Remarks on Measurable Sets and Functions

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A. J. Goldman (On measurable sets and functions, J. Res. NBS 69B (Math. and Math. Phys.) Nos. 1 and 2, 99–100 (1965)) conjectured that the Borel sets are characterized by their property of having measurable inverse images under all Lebesgue measurable functions; here it is pointed out that the existence of analytic non-Borel sets refutes this and a related conjecture. Also an error in Goldman’s Theorem 2 is corrected.

Key Words: Measure, integration, real function.

We deal exclusively with subsets of the real line \( R \), and with real-valued functions having \( R \) as domain. Let \( (BS) \) and \( (BF) \) denote the respective families of Borel sets and Borel-measurable functions, while \( (LS) \) and \( (LF) \) denote the respective families of Lebesgue-measurable sets and functions. Then \( f \in (LF) \) if and only if

\[
f^{-1}(B) \in (LS) \quad \text{for all} \quad B \in (BS). \tag{1}
\]

Recently Goldman \(^2\) asked whether (1) characterized \((BS)\), in the sense of the following

**Conjecture:** If \( S \) is not in \((BS)\), then there is an \( f \in (LF) \) such that \( f^{-1}(S) \) is not in \((LS)\).

We can disprove this conjecture as follows. Let

\[
n = \{n(1), n(2), \ldots \}
\]

be generic notation for an infinite sequence of positive integers. If \( \hat{\mathcal{A}} \) is a family of sets, then any set

\[
\bigcup_{n \in \mathbb{N}} F(n(1), \ldots , n(r)),
\]

where each \( F(n(1), \ldots , n(r)) \in \hat{\mathcal{A}} \), is said to be “obtained from \( \hat{\mathcal{A}} \) by operation \((\mathcal{A})\)”. If \( \mathcal{A}(\hat{\mathcal{A}}) \) consists of all sets obtainable from \( \hat{\mathcal{A}} \) by operation \((\mathcal{A})\), then for any function \( f \),

\[
f^{-1}(\mathcal{A}(\hat{\mathcal{A}})) = \mathcal{A}(f^{-1}(\hat{\mathcal{A}})). \tag{2}
\]

When \( \hat{\mathcal{A}} = (BS) \), \( \mathcal{A}(\hat{\mathcal{A}}) \) is called the class of analytic sets, and it is known \(^3\) that

\[(BS) \subset \mathcal{A}(BS) \quad \text{but} \quad (BS) \neq \mathcal{A}(BS). \tag{3}\]

For any \( f \in (LF) \), it follows from (1) and (2) that

\[
f^{-1}(\mathcal{A}(BS)) \subset \mathcal{A}(LS) \tag{4}.
\]

It is also known \(^4\) that \((LS)\) is closed under operation \((\mathcal{A})\), so that (4) implies

\[
f^{-1}(\mathcal{A}(BS)) \subset (LS) \quad \text{for all} \quad f \in (LF). \tag{5}\]

Considering \( S \in \mathcal{A}(BS) - (BS) \), as permitted by (3), we are led via (5) to a contradiction of the conjecture.

Denote functional composition by an asterisk \((f \ast g)(x) = f(g(x))\), and let \((LCF)\) be the class of functions \( f \) such that

\[
ge \in (LF) \quad \text{implies} \quad f \ast g \in (LF). \tag{6}
\]

Goldman (Theorem 4, op cit) also showed that we should have

\[(BF) = (LCF) \tag{6}\]

if the Conjecture were true. That (6) fails together with the Conjecture can be proved by choosing as \( f \) the characteristic function of some \( S \in \mathcal{A}(BS) - (BS) \); clearly \( f \) is not in \((BF)\), but for any \( B \in (BS) \) we have \( f^{-1}(B) \) a member of \( \mathcal{A}(BS) \), namely \( R \) or \( \phi \) or \( S \) or \( R - S \).

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\(^4\) K. Kuratowski, op. cit., p. 64.
so that for any $g \in (LF)$ it follows from (5) that

$$(f * g)^{-1}(B) = g^{-1}(f^{-1}(B)) \in (LS),$$

proving $f * g \in (LF)$ and hence $f \in (LCF)$.

Thus the problem of finding a satisfactory characterization of $(LCF)$ remains open. If $(QS)$ is the class of sets $Q$ such that

$$g^{-1}(Q) \in (LS) \quad \text{for all } g \in (LF),$$

then $f \in (LCF)$ if and only if

$$f^{-1}(B) \in (QS) \quad \text{for all } B \in (BS).$$

Hence characterizing $(LCF)$ is closely related to characterizing $(QS)$.

Finally, Goldman’s Theorem 2 (op cit) should be amended to read as follows:

**Theorem:** For any $B \in (BS)$ and $L \in (LS)$, with sole exceptions $(B = \phi, L \neq \phi)$ and $(B = R, L \neq R)$, there is an $f \in (LF)$ such that $L = f^{-1}(B)$.

**Proof:** If $B = \phi$ and $L = \phi$, or $B = R$ and $L = R$, then any $f \in (LF)$ will do. If $B = \phi$ and $L \neq \phi$, or $B = R$ and $L \neq R$, then no $f$ will do. Finally, if $B \neq \phi$ and $B \neq R$, then we can define $f$ on $L$ so that $f(L) \subset B$, and on $R - L$ so that $f(R - L) \subset R - B$. 

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