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# A GENERAL FORMULA FOR THE COMPUTATION OF COLORIMETRIC PURITY

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ABSTRACT

The formula for purity is called general because if the trilinear coordinates of any color referred to any given set of primary color processes and any basic stimulus be given, the colorimetric purity of this stimulus may be computed with respect to any heterogeneous stimulus as the fixed stimulus of homo-heterogeneous analysis. The derivation of the formula is given and five special cases of historical and practical interest are worked out in order to demonstrate that previous formulas for the computation of colorimetric purity and an allied ratio are special cases of this one. The connection is also given between colorimetric purity and the allied ratio variously known as saturation, saturation fraction, and Sättigung.

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## I. INTRODUCTION

The definition of colorimetric purity in American literature and the methods of computing it have been fairly uniform, perhaps due to the comparative recency of its introduction in this country. Abroad, however, a number of allied concepts are variously known by this name, and this concept given, in addition, various other names. Discussion of these concepts, and indeed attempts to provide nomenclature adequate for discussion, have been handicapped by lack of more than fragmentary knowledge of the properties of the concepts, their interrelations and their dependence on choice of coordinate system. It is the purpose of the present paper to supply this lack by deriving a general formula which embraces all these allied concepts in a general system of coordinates.

## II. THE PROBLEM

The tristimulus mode of specifying colors has long been recognized as the fundamental one. According to this mode a color is specified by giving the amounts of three stimuli whose mixture is requisite to produce it. The particular three stimuli in terms of which other stimuli are expressed according to the colors they evoke are called primary stimuli and the colors evoked by them are called primary colors. The choice of primary stimuli is virtually unlimited; for example, many groups of three spectrum stimuli may be chosen, or stimuli produced by means of source-and-filter combinations may be

substituted for any or all of the spectrum stimuli. The change from one set of primaries to another is merely equivalent to a change in coordinate system. The formula for colorimetric purity to be derived holds for any choice of primaries; that is, the transformation from one coordinate system to another is implied in the formula itself.

Let $\rho$, $\gamma$, $\beta$ be the O. S. A. "excitation" curves. These functions of wave length specify the spectrum colors in terms of a certain choice of primaries according to an hypothetical observer whose characteristics are known to approximate satisfactorily those of average, normal vision.[1] The O. S. A. "excitation" curves are distinguished by their simplicity from the infinite number of other sets of curves which may represent the characteristics of the same hypothetical observer; that is, none of the ordinates of these distribution curves is less than zero, and many are zero. Any one of these infinite number of other sets of curves, say $\chi$, $\psi$, $\zeta$, may be computed from $\rho$, $\gamma$, $\beta$ as follows:

$$\left.\begin{aligned} \chi &\equiv K_1\rho + K_2\gamma + K_3\beta \\ \psi &\equiv K_4\rho + K_5\gamma + K_6\beta \\ \zeta &\equiv K_7\rho + K_8\gamma + K_9\beta \end{aligned}\right\} \tag{1}$$

where $K_1$ to $K_9$ are arbitrary constants such that:

$$\begin{vmatrix} K_1 K_2 K_3 \\ K_4 K_5 K_6 \\ K_7 K_8 K_9 \end{vmatrix} \neq 0 \tag{2}$$

Define:

$$\left.\begin{aligned} \bar{\chi} &\equiv \int_0^\infty \chi ET d\lambda \\ \bar{\psi} &\equiv \int_0^\infty \psi ET d\lambda \\ \bar{\zeta} &\equiv \int_0^\infty \zeta ET d\lambda \end{aligned}\right\} \tag{3}$$

where $E$ is the spectral distribution of energy of the source used and $T$ is the spectral transmission (or spectral reflectance in case of a reflecting surface) of the sample which the source illuminates. The product, $ET$, as a function of wave length is the spectral distribution of energy, therefore, of the stimulus whose colorimetric purity is to be computed [2] by recourse to the mixture data of $\rho$, $\gamma$, $\beta$. The numbers, $\bar{\chi}$, $\bar{\psi}$, $\bar{\zeta}$, may be used to specify the color evoked by the stimulus specified by $ET$ because all stimuli for which $\bar{\chi}$, $\bar{\psi}$, $\bar{\zeta}$ are identical produce identical colors within the visual mechanism specified by $\rho$, $\gamma$, $\beta$.[3]

Define:

$$\left.\begin{aligned} X &\equiv \bar{\chi}/(\bar{\chi}+\bar{\psi}+\bar{\zeta}) \\ Y &\equiv \bar{\psi}/(\bar{\chi}+\bar{\psi}+\bar{\zeta}) \\ Z &\equiv \bar{\zeta}/(\bar{\chi}+\bar{\psi}+\bar{\zeta}) \end{aligned}\right\} \tag{4}$$

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[1] For the sake of being specific we take the O. S. A. "excitation" curves in the form extrapolated by Priest and Gibson (J. Opt. Soc. Am. and Rev. Sci. Inst., vol. 10, p. 230, 1925); but we might nearly as well have taken the O. S. A. "excitation" curves in their original form (J. Opt. Soc. Am. and Rev. Sci. Inst., vol. 6, p. 548, 1922) or some other set of curves which yield a satisfactory approximation to average, normal vision.

[2] If the stimulus whose colorimetric purity is to be computed were radiant energy direct from a light source, a single symbol would naturally be used for $ET$. Since most stimuli whose purities are to be computed are reflected or transmitted energies, it is convenient to use the product, $ET$.

[3] D. B. Judd, Reduction of Data on Mixture of Color Stimuli, B. S. Jour. Research, vol. 4, pp. 515–548; 1930.

$X$, $Y$, $Z$ are called the trilinear coordinates of the spectrum; they refer to some set of three primary processes and some basic stimulus [4] which may be inferred from the values of the constants, $K_1$ to $K_9$, of relations (1) and (2).

Define:

$$\left.\begin{aligned} x &\equiv \bar{\chi}/(\bar{\chi}+\bar{\psi}+\bar{\zeta}) \\ y &\equiv \bar{\psi}/(\bar{\chi}+\bar{\psi}+\bar{\zeta}) \\ z &\equiv \bar{\zeta}/(\bar{\chi}+\bar{\psi}+\bar{\zeta}) \end{aligned}\right\} \tag{5}$$

The numbers, $x$, $y$, $z$, are called the trilinear coordinates of the color evoked by the stimulus of spectral energy distribution, $ET$; these trilinear coordinates are referred to the same primary color processes and basic stimulus as $X$, $Y$, $Z$ of relation (4).

When the energy of a stimulus is confined to a range of wave lengths, $\lambda-\Delta\lambda$ to $\lambda+\Delta\lambda$, so restricted that reduction of $\Delta\lambda$ makes no difference in the chromaticity of the evoked color, this stimulus is called homogeneous radiant energy of wave length, $\lambda$. Homogeneous radiant energy, $E_\lambda$, may be colorimetrically specified from relation (3) by the definite integrals:

$$\int_{\lambda-\Delta\lambda}^{\lambda+\Delta\lambda} \chi E_\lambda d\lambda,\ \int_{\lambda-\Delta\lambda}^{\lambda+\Delta\lambda} \psi E_\lambda d\lambda,\ \int_{\lambda-\Delta\lambda}^{\lambda+\Delta\lambda} \zeta E_\lambda d\lambda,$$

which for brevity are written hereafter as: $\bar{\chi}_h$, $\bar{\psi}_h$, $\bar{\zeta}_h$. Note that these three values are proportional to: $\chi$, $\psi$, $\zeta$, for the wave length, $\lambda$.

Heterogeneous radiant energy of spectral distribution, $E_w$, is specified colorimetrically from relation (3) by the definite integrals:

$$\int_0^\infty \chi E_w d\lambda,\ \int_0^\infty \psi E_w d\lambda,\ \int_0^\infty \zeta E_w d\lambda,$$

which are abbreviated hereafter as: $\bar{\chi}_w$, $\bar{\psi}_w$, $\bar{\zeta}_w$. The subscript, $w$, is chosen because it suggests that the heterogeneous radiant energy is, as is usual, such as to evoke a "white" or "neutral" color.

The mixture of homogeneous radiant energy, $E_\lambda$, with heterogeneous radiant energy, $E_w$, is specified colorimetrically by the three sums of these definite integrals:

$$\bar{\chi}_h+\bar{\chi}_w,\ \bar{\psi}_h+\bar{\psi}_w,\ \bar{\zeta}_h+\bar{\zeta}_w,$$

that is, for the mixture:

$$\left.\begin{aligned} \bar{\chi} &= \bar{\chi}_h+\bar{\chi}_w \\ \bar{\psi} &= \bar{\psi}_h+\bar{\psi}_w \\ \bar{\zeta} &= \bar{\zeta}_h+\bar{\zeta}_w \end{aligned}\right\} \tag{6}$$

[4] The basic stimulus as defined by Priest is any stimulus whose color is represented at the center of the Maxwell triangle; that is, any stimulus whose color has the trilinear coordinates (see relation (5) ), $x=y=z=1/3$. From relation (3) we see that the basic stimulus is any stimulus whose spectral energy distribution $E_b$, satisfies the condition: $\int_0^\infty \chi E_b d\lambda=\int_0^\infty \psi E_b d\lambda=\int_0^\infty \zeta E_b d\lambda$. To specify a basic stimulus is a convenient way to specify the relative scale magnitudes of $\chi$, $\psi$, $\zeta$; these magnitudes could be just as rigorously specified by giving the spectral energy distribution, $E_x$, for which the numbers: $\int_0^\infty \chi E_x d\lambda$, $\int_0^\infty \psi E_x d\lambda$, $\int_0^\infty \zeta E_x d\lambda$, have any given relative values.

Hence, from relation (5), the trilinear coordinates of the color evoked by this mixture are:

$$\left.\begin{aligned} x&=\frac{\bar{\chi}_h+\bar{\chi}_w}{\bar{\chi}_h+\bar{\psi}_h+\bar{\zeta}_h+\bar{\chi}_w+\bar{\psi}_w+\bar{\zeta}_w}\\ y&=\frac{\bar{\psi}_h+\bar{\psi}_w}{\bar{\chi}_h+\bar{\psi}_h+\bar{\zeta}_h+\bar{\chi}_w+\bar{\psi}_w+\bar{\zeta}_w}\\ z&=\frac{\bar{\zeta}_h+\bar{\zeta}_w}{\bar{\chi}_h+\bar{\psi}_h+\bar{\zeta}_h+\bar{\chi}_w+\bar{\psi}_w+\bar{\zeta}_w}\end{aligned}\right\} \tag{7}$$

The trilinear coordinates, $x_w$, $y_w$, $z_w$, of the color evoked by the heterogeneous component of the mixture are:

$$\left.\begin{aligned} x_w&=\bar{\chi}_w/(\bar{\chi}_w+\bar{\psi}_w+\bar{\zeta}_w)\\ y_w&=\bar{\psi}_w/(\bar{\chi}_w+\bar{\psi}_w+\bar{\zeta}_w)\\ z_w&=\bar{\zeta}_w/(\bar{\chi}_w+\bar{\psi}_w+\bar{\zeta}_w)\end{aligned}\right\} \tag{8}$$

Define:

$$L_h\equiv\bar{\rho}_hL_r+\bar{\gamma}_hL_g+\bar{\beta}_hL_b \tag{9}$$

where $L_r$, $L_g$, $L_b$ are any constants not all zero. $\bar{\rho}_h$, $\bar{\gamma}_h$, $\bar{\beta}_h$ represent definite integrals analogous to those represented by $\bar{\chi}_h$, $\bar{\psi}_h$, $\bar{\zeta}_h$. We use the symbol, $L_h$ (luminosity), because when $L_r/(L_r+L_g+L_b)=0.45$, $L_g/(L_r+L_g+L_b)=0.54$, and $L_b/(L_r+L_g+L_b)=0.01$, or values not greatly differing from these, $L_h$ varies according to wave length very closely as the luminosity of the equal-energy spectrum determined experimentally.[5] We refrain from calling $L_h$ the luminosity because $L_h$ deserves such a name only for the special case just mentioned. Although this special case commands the major interest because of the possibility of determining it directly by experiment[6] some attention has been given another case, $L_r=L_g=L_b$, as will appear later.

It has been shown that:[7]

$$L_h=\bar{\chi}_hL_x+\bar{\psi}_hL_y+\bar{\zeta}_hL_z \tag{9a}$$

if:

$$\left.\begin{aligned} L_x&=\frac{1}{\Delta}\left[L_r\begin{vmatrix}K_5&K_8\\K_6&K_9\end{vmatrix}+L_g\begin{vmatrix}K_7&K_4\\K_9&K_6\end{vmatrix}+L_b\begin{vmatrix}K_4&K_7\\K_5&K_8\end{vmatrix}\right]\\ L_y&=\frac{1}{\Delta}\left[L_r\begin{vmatrix}K_8&K_2\\K_9&K_3\end{vmatrix}+L_g\begin{vmatrix}K_1&K_7\\K_3&K_9\end{vmatrix}+L_b\begin{vmatrix}K_7&K_1\\K_8&K_2\end{vmatrix}\right]\\ L_z&=\frac{1}{\Delta}\left[L_r\begin{vmatrix}K_2&K_5\\K_3&K_6\end{vmatrix}+L_g\begin{vmatrix}K_4&K_1\\K_6&K_3\end{vmatrix}+L_b\begin{vmatrix}K_1&K_4\\K_2&K_5\end{vmatrix}\right]\end{aligned}\right\} \tag{9b}$$

where $\Delta\equiv\begin{vmatrix}K_1&K_2&K_3\\K_4&K_5&K_6\\K_7&K_8&K_9\end{vmatrix}$, the constants, $K_1$ to $K_9$, being those of relations (1) and (2).

[5] D. B. Judd, Chromatic Visibility Coefficients by the Method of Least Squares, J. Opt. Soc. Am. and Rev. Sci. Inst., vol. 10, pp. 635-651; 1925.

[6] See footnote 9, p. 831, and footnote 11, p. 832.

[7] See footnote 3, p. 828.

Define:

$$L_m \equiv \bar{\chi} L_x + \bar{\psi} L_y + \bar{\zeta} L_z \tag{10}$$

The subscript, $m$, may be read, "of the mixture," and the analogous subscript, $h$, of relations (9) and (9a) may be read, "of the homogeneous component."

The ratio, $L_h/L_m$, is of particular interest in colorimetry because in the special case[8] where it is a ratio of luminosities it may be determined experimentally[9] by the methods of heterochromatic photometry, and it has been given a special name, the colorimetric purity. We may now state the problem.

Given: (1) Values of the trilinear coordinates, $x$, $y$, $z$, of a color produced by the mixture of a homogeneous and a heterogeneous component, (2) the trilinear coordinates, $x_w, y_w, z_w$, of the color evoked by the heterogeneous component alone, (3) the trilinear coordinates, $X$, $Y$, $Z$, of the color evoked by the homogeneous component alone, (4) the constants, $K_1$ to $K_9$, specifying the primaries and basic stimulus of the distribution curves as related to the O. S. A. "excitation" curves, and (5) the constants, $L_r$, $L_g$, $L_b$; required: the ratio, $L_h/L_m$.

## III. THE SOLUTION

The derivation of the general formula has been foreshadowed by the derivation of a special case of it.[10]

From relation (9a) and definition (10) we have:

$$\frac{L_h}{L_m} = \frac{\bar{\chi}_h L_x + \bar{\psi}_h L_y + \bar{\zeta}_h L_z}{\bar{\chi} L_x + \bar{\psi} L_y + \bar{\zeta} L_z}$$

which, in virtue of relation (6), becomes:

$$\frac{L_h}{L_m} = \frac{\bar{\chi}_h L_x + \bar{\psi}_h L_y + \bar{\zeta}_h L_z}{\bar{\chi}_h L_x + \bar{\psi}_h L_y + \bar{\zeta}_h L_z + \bar{\chi}_w L_x + \bar{\psi}_w L_y + \bar{\zeta}_w L_z}$$

After the factor, $\bar{\chi}_h + \bar{\psi}_h + \bar{\zeta}_h$, abbreviated as $\sigma$, has been divided out, this ratio becomes, from definition (4):

$$\frac{L_h}{L_m} = \frac{XL_x + YL_y + ZL_z}{XL_x + YL_y + ZL_z + \frac{\bar{\chi}_w}{\sigma} L_x + \frac{\bar{\psi}_w}{\sigma} L_y + \frac{\bar{\zeta}_w}{\sigma} L_z} \tag{11}$$

From definition (4) and relations (7) and (8) it follows at once that:

$$\left.\begin{aligned} \bar{\chi}_w/\sigma &= x_w(x-X)/(x_w-x) \\ \bar{\psi}_w/\sigma &= y_w(y-Y)/(y_w-y) \\ \bar{\zeta}_w/\sigma &= z_w(z-Z)/(z_w-z) \end{aligned}\right\} \tag{12}$$

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[8] See relation (9) et. seq.

[9] I. G. Priest, Apparatus for the Determination of Color in Terms of Dominant Wave-Length, Purity, and Brightness, J. Opt. Soc. Am. and Rev. Sci. Inst., vol. 8, pp. 173–200, 1924.

[10] D. B. Judd, The Computation of Colorimetric Purity, J. Opt. Soc. Am. and Rev. Sci. Inst., vol. 13, pp. 136–138; 1926.

By substituting these three values in relation (11) a solution of the problem is obtained; but this solution may be expressed more simply by taking account of the collinearity condition:[11]

$$\frac{x-X}{x_w-x}=\frac{y-Y}{y_w-y}=\frac{z-Z}{z_w-z} \tag{13}$$

which also follows from (4), (7), and (8); it states that the points $(X,Y,Z)$, $(x,y,z)$, and $(x_w, y_w, z_w)$, lie on the same straight line. In virtue of relation (13), relation (11) may be written:

$$\frac{L_h}{L_m}=\frac{XL_x+YL_y+ZL_z}{XL_x+YL_y+ZL_z+F(x_wL_x+y_wL_y+z_wL_z)} \tag{14}$$

where $F=(x-X)/(x_w-x)=(y-Y)/(y_w-y)=(z-Z)/(z_w-z)$. Relation (14) is a solution of the problem since $X$, $Y$, $Z$, $x$, $y$, $z$, $x_w$, $y_w$, $z_w$ are known and $L_x$, $L_y$, $L_z$ may be found from relation (9b), since the constants, $K_1$ to $K_9$, and $L_r$, $L_g$, $L_b$ are given.

## IV. FIVE SPECIAL CASES

1. For routine computation of colorimetric purity at the National Bureau of Standards it has been customary to choose constants $K_1$ to $K_9$, all zero except $K_1$, $K_5$, and $K_9$ (which retains all the simplicity of the O. S. A. "excitation" curves), and so to adjust $K_1$, $K_5$, and $K_9$ that the heterogeneous stimulus is represented at the center of the Maxwell triangle; that is, the heterogeneous stimulus is chosen as the basic stimulus.[12] Hence the constants are taken so that

$$\int_0^\infty \chi E_w d\lambda=\int_0^\infty \psi E_w d\lambda=\int_0^\infty \zeta E_w d\lambda$$

which, from this special case of relation (1), may be written

$$K_1\int_0^\infty \rho E_w d\lambda=K_5\int_0^\infty \gamma E_w d\lambda=K_9\int_0^\infty \beta E_w d\lambda$$

$L_x$, $L_y$, and $L_z$ are chosen so that $L_h$ of relation (9a) may be taken to represent the luminosity of the equal energy spectrum;[13] hence we may write colorimetric purity, $p$, instead of $L_h/L_m$ in relation (14) which becomes; since $x_w=y_w=z_w=1/3$:

$$p=\frac{XL_x+YL_y+ZL_z}{XL_x+YL_y+ZL_z+F(L_x+L_y+L_z)/3}$$

Since, as is permissible (see relation (9) et seq.), it is customary to set $L_x+L_y+L_z=1$, a further simplification results:

$$p=\frac{XL_x+YL_y+ZL_z}{XL_x+YL_y+ZL_z+F/3} \tag{15}$$

[11] I. G. Priest, The Computation of Colorimetric Purity, J. Opt. Soc. Am. and Rev. Sci. Inst., vol. 9, p. 520; 1924. D. B. Judd, The Computation of Colorimetric Purity, J. Opt. Soc. Am. and Rev. Sci. Inst., vol. 13, p. 138; 1926.

[12] See footnote 4, p. 829.

[13] They may be found from relation (9b) if $L_r$, $L_g$, and $L_b$ are appropriately chosen (see relation (9) et seq.), or they may be found de novo.

where $F/3 = (x-X)/(1-3x) = (y-Y)/(1-3y) = (z-Z)/(1-3z)$. Relation (15) may be recognized as a restatement of the Judd *R*, Judd *G*, and Judd *B* forms of purity formula[14] with obvious changes in symbology. It is also identical with other forms of purity formula previously derived (Ives, Priest, Tuckerman). A formula developed by Froelich[15] is identical with relation (15) providing $L_h$ of relation (9a) be taken equal to her $L_\lambda$, and providing her ratio, $a$, determined graphically by measuring two distances on the mixture diagram, be given its analytic equivalent, $1/(1+F)$. Two formulas developed by Guild[16] also involve ratios of lengths on the mixture diagram, the ratio in the first formula given (Guild's, p. 161) being equal to $1/(1+F)$ and that in the second formula (p. 162) being equal to $1/F$. When these substitutions are made, the first formula can readily be shown to be identical with the Ives form, which has already been proven identical with relation (15); the second formula becomes relation (15) directly.

It may be remarked that the various forms of purity formula with this convention of making the heterogeneous stimulus the basic stimulus of the system arise chiefly because (1) the coordinates of any pair of the three collinear points may be taken to form the luminosity sums, as, for example, in relation (15), which takes the coordinates of the spectrum point and the "white" point; (2) the ratio of lengths may be expressed as the ratio of the segments or as the ratio of either segment to the total, as, for example, in relation (15), which uses $F$, the ratio of the segments; and (3) this ratio of lengths may be evaluated from any parallel projection of the line, as, for example, in relation (15), which evaluates the ratio, $F$, from projections onto the axes of a mixture diagram in rectangular coordinates or onto the bisectors of the angles of the equilateral triangle, or as, for another example, in Ives's formulas, which evaluate a ratio of lengths from projections onto the sides of the equilateral triangle, or as in the formulas of Froelich and Guild, which evaluate the ratios directly from the lengths themselves. The various forms of purity formula are all useful, the particular form of purity formula to be chosen depending on the nature of the problem.

2. Another choice of primary processes which tends more toward convenience for those who think in terms of the opponent-colors theory of vision (Hering) is that which takes "neutral" as one of the three primary processes. The chromatic aspect of the response is then described solely by the other two components which have negative as well as positive values, such as a red process, whose negative is green, and a yellow process, whose negative is blue. For such a choice, if further the heterogeneous stimulus be such as to evoke the "neutral" color, and if the "neutral" color be chosen as the third primary whose distribution in the spectrum is given by $\zeta$, the constants, $K_1$ to $K_9$, of relation (1) must satisfy the condition:

$$\int_0^\infty \chi E_w d\lambda = \int_0^\infty \psi E_w d\lambda = 0$$

[14] See footnote 10, p. 831.

[15] Clara L. Froelich, Algebraic Methods for the Calculation of Color Mixture Transformation Diagrams, J. Opt. Soc. Am. and Rev. Sci. Inst., vol. 9, p. 39; 1924.

[16] J. Guild, The Geometrical Solution of Colour Mixture Problems, Trans. Opt. Soc., vol. 26, pp. 161–162; 1924–25.

which from relation (1) is equivalent to:

$$K_1\int_0^\infty \rho E_w d\lambda + K_2\int_0^\infty \gamma E_w d\lambda + K_3\int_0^\infty \beta E_w d\lambda = 0$$

$$K_4\int_0^\infty \rho E_w d\lambda + K_5\int_0^\infty \gamma E_w d\lambda + K_6\int_0^\infty \beta E_w d\lambda = 0$$

From relation (8) and the definitions of $\overline{\chi_w}$, $\overline{\psi_w}$, $\overline{\zeta_w}$, it follows that $x_w = y_w = 0$ and $z_w = 1$. Relation (14) becomes:

$$\frac{L_h}{L_m} = \frac{XL_x + YL_y + ZL_z}{XL_x + YL_y + ZL_z + FL_z} \tag{16}$$

where

$$F = (x - X)/(-x) = (y - Y)/(-y) = (z - Z)/(1 - z)$$

A further simplification is possible if $K_7$, $K_8$, and $K_9$ be chosen equal, respectively, to $L_r$, $L_g$, and $L_b$. From relation (9b) it follows that $L_x = L_y = 0$. And if still further we arbitrarily set $L_z = 1$, as is permissible, we may write:

$$L_h/L_m = Z/(Z+F)$$

If, further, $L_r$, $L_g$, and $L_b$, be chosen so that the ratio may be considered a luminosity ratio as in case (1), this relation may be written as a very simple formula for colorimetric purity:

$$p = (1 - 1/z)/(1 - 1/Z) \tag{17}$$

Since Schrödinger's "vierfarbentheorie" curves [17] satisfy all the conditions under which relation (17) was derived, this relation may be taken for colorimetric purity if the trilinear coordinates $X$, $Y$, $Z$ and $x$, $y$, $z$ are computed from those curves. The simplicity of the formula raises the question whether Schrödinger's choice of primary color processes, which were chosen because of their interest for psycho-physiological speculation, might not be a favorable choice purely for the purpose of simplifying the computation of colorimetric purity. This question is discussed presently.

3. Some attention has been given to the ratio, $L_h/L_m$, in the special case for which $L_x = L_y = L_z$,[18] perhaps chiefly because of the relative ease with which it may be computed. Since, from relation (4), $X + Y + Z = 1$, relation (14) for this special case may be written:

$$L_h/L_m = 1/(1+F)$$

or

$$L_h/L_m = (x_w - x)/(x_w - X) = (y_w - y)/(y_w - Y) = (z_w - z)/(z_w - Z) \tag{18}$$

Inspection of relation (18) shows that $L_h/L_m$ for this case is the distance on the mixture diagram from the point ($x$, $y$, $z$) to the "white" point ($x_w$, $y_w$, $z_w$), divided by the distance from the "spectrum" point ($X$, $Y$, $Z$) to the "white" point; hence the ratio may in this case be determined quite easily from the mixture diagram directly.

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[17] E. Schrödinger, Über das Verhältnis der Vierfarbenzur Dreifarbentheorie, Sitz. Akad. Wiss., Wien, IIa, vol. 134, p. 479; 1925.

[18] W. Dziobek and M. Pirani, Normung von Signalgläsern, Proc. International Congress on Illumination, Saranac Inn, pp. 818–833; 1928. I. Runge, Die Einheitsmengen im Maxwell-Helmholtzschen Farbdreieck und die Bestimmung der Farbsättigung, Zs. f. Instrumentenk., vol. 49, pp. 600–603; 1929. S. Rösch, Darstellung der Farbenlehre für die Zwecke des Mineralogen, Fortschritte d. Mineralogie, Kristallographie u. Petrographie, vol. 30, pp. 135–136; 1929.

It should be noted, however, that $L_h/L_m$ in this case is not necessarily the colorimetric purity because $L_x$, $L_y$, and $L_z$ have not been especially chosen to make $L_h$ of relation (9a) proportional to the luminosity of the equal-energy spectrum. It should also be noted that this special case loses the characteristic which is the probable cause of its existence if a different choice of primaries or basic stimulus be made; that is, with certain trivial exceptions, the coefficients, $L_x$, $L_y$, $L_z$, in another system of coordinates would no longer be equal; hence this special case of the ratio $L_h/L_m$ would be just as hard to compute as colorimetric purity or any other special case of relation (14).[19] The condition for the exceptions may be seen by inspection of relation (9b).

4. In order to facilitate the computation of colorimetric purity, Runge[20] has suggested another choice of constants, $K_1$ to $K_9$. For this choice the simple formula (18) just derived results in the special case of $L_h/L_m$ which has been defined as the colorimetric purity. It is interesting to see how the choice of constants, $K_1$ to $K_9$, suggested by Runge insures that relation (18) results in colorimetric purity.

Take $L_r$, $L_g$, $L_b$ such that $L_h$ of relation (9) gives the distribution of luminosity in the equal-energy spectrum. This choice permits us to substitute $p$ for $L_h/L_m$. Then the constants, $K_1$ to $K_9$, are chosen, all zero except $K_1$, $K_5$, $K_9$, which are taken as $L_r$, $L_g$, $L_b$, respectively. From relation (9b) it will be discovered that, for this choice, $L_x = L_y = L_z = 1$; hence it follows that relation (18) in this special case gives colorimetric purity.

According to this plan, however, the basic stimulus of the $\rho\gamma\beta$ system is taken as the heterogeneous stimulus in the $\chi\psi\zeta$ system, necessitating that:

$$\int_0^\infty \rho E_w d\lambda = \int_0^\infty \gamma E_w d\lambda = \int_0^\infty \beta E_w d\lambda$$

which, from relation (1), is equivalent to:

$$(1/L_r)\int_0^\infty \chi E_w d\lambda = (1/L_g)\int_0^\infty \psi E_w d\lambda = (1/L_b)\int_0^\infty \zeta E_w d\lambda$$

whence, from relation (8), it is evident that:

$$x_w = L_r/(L_r + L_g + L_b)$$
$$y_w = L_g/(L_r + L_g + L_b)$$
$$z_w = L_b/(L_r + L_g + L_b)$$

Since $L_b$ in this case is about one-fiftieth of either $L_r$ or $L_g$, it is evident that the "white" point $(x_w, y_w, z_w)$ falls relatively near one side of the Maxwell triangle.[21] This consequence of Runge's choice of coordinate system is emphasized by setting, as is usual, $L_r + L_g + L_b = 1$, which permits relation (18) for this special case to be written:

$$p = (L_r - x)/(L_r - X) = (L_g - y)/(L_g - Y) = (L_b - z)/(L_b - Z) \qquad (19)$$

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[19] Another way of making the same statement is that if $L_x$, $L_y$, and $L_z$ in another system be made equal in general, the number defined by such a choice will not be identical with $L_h/L_m$ in this special case.

[20] See footnote 18, p. 834.

[21] See Runge's fig. 2 (footnote 18, p. 834).

5. Another coordinate system has since been proposed by Judd [22] to facilitate the computation of colorimetric purity. This system avoids the close approach of the spectrum locus to the "white" point.

Take $K_2=K_3=K_7=K_8=0$, $K_4=L_r$, $K_5=L_g$, and $K_6=L_b$, where $L_r$, $L_g$, $L_b$, as before, are chosen so that $L_h$ of relation (9) gives the distribution of luminosity in the equal-energy spectrum. As in case (2), it is discovered from relation (9b) that two of the three numbers, $L_x$, $L_y$, $L_z$, are zero; that is, in this case, $L_x=L_z=0$, $L_y=1$.

$K_1$, $K_9$ and $L_r+L_g+L_b$ are chosen so as to make $E_w$ the basic stimulus, which, from relation (1), requires that:

$$K_1\int_0^\infty \rho E_w d\lambda = K_9\int_0^\infty \beta E_w d\lambda = L_r\int_0^\infty \rho E_w d\lambda + L_g\int_0^\infty \gamma E_w d\lambda + L_b\int_0^\infty \beta E_w d\lambda$$

Since, by this choice, $x_w=y_w=z_w=1/3$, relation (14) may be written for $y\neq 1/3$:

$$p=(3-1/y)/(3-1/Y) \tag{20}$$

An equally simple form may be written for $y=1/3$.[22]

Relation (17) is just as simple for routine computation as relation (20) provided the trilinear coordinates ($x$, $y$, $z$) of the mixture are at hand. The distribution curves leading up to relation (17), however, are less simple for routine computation of the coordinates, $x$, $y$, $z$, than those leading up to relation (20) because they embody more ordinates different from zero and because they embody negative as well as positive ordinates; hence relation (20), considering the entire computation of colorimetric purity from spectral distribution of energy, is more advantageous than relation (17).

Relation (20) is not quite so convenient for routine computation as relation (19), since it involves the finding of two reciprocals in addition to the two subtractions and one division indicated in relation (19). However, the ease with which $X$, $Y$, $Z$ may be found from $x$, $y$, $z$ within the coordinate system of relation (20) is generally considerably greater than for the coordinate system of relation (19) because the coordinate system of relation (20) avoids the close approach to the spectrum locus of the "white" point ($x_w$, $y_w$, $z_w$).[23]

If any coordinate system is to be chosen for the specific purpose of facilitating the computation of colorimetric purity, it would seem that the adoption of that leading to relation (20) would be the most advantageous in routine computation.

## V. PURITY AND SATURATION

Discussion of the terms "purity" and "saturation" is facilitated by the introduction of a ratio, $L_a/L_m$, which is in all respects save one the same as $L_h/L_m$. Since this ratio is of considerably less importance than $L_h/L_m$, there is little interest in showing how it may be computed from one set of primaries after having been defined in terms of another, as was done with $L_h/L_m$. It suffices to write a relation analogous to relation (14) which will serve to define $L_a/L_m$,

[22] See footnote 3, p. 828.
[23] Compare Runge's fig. 2 (see footnote 18, p. 834) with Judd's fig. 5 (see footnote 3, p. 828).

where the subscript, $a$, is to be read "absolute" (or "basic," according to Priest) and to be understood to refer to a color represented on the edge of the Maxwell triangle:

$$\frac{L_a}{L_m}=\frac{X_aL_x+Y_aL_y+Z_aL_z}{X_aL_x+Y_aL_y+Z_aL_z+F_a(x_wL_x+y_wL_y+z_wL_z)} \qquad (21)$$

where $F_a\equiv(x-X_a)/(x_w-x)$ or $(y-Y_a)/(y_w-y)$ or $(z-Z_a)/(z_w-z)$. Relation (21) differs from relation (14) only by the substitution of $X_a$, $Y_a$, $Z_a$, respectively, for $X$, $Y$, $Z$. Since $L_h/L_m$ expresses by its approach to unity the degree of approach of the color $(x, y, z)$ to the color $(X, Y, Z)$, the ratio, $L_a/L_m$, similarly gives the degree of approach of the color $(x, y, z)$ to the color $(X_a, Y_a, Z_a)$. The color $(X, Y, Z)$, used as a reference in the one case, may be evoked by some portion of the spectrum; in the second case the color $(X_a, Y_a, Z_a)$, used as a reference, is represented on an edge of the Maxwell triangle; we know, therefore, that one of $X_a$, $Y_a$, $Z_a$ is zero.

We have seen that two special cases of the ratio, $L_h/L_m$, are of interest, the one in which $L_x=L_y=L_z$ and the one in which $L_x$, $L_y$, $L_z$ are chosen, so that $L_h$ of relation (9a) may be taken as the luminosity of the equal-energy spectrum. These two special cases also command all the attention given to the ratio $L_a/L_m$.

The first case yields a formula analogous to relation (18), which becomes for $x_w=y_w=z_w=1/3$:

$$L_a/L_m=(1-3x)/(1-3X_a)=(1-3y)/(1-3Y_a)=(1-3z)/(1-3Z_a)$$

which may be written:

$$L_a/L_m=1-3c \qquad (22)$$

where $c$ is $x$ or $y$ or $z$, according as $X_a$ or $Y_a$ or $Z_a$ is zero. This ratio was defined by Exner [24] as "Sättigung," for the coordinate system based on the Grund-Empfindungs-Curven of König and Dieterici.[25] Exner's definition has been followed by a number of European workers. It has been recently restated as a "not wholly unnatural measure of saturation" by Schrödinger.[26]

The second case (for $x_w=y_w=z_w=1/3$) has been given by Martin[27] as one of three variables for specifying color stimuli; the ratio, $L_a/L_m$, is defined by a numerical example referring to Abney's "sensation" curves and is called the "saturation fraction." This second case was also mentioned later by Priest [28] incidental to the derivation of a formula for colorimetric purity; he named the ratio, $L_a/L_m$, for this case the "absolute purity." Still later this ratio was defined by Haschek,[29] who called it "Sättigung."

Probably two reasons may account for choosing as a reference the edges of the Maxwell triangle as in the ratio, $L_a/L_m$. The first is merely that the resulting formula is simpler because one of $X_a$, $Y_a$, $Z_a$ is always zero; this is a logical extension of the motive which prob-

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[24] F. Exner, Sitz. Akad. Wiss., Wien, I, vol. 119, p. 233; 1910.

[25] A. König and C. Dieterici, Die Grundempfindungen in normalen und anomalen Farbensystemen und ihre Intensitätsvertheilung im Spectrum, Zs. f. Psych. u. Physiol. d. Sinnesorgane, 4, pp. 241–347; 1893; or see A. König, Ges. Abh., Leipzig, Barth, pp. 214–321; 1903.

[26] E. Schrödinger, Müller-Pouillets Lehrbuch d. Physik, 2d ed., vol. 2, pp. 482–484; 1926.

[27] L. C. Martin, Colour and Methods of Colour Reproduction, London, Blackie, p. 133; 1923.

[28] I. G. Priest, The Computation of Colorimetric Purity, J. Opt. Soc. Am. and Rev. Sci. Inst. vol. 9, p. 509; 1924.

[29] E. Haschek, Quantitative Beziehungen in der Farbenlehre, Sitz. Akad. Wiss., Wien, IIa, vol. 136, pp. 461–468; 1927.

ably led up to relation (18) and applies with particular force to the first case which results in the simple relation (22). The second reason is the suspicion, or the delusion, or, perhaps, the hope that the edges of the Maxwell triangle of the particular coordinate system chosen possess a fundamental theoretical significance by virtue of which they deserve the dignity of being chosen as reference lines. If the primary color processes of the coordinate system were known to be processes functionally distinct somewhere within the visual mechanism, then it could be argued that every color process represented by a point outside the Maxwell triangle would be nonexistent, or imaginary. The edges of the triangle would then represent the colors which differ from an achromatic color of the same brilliance by the maximum amount possible, and hence they would form a rather natural reference locus. However, the argument by which the König Grund-Empfindungs-Curven are taken to give the distributions of the true primary processes is by no means complete. The primaries chosen by König satisfy two criteria: (1) They are a particular "red," "green," and "blue" whose hues are close to three of the four psychologically unitary hues, that is, the hues are nearly Urrot, Urgrün, and Urblau; and (2) the distribution curves for these three primary processes may be used not only to describe the mixture relations of normal vision when used all three together, but also, by disregarding the one or the other of the "red" and "green" curves, the two most common types of partial color-blindness may be described.

Now it is not necessary to assume that one distribution curve must be disregarded to account for partial color-blindness; this is merely the simplest assumption.[30] Furthermore, the König Grund-Empfindungs-Curven are not the only set of curves which satisfy the two conditions given; they are merely the simplest set which satisfy them; that is, the Maxwell triangle which they yield approaches as closely as possible to the spectrum locus as is consistent with the criteria just mentioned. Other sets of distribution curves satisfying the two conditions satisfied in the simplest possible way by the König curves would yield Maxwell triangles whose sides departed to a greater or less extent from the spectrum locus. Now if it should be desired to choose the particular triangle of these possible triangles whose sides represent the colors of the maximum saturation possible, it would be found that so far little evidence has been recorded to guide the choice. It is rather certain, however, that any triangle which touches the spectrum locus is not the right choice, because it is an accepted fact that the colors of the spectrum viewed by a chromatically rested eye are not as saturated as the colors of the spectrum viewed by an eye adapted to the complementary color. But in all the triangles used by those proposing the ratio, $L_a/L_m$, as of fundamental importance, the spectrum locus does touch at least one side of the triangle. Hence our knowledge of the true physiological primaries, though much too incomplete to determine them, is sufficient to make certain that the coordinate systems thus far used for the ratio, $L_a/L_m$, fail to give that ratio a single vestige of theoretical importance.

It may be concluded, then, that the interest attached to the ratio, $L_a/L_m$, like that attached to the ratio, $L_h/L_m$, is a purely practical

[30] For the general form of dichromasy, see H. v. Helmholtz, Physiol. Optik, 2d ed., Leipsig, Voss. pp. 458–462; 1896.

one; either ratio for any choice of $L_x$, $L_y$, $L_z$ will serve as one of three variables by which color stimuli may be classified according to the colors they evoke in the visual mechanism defined by $\rho$, $\gamma$, $\beta$. The special case of $L_h/L_m$ when $L_x$, $L_y$, $L_z$ are so chosen that it is a ratio of luminosities is, of course, of considerably greater interest than any other special case of either $L_h/L_m$ or $L_a/L_m$, because this ratio alone can be evaluated by direct measurement.[31] To this special case, therefore, it has seemed expedient to give the name "colorimetric purity," $p$; and since this ratio serves all the purposes served by any other special case of either $L_h/L_m$ or $L_a/L_m$, there would seem to be little justification for paying any further attention to them. When the computation of colorimetric purity was not well understood it was natural that allied entities whose derivation seemed simpler should spring up. It should be evident, however, that methods of computation of colorimetric purity are now available which compare so favorably in simplicity with any of those for any of the allied entities that the argument of mere ease of computation is no longer applicable.

It has become customary in some quarters, however, (Runge, Rösch) [32] to speak of purity and saturation (Reinheit and Sättigung) interchangeably and to remark that purity (or saturation) has been computed in four different ways. If we adopt the view that at least these four special cases deserve attention, it would seem conducive to clarity in discussion to adopt separate names for them. Terms proposed by Priest [33] for these four special cases, together with an indication of their derivation, appear in Table 1. It will be noted that, according to this terminology, the most important of the concepts, colorimetric purity, would be renamed "spectral brightness purity." In spite of the fact that previous writers have frequently applied the name "saturation" ("Sättigung") to one or another of these concepts, the suggested terminology does not include that term. This conforms to the recommendations of the O. S. A. committee on colorimetry,[34] which reserve the term "saturation" to apply to the response to a color stimulus rather than to the stimulus itself. The concepts thus far dealt with may be taken as stimulus terms, because they merely serve to group together stimuli which evoke identical colors regardless of the chromatic condition of the visual mechanism.

If, however, we could compute from the trilinear coordinates of a color a number, $s$, satisfying the following conditions:

(a) $s = 0$ for an achromatic color,

(b) $s_1 = s_2$ for two colors appearing to be equally saturated,

then it would seem reasonable to name this number the "saturation," in which case there would be at hand a quantitative means of evaluating the concept "saturation" which is now only qualitatively defined. Now both the ratios, $L_a/L_m$ and $L_h/L_m$, satisfy condition (a), providing the heterogeneous stimulus whose color is represented at the point $(x_w, y_w, z_w)$ be so chosen as to evoke an achromatic color; but neither $L_a/L_m$ nor $L_h/L_m$ has been shown to satisfy condition (b). In fact, the failure of one special case of one of these

[31] See footnote 9, p. 831.
[32] See footnote 18, p. 834.
[33] Letter to Dr. M. Pirani, Aug. 21, 1929.
[34] L. T. Troland, Report of Committee on Colorimetry for 1920–21, J. Opt. Soc. Am. and Rev. Sci. Inst., vol. 6, pp. 531–538; 1922.

ratios, $L_a/L_m$, for $L_x = L_y = L_z$ (Exner's "Sättigung," or basic excitation purity referred to the König fundamentals as primaries) to satisfy condition (b) has been proven experimentally by Seitz.[35] Other special cases which have been proposed do not differ from this case in such a way as to repair this deficiency; and, indeed, we may conclude from Seitz's work that no special case of either ratio satisfies condition (b) at all satisfactorily; so neither of these ratios deserve the name "saturation."

A number which may prove to satisfy fairly closely both conditions (a) and (b) is $\int_0^p (dE/dp)dp$, where $p$ is the colorimetric purity [36] of the stimulus and $dE/dp$ is the reciprocal of the least difference in purity perceptible as a function of purity when the change in purity is made at constant brightness along some specified path on the color triangle.[37] $\int_0^p (dE/dp)dp$ may be described as the number of least perceptible differences between a color whose stimulus is of purity, $p$, and the color of the same brilliance whose stimulus is of purity, $p = 0$. Since the saturation of a color depends nearly if not quite as much on the momentary condition of the visual mechanism as upon the stimulus itself, it is evident that we may expect the number, $\int_0^p (dE/dp)dp$, to satisfy conditions (a) and (b) only when the visual mechanism is in such a state that the stimulus for which $p = 0$ evokes an achromatic color. Since the saturation of the color evoked by a stimulus is dependent on the brightness as well as the purity of the stimulus, it is further evident that we may hope for $\int_0^p (dE/dp)dp$ to satisfy condition (b) only for the brightness to which $dE/dp$ applies. Whether with these restrictions this number really may be made to satisfy conditions (a) and (b) and deserves the name "saturation" is a matter which is plainly of considerable complexity; it has not yet been decided. An extensive quantity of experimental data has yet to be amassed before this definition of "saturation" may be more than tentatively accepted.

Before this definition, if acceptable, can be of convenient application, a way must be found of computing $dE/dp$ from the trilinear coordinates of the color whose stimulus has a purity, $p$; but at least a start has already been made in this direction. Schrödinger [38] has derived an expression for $dE/dp$ from theoretical ground; but this expression apparently has not yet been checked against experimental results. Judd [39] has discovered empirically an expression which agrees fairly well with such experimental results as are available, but the agreement is not sufficiently striking to lead to the belief that the expression is of more than temporary theoretical interest.

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[35] W. Seitz, Über die Definition der Sättigung einer Farbe nach Helmholtz und Exner und über das Ostwaldsche Farbensystem, Zs. f. Sinnesphysiol., II, vol. 54, pp. 146–158; 1922; or see Phys. Zs., vol. 23, pp. 297–301; 1922.

[36] We define this number in terms of $p$, but the definition could be equally well given by substituting for $p$ any special case of either $L_a/L_m$ or $L_h/L_m$.

[37] For instance, at constant dominant wave length as adopted in defining this number by Jones and Lowry (L. A. Jones and E. M. Lowry, Retinal Sensibility to Saturation Differences, J. Opt. Soc. Am. and Rev. Sci. Inst., vol. 13, pp. 25–34; 1926), who identified it with "saturation," or at constant hue, or along such a path that this number is a minimum, as adopted by Schrödinger in defining "Sättigung" (E. Schrödinger, Müller-Pouillets Lehrb. d. Physik, 2d ed., vol. 2, pp. 555-558; 1926). Schrödinger makes the not implausible assumption that this path coincides exactly with the path of constant hue. In general, there is quite a definite difference between the path of constant dominant wave length and the path of constant hue, but we need not here choose between them for the purpose of this definition, because there is no experimental evidence that the number defined on the one basis would differ at all importantly from that defined on the other.

[38] E. Schrödinger, Grundlinien einer Theorie der Farbenmetrik im Tagessehen, Ann. d. Physik (4), vol. 63, pp. 483–520; 1920; see also footnote 37, p. 840.

[39] D. B. Judd, Purity and Saturation, A Saturation Scale for Yellow, J. Opt. Soc. Am. and Rev. Sci. Inst., vol. 14, p. 470; 1927; Saturation of Colors Determined from the Visual Response Functions, J. Opt. Soc. Am. and Rev. Sci. Inst., vol. 16, p. 115; 1928.

It is evident, therefore, that our knowledge of vision still falls considerably short of a general quantitative evaluation of the response entity, saturation.

Table 1.—*Terminology suggested by Priest*

| Quantity to be named | | | |
|---|---|---|---|
| $L_a/L_m$ for $L_x=L_y=L_z$ | $L_h/L_m$ for $L_x=L_y=L_z$ | $L_a/L_m$ for $\bar{x}_h L_x+\bar{y}_h L_y+\bar{z}_h L_z$ equal to the luminosity of the equal-energy spectrum | $L_h/L_m$ for $\bar{x}_h L_x+\bar{y}_h L_y+\bar{z}_h L_z$ equal to the luminosity of the equal-energy spectrum |
| DESCRIPTION | | DESCRIPTION | |
| Hueful fraction of excitation relative to the sides of the triangle. | Hueful fraction of excitation relative to the spectrum. | Hueful fraction of brightness relative to the sides of the triangle. | Hueful fraction of brightness relative to the spectrum. |
| SUGGESTED TERM | | SUGGESTED TERM | |
| Basic excitation purity. | Spectral excitation purity. | Basic brightness purity. | Spectral brightness purity. |

Washington, D. C., May, 1931.