})' \preceq b_{S} a_{S}. \tag{2.7}
\]
We prove only (2.6), the other inequality following by symmetry. Since \( X \) is nonnegative, \( (u_{S} X)' \preceq (u_{S}') \preceq a \).

It follows that
\[
(a_{i} a_{j})' \preceq b_{i} a_{i} b_{j} \preceq b_{i} \lambda a_{i} b_{j}, \tag{2.8}
\]
for a nonnegative matrix \( B \), then \( \mu \) is at most the largest eigenvalue of \( B \). Applying this to (3.2), and observing that \( 1 - \sum c_{j} \) is the largest eigenvalue of \( A_{c} \), we get the first inequality of Theorem B. The second is proved in an analogous manner.

### 4. Proof of Theorem C

Let \( y \) be any real vector such that \( y' A \geq 0 \), and \( z' = y' A \).

Then, with \( B = A^{-1} \),
\[
(4.1) \quad \lambda \max \frac{|y_{j}|}{z'B u} = \max \frac{|z'B_{j}|}{z'B u}.
\]
where \( u = (1, \ldots, 1) \). But \( A u = u \) implies \( B u = u \), and numerator and denominator of the right side of (4.1) are homogeneous in \( z \). Hence,
\[
\max \frac{|z'B_{j}|}{z'B u} = \max \frac{|z'B_{j}|}{z'B u} \tag{4.2}
\]
Let \( i_{0}, j_{0} \) be such that \( |b_{i_{0}, j_{0}}| = \max |b_{i, j}| \). Then
\[
|z'B_{j}| \leq \sum_{j} |z_{i} b_{i,j}| \leq \sum_{i} |z_{i} b_{i,j}| \leq \max |b_{i,j}| \leq |b_{i_{0},j_{0}}|, \tag{4.3}
\]
Consequently, the right-hand side of (4.2) is at most \( |b_{i_{0},j_{0}}| \). But this number is achieved if \( z \) is the vector with 1 in position \( i_{0} \) and 0 everywhere else. Consequently, the right-hand side of (4.2) is \( |b_{i_{0},j_{0}}| \), which combines with (4.1) to prove Theorem C. Noting that
\[
1 \geq |\text{cofactor of } i_{0}, j_{0} \text{ in } A| = |b_{i_{0},j_{0}}| \cdot |\det A|,
\]
one can deduce an alternative proof of [1], Theorem 2. Incidentally, it is manifest that essentially the same arguments also prove:

\[
A_{c} = (a_{i} - c_{j}). \quad \text{Since} \quad (v, u_{v}) = 0, v'A_{c} = v'A. \quad \text{Hence,} \quad (3.1) \quad \lambda v' = v'A = v'A.
\]
Taking absolute values in (3.1), we have
\[
(3.2) \quad |\lambda| |v_{j}| \leq \sum |v_{i}| |a_{i} - c_{j}|.
\]
But \( A_{c} \) is a nonnegative matrix, and it is well-known (see [5]) that if a nonnegative nonzero vector \( x \) and a nonnegative number \( \mu \) satisfy
\[
\mu x_{j} \leq \sum_{i} x_{i} b_{ij}
\]
for a nonnegative matrix \( B \), then \( \mu \) is at most the largest eigenvalue of \( B \). Applying this to (3.2), and observing that \( 1 - \sum c_{j} \) is the largest eigenvalue of \( A_{c} \), we get the first inequality of Theorem B. The second is proved in an analogous manner.

### 3. Proof of Theorem B

If \( \lambda \neq 1 \), and \( \lambda v' = v'A \), then \( (v, u_{v}) = 0 \), since \( A_{u} = 1 u_{v} \). Let \( c_{j} = \min a_{ij} \). Consider the matrix
\[
A_{c} = (a_{i} - c_{j}). \quad \text{Since} \quad (v, u_{v}) = 0, v'A_{c} = v'A. \quad \text{Hence,} \quad (3.1) \quad \lambda v' = v'A = v'A.
\]
\[
\begin{align*}
\max b_{ij} &= \max_{i,j} \frac{\max y_j}{y_j > 0} \sum_{j} y_j \\
\min b_{ij} &= \min_{i,j} \frac{\min y_j}{y_j > 0} \sum_{j} y_j
\end{align*}
\]

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5. References


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