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# Exact Inductance Equations for Rectangular Conductors With Applications to More Complicated Geometries

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Exact equations are given for the calculation of the self-inductance of rectangular conductors and of the mutual inductance between combinations of parallel filaments, thin tapes and rectangular conductors. A general procedure is also given for calculating the self-inductance of complicated geometries by dividing the geometry into simple elements whose inductances can be calculated. This general procedure is valid for conductors having nonuniform, as well as uniform current densities.

## 1. Introduction

It is the purpose of this paper to give a number of exact equations for the self- and mutual inductance of rectangular conductors, and also to show how these equations may be combined to obtain exact inductance solutions for more complicated geometries. These equations were derived during a recent investigation of thin tapes as possible high frequency inductance standards.

A number of approximate equations exist for calculating the inductance of rectangular conductors [1–5].[1] The usual method is to calculate the mutual inductance between two filaments spaced a distance apart equal to the geometric-mean-distance [6] of the conductor or conductors. This method assumes that the length is much greater than the other dimensions. However, when the length is no larger than ten times the next largest dimension, the error from this method may be as large as several percent. It is for those cases where a high degree of accuracy is needed that the equations in this paper have been derived.

## 2. Mutual Inductance Calculations

In general, the mutual inductance between two conductors will be a function of the current distribution in each conductor. A conductor having a constant cross-sectional area [2] along its length may be thought of as a bundle of parallel filaments, each having a cross-sectional area $dA$ and carrying a current $JdA$. The current density, $J$, is assumed to be constant along the length of each filament but to vary from filament to filament. The mutual inductance, $M$, between two conductors having constant cross-sectional areas $A_1$ and $A_2$ and carrying currents $I_1$ and $I_2$ may be derived from energy considerations and is

$$M=\frac{1}{I_1 I_2}\int_{A_1}\int_{A_2} M_{12}J_1J_2dA_1dA_2, \tag{1}$$

where $M_{12}$ is the mutual inductance between a filament carrying a current $J_1dA_1$ in the first conductor and a filament carrying a current $J_2dA_2$ in the second conductor. Since it is assumed that the current is constant along the length of each filament, the mutual inductance between

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[1] Figures in brackets indicate the literature references at the end of this paper.
I.e., tne cross-section dimensions do not change.

any two of the filaments is given by Neumann's formula,

$$M_{12}=0.001 \int_{l_1} \int_{l_2} \frac{\vec{dl_1} \cdot \vec{dl_2}}{r}. \tag{2}$$

The distance $r$ is between two elements of length $dl_1$ and $dl_2$ which are along two filaments having total length $l_1$ and $l_2$, respectively. For dimensions in centimeters, $M_{12}$ is in microhenrys. Since inductance cannot be calculated or specified except for a closed circuit, the integrals of (2) are usually over a closed path. However, the self- and mutual inductance of parts of a closed circuit (such as those to follow) may be calculated, provided that the total inductance of the circuit includes the contribution from each part.

If the current density is constant throughout each conductor, (1) reduces to

$$M=\frac{1}{A_1A_2} \int_{A_1} \int_{A_2} M_{12} dA_1 dA_2 \tag{3}$$

which is independent of the current in either conductor and a function of only their dimensions. Using (2) and (3), the mutual inductance between any two conductors having constant cross-sectional areas and uniform (but not necessarily equal) current densities may be evaluated.

Results obtained from (2) and (3) are quite useful even though the current densities are not uniform. In many cases, the conductors may be subdivided into elements small enough so that each element may be considered as having a uniform current density. The mutual inductances between the smaller elements are then properly summed to get the total mutual inductance. Results from (2) and (3) are also useful when the cross-sectional area of a conductor is not uniform along its length. In this case the conductor is subdivided into sections, each of which has a constant cross-sectional area along its length. The mutual inductances between sections are then properly summed to get the total mutual inductance. The technique of subdividing conductors into smaller elements whose inductance can be calculated is developed at the end of this paper.

### 2.1. Mutual Inductance Between Parallel Filaments

The first step in calculating the mutual inductance between parallel tapes or bars is to calculate the mutual inductance between parallel filaments using Neumann's formula. The mutual inductance, $M_f$, between two parallel filaments of length $l_1$ and $l_2$ spaced in any relative position is

$$M_f=0.001 \left[ z \ln (z+\sqrt{z^2+\rho^2}) - \sqrt{z^2+\rho^2} \right]^{l_3-l_1,\, l_3+l_2}_{l_2+l_3-l_1,\, l_3} (z) \tag{4}$$

where

$$\left[ f(z) \right]^{s_1,\, s_3}_{s_2,\, s_4} (z) \equiv \sum_{k=1}^{4} (-1)^{k+1} f(s_k).$$

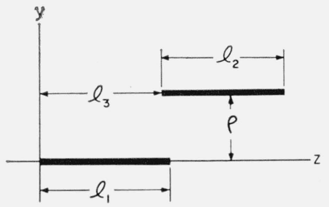


FIGURE 1. *Two parallel filaments whose mutual inductance is given by eq (4).*

The dimensions are defined in figure 1. If the left end of $l_2$ is to the left of the $y$-axis, the value of $l_3$ will be negative. The value of $M_f$ is in microhenrys for dimensions in centimeters. This one equation covers all possible positions of two parallel filaments. It is given in its expanded form for special cases elsewhere in the literature [1, 7].

### 2.2. Mutual Inductance Between a Thin Tape and a Filament

The mutual inductance, $M_{tf}$, between a filament and a thin tape parallel to the filament such as shown in figure 2 is obtained by putting $M_f$ into (3) and integrating over the width of the thin tape.

$$M_{tf}=\frac{1}{a}\int_0^a M_f dx, \tag{5}$$

where $\rho^2=P^2+(E-x)^2$ in $M_f$.

Since $z$ in $M_f$ is not a function of $x$ (that is, the limits of integration $s_1$, $s_2$, $s_3$, and $s_4$ are independent of $x$), it is unnecessary to expand $M_f$, and the two terms inside of the brackets of (4) can be integrated as they are. This integration gives

$$M_{tf}=\frac{0.001}{a}\left[\left[\frac{z^2-P^2}{2}\ln\left(x+\sqrt{x^2+P^2+z^2}\right)+xz\ln\left(z+\sqrt{x^2+P^2+z^2}\right)\right.\right.$$
$$\left.\left.-Pz\,\mathrm{Tan}^{-1}\frac{xz}{P\sqrt{x^2+P^2+z^2}}-\frac{x}{2}\sqrt{x^2+P^2+z^2}\right]_{E-a}^{E}(x)\right]_{l_3+l_2-l_1,\ l_3}^{l_3-l_1,\ l_3+l_2}(z) \tag{6}$$

where

$$\left[\left[f(x,z)\right]_{q_2}^{q_1}(x)\right]_{s_2,\ s_4}^{s_1,\ s_3}(z)\equiv\sum_{i=1}^{2}\sum_{k=1}^{4}(-1)^{i+k}f(q_i,s_k).$$

The positive directions for $E$, $P$, and $l_3$ are taken as the positive directions of the $x$, $y$, and $z$ axes, respectively. If the back end of the filament is in any octant other than that shown in figure 2, some or all of the values of $E$, $P$, and $l_3$ will be negative. In this equation and in those following, the principal value of the inverse tangent is to be used.

### 2.3. Mutual Inductance Between Two Thin Tapes

To obtain the mutual inductance, $M_t$, between two parallel thin tapes of zero thickness, such as shown in figure 3, $M_f$ is integrated over all filaments in both tapes.

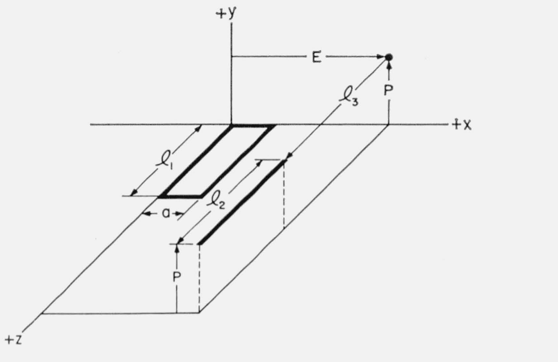


FIGURE 2. *A filament parallel to a thin tape whose mutual inductance is given by eq (6).*

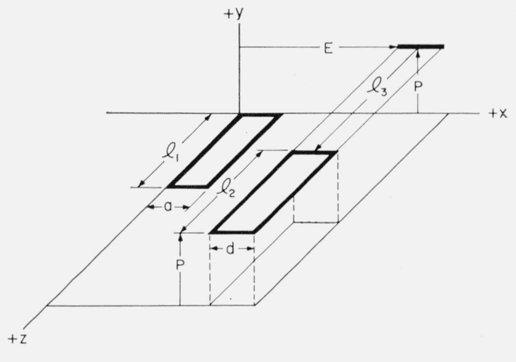


FIGURE 3. *Two parallel thin tapes whose mutual inductance is given by eq (8).*

$$M_t=\frac{1}{ad}\int_E^{E+d}\int_0^a M_f dx_1 dx_2 \tag{7}$$

where $\rho^2=P^2+(x_2-x_1)^2$ in $M_f$. This integration gives

$$M_t=\frac{0.001}{ad}\Bigg[\Bigg[\frac{x^2-P^2}{2}z\ln\left(z+\sqrt{x^2+P^2+z^2}\right)+\frac{z^2-P^2}{2}x\ln\left(x+\sqrt{x^2+P^2+z^2}\right)$$

$$-\frac{1}{6}(x^2-2P^2+z^2)\sqrt{x^2+P^2+z^2}-xPz\,\mathrm{Tan}^{-1}\frac{xz}{P\sqrt{x^2+P^2+z^2}}\Bigg]_{E+d-a,\,E}^{E-a,\,E+d}(x)\Bigg]_{l_3+l_2-l_1,\,l_3}^{l_3-l_1,\,l_3+l_2}(z), \tag{8}$$

where

$$\left[\left[f(x,z)\right]_{q_2,\,q_4}^{q_1,\,q_3}(x)\right]_{s_2,\,s_4}^{s_1,\,s_3}(z)\equiv\sum_{i=1}^{4}\sum_{k=1}^{4}(-1)^{i+k}f(q_i,s_k).$$

The distance $E$ is measured from the $yz$ plane to the left edge of the second tape. If the back left corner of the second tape is in any octant other than that shown in figure 3, some or all of the values of $E$, $P$, and $l_3$ will be negative.

The mutual inductance, $M_{t\perp}$, between two thin tapes whose axes are parallel but whose widths are perpendicular to one another such as shown in figure 4 is

$$M_{t\perp}=\frac{1}{ac}\int_P^{P+c}\int_0^a M_f dx dy=\frac{1}{c}\int_P^{P+c}M_{tf}dy, \tag{9}$$

where $P$ in $M_{tf}$ has been replaced by the variable $y$. This integration gives

$$M_{t\perp}=\frac{0.001}{ac}\Bigg[\Bigg[\Bigg[\left(\frac{z^2}{2}-\frac{y^2}{6}\right)y\ln\left(x+\sqrt{x^2+y^2+z^2}\right)+\left(\frac{z^2}{2}-\frac{x^2}{6}\right)x\ln\left(y+\sqrt{x^2+y^2+z^2}\right)$$

$$+xyz\ln\left(z+\sqrt{x^2+y^2+z^2}\right)-\frac{xy}{3}\sqrt{x^2+y^2+z^2}-\frac{z^3}{6}\mathrm{Tan}^{-1}\frac{xy}{z\sqrt{x^2+y^2+z^2}}$$

$$-\frac{x^2z}{2}\mathrm{Tan}^{-1}\frac{yz}{x\sqrt{x^2+y^2+z^2}}-\frac{y^2z}{2}\mathrm{Tan}^{-1}\frac{xz}{y\sqrt{x^2+y^2+z^2}}\Bigg]_{E-a}^{E}(x)\Bigg]_{P}^{P+c}(y)\Bigg]_{l_3+l_2-l_1,\,l_3}^{l_3-l_1,\,l_3+l_2}(z) \tag{10}$$

where

$$\left[\left[\left[f(x,y,z)\right]_{q_2}^{q_1}(x)\right]_{r_2}^{r_1}(y)\right]_{s_2,\,s_4}^{s_1,\,s_3}(z)\equiv\sum_{i=1}^{2}\sum_{j=1}^{2}\sum_{k=1}^{4}(-1)^{i+j+k+1}f(q_i,r_j,s_k).$$

The distance $P$ is measured from the $xz$ plane to the bottom of the second tape.

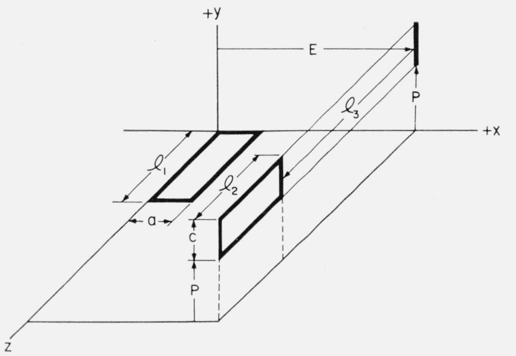


FIGURE 4. *Two thin tapes whose axis are parallel but whose widths are perpendicular.*
The mutual is given by eq (10).

## 2.4. Mutual Inductance Between a Bar and a Filament

The mutual inductance, $M_{bf}$, between a rectangular bar and a filament parallel to the bar such as shown in figure 5 is obtained by integrating $M_{tf}$ over all thin tapes in the bar. Equation (3) gives

$$M_{bf}=\frac{1}{ab}\int_0^b\int_0^a M_f dx dy=\frac{1}{b}\int_{P-b}^{P} M_{tf} dY, \tag{11}$$

where $P$ in $M_{tf}$ is now replaced by the variable $Y\equiv P-y$. The integration of (11) is identical to the integration of (9) except for the limits of integration and the constant.

$$M_{bf}=\frac{0.001}{ab}\left[\left[\left[f(x,y,z)\right]_{E-a}^{E}(x)\right]_{P-b}^{P}(y)\right]_{l_3+l_2-l_1,\ l_3}^{l_3-l_1,\ l_3+l_2}(z), \tag{12}$$

where $f(x, y, z)$ is the expression within the inner brackets of (10). The distance $P$ in (12) is measured from the $xz$ plane to the filament.

## 2.5. Mutual Inductance Between Rectangular Bars

To obtain the mutual inductance, $M_b$, between two rectangular bars, $M_t$ is integrated over all thin tapes in both bars.

$$M_b=\frac{1}{bc}\int_P^{p+c}\int_0^b M_t dy_1 dy_2, \tag{13}$$

where $P$ in $M_t$ is replaced by the variable $(y_2-y_1)$. Since the $x$ and $z$ limits of integration on $M_t$ are independent of $y$, the four terms inside the brackets of (8) may be integrated as they are. After a lengthy integration, (13) yields an exact expression for the mutual inductance between two parallel rectangular bars spaced in any relative position.

$$\begin{aligned}M_b=\frac{0.001}{abcd}\Bigg[\Bigg[\Bigg[&\left(\frac{y^2z^2}{4}-\frac{y^4}{24}-\frac{z^4}{24}\right)x\ln\left(\frac{x+\sqrt{x^2+y^2+z^2}}{\sqrt{y^2+z^2}}\right)+\left(\frac{x^2z^2}{4}-\frac{x^4}{24}-\frac{z^4}{24}\right)y\ln\left(\frac{y+\sqrt{y^2+z^2+x^2}}{\sqrt{z^2+x^2}}\right)\\&+\left(\frac{x^2y^2}{4}-\frac{x^4}{24}-\frac{y^4}{24}\right)z\ln\left(\frac{z+\sqrt{z^2+x^2+y^2}}{\sqrt{x^2+y^2}}\right)+\frac{1}{60}(x^4+y^4+z^4-3x^2y^2-3y^2z^2-3z^2x^2)\sqrt{x^2+y^2+z^2}\\&-\frac{xyz^3}{6}\operatorname{Tan}^{-1}\frac{xy}{z\sqrt{x^2+y^2+z^2}}-\frac{xy^3z}{6}\operatorname{Tan}^{-1}\frac{xz}{y\sqrt{x^2+y^2+z^2}}\\&-\frac{x^3yz}{6}\operatorname{Tan}^{-1}\frac{yz}{x\sqrt{x^2+y^2+z^2}}\Bigg]_{E+d-a,\ E}^{E-a,\ E+d}(x)\Bigg]_{P+c-b,\ P}^{P-b,\ P+c}(y)\Bigg]_{l_3+l_2-l_1,\ l_3}^{l_3-l_1,\ l_3+l_2}(z)\end{aligned} \tag{14}$$

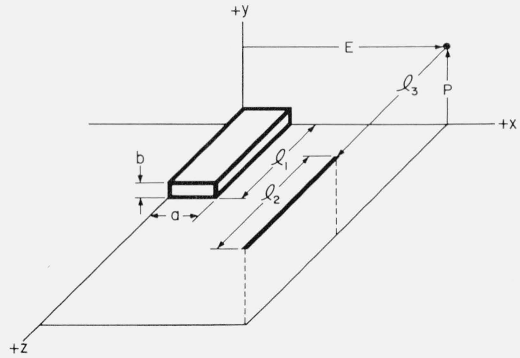


FIGURE 5. *A filament parallel to a rectangular bar whose mutual inductance is given by eq (12).*

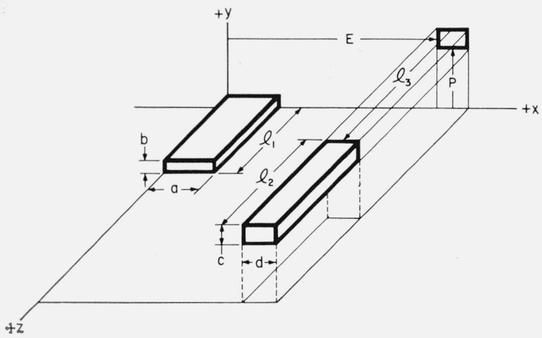


FIGURE 6. *Two parallel rectangular bars whose mutual inductance is given by eq (14).*

where $$\left[\left[\left[f(x, y, z)\right]_{q_2, q_4}^{q_1, q_3}(x)\right]_{r_2, r_4}^{r_1, r_3}(y)\right]_{s_2, s_4}^{s_1, s_3}(z) \equiv \sum_{i=1}^{4}\sum_{j=1}^{4}\sum_{k=1}^{4}(-1)^{i+j+k+1}f(q_i, r_j, s_k).$$

The dimensions are defined in figure 6. Either bar may be placed so that its back lower left corner is at the origin. The second bar may fall in any of the octants. If the back lower left corner of the second bar falls in any octant other than that shown in figure 6, some or all of the values of $E$, $P$, and $l_3$ will be negative. The distance $E$ is measured from the $yz$ plane to the left side of the second bar; $P$ is measured from the $xz$ plane to the bottom of the second bar, and $l_3$ is measured from the $xy$ plane to the back of the second bar.

The mutual inductance equations (4) through (14) have intentionally been left in their rather complicated form because it is in this form that one can most efficiently program them on a computer. For most cases, considerable effort is required to evaluate these expressions without the aid of a computer. If the dimensions vary greatly in magnitude, the positive terms may very nearly cancel with the negative terms, and care must be taken to calculate each term to an accuracy sufficient to assure the required accuracy in the result.

It should also be noted that if either $x$ or $y$ or $z$ approaches zero, all inverse tangents in (6) through (14) go to zero.[3] If any two of the variables $x$, $y$, and $z$ approach zero, all terms in (14) go to zero except the square root term.

## 3. Self-Inductance Calculations

The self-inductance of a conductor is a special case of the mutual inductance between two conductors. We can think of the self-inductance of a conductor as the mutual inductance between two identical conductors which coincide with each other. The discussion on mutual inductance calculations may then be applied to self-inductance calculations. The self-inductance, $L$, of a conductor having a constant cross-sectional area, $A$, and carrying a current $I$ is given by (1) where the two integrations are now over the same conductor.

$$L = \frac{1}{I^2}\int_A\int_A M_{12}J_1J_2dA_1dA_2. \tag{16}$$

The mutual inductance, $M_{12}$, between two filaments in the conductor is given by (2) if the current $JdA$ in each filament is constant along the length of the filament.

If the current density is constant throughout the conductor, (16) reduces to

$$L = \frac{1}{A^2}\int_A\int_A M_{12}dA_1dA_2, \tag{17}$$

which is a function only of the dimensions of the conductor.

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[3] In eqs (6) and (8), $y=P$.

For the self-inductance, $L_t$, of a thin tape of width "$a$" and length "$l$", (17) gives

$$L_t = \frac{1}{a^2} \int_0^a \int_0^a M_f \, dx_1 dx_2,$$

where $\rho^2 = (x_2 - x_1)^2$ in $M_f$. This integration gives

$$L_t = \frac{0.002}{3a^2} \left[ 3a^2 l \ln \frac{l + \sqrt{l^2 + a^2}}{a} - (l^2 + a^2)^{3/2} + 3l^2 a \ln \frac{a + \sqrt{a^2 + l^2}}{l} + l^3 + a^3 \right]. \tag{18}$$

This result may also be obtained from (8), the expression for the mutual between two thin tapes, by setting $d = a$ and $E = P = l_3 = 0$.

The self-inductance, $L_b$, of a rectangular bar of sides "$a$" and "$b$" and length "$l$" is most readily obtained from (14), the expression for the mutual inductance between rectangular bars, by setting $d = a$, $c = b$, and $E = P = l_3 = 0$. These substitutions give

$$L_b = \frac{0.001}{a^2 b^2} \left[ \left[ \left[ f(x, y, z) \right]_{0,\,0}^{-a,\,a} (x) \right]_{0,\,0}^{-b,\,b} (y) \right]_{0,\,0}^{-l,\,l} (z), \tag{19}$$

where $f(x, y, z)$ is the expression within the inner brackets of (14). It can be shown that $f(x, y, z)$ is an even function of $x$, of $y$, and of $z$. This result allows one to use the absolute value of the limits of integration, thereby reducing (19) to

$$L_b = \frac{0.008}{a^2 b^2} \left[ \left[ \left[ f(x, y, z) \right]_0^a (x) \right]_0^b (y) \right]_0^l (z), \tag{20}$$

where

$$\left[ \left[ \left[ f(x, y, z) \right]_{q_2}^{q_1} (x) \right]_{r_2}^{r_1} (y) \right]_{s_2}^{s_1} (z) \equiv \sum_{i=1}^{2} \sum_{j=1}^{2} \sum_{k=1}^{2} (-1)^{i+j+k+1} f(q_i, r_j, s_k)$$

For bars with a thickness-to-width ratio less than 0.1, the self-inductance $L_b$ may be calculated from the empirical equation

$$L_b = L_t - Kl \frac{b}{a} 10^{-3}, \tag{21}$$

where $L_t$ is calculated from (18). The values of $K$ given in table 1 are sufficiently accurate to calculate $L_b$ to an uncertainty of 1 ppm for the range covered by the table. These values were obtained by forcing (21) to give results equal to that obtained from (20).

TABLE 1. *Values of* K, $\mu$h/*cm*

| $b/a$ \ $l/a$ | 2 | 5 | 10 | 20 | 50 | 100 |
|---|---|---|---|---|---|---|
| 0.002 | 2.087 | 2.088 | 2.088 | 2.089 | 2.089 | 2.089 |
| .005 | 2.077 | 2.080 | 2.081 | 2.082 | 2.082 | 2.082 |
| .010 | 2.0633 | 2.0686 | 2.0703 | 2.0712 | 2.0717 | 2.0719 |
| .015 | 2.0508 | 2.0581 | 2.0605 | 2.0617 | 2.0624 | 2.0630 |
| .020 | 2.0392 | 2.0482 | 2.0513 | 2.0529 | 2.0538 | 2.0542 |
| .030 | 2.0176 | 2.0300 | 2.0342 | 2.0364 | 2.0376 | 2.0377 |
| .040 | 1.9978 | 2.0131 | 2.0184 | 2.0210 | 2.0226 | 2.0232 |
| .050 | 1.9792 | 1.9973 | 2.0035 | 2.0066 | 2.0085 | 2.0091 |
| .060 | 1.9616 | 1.9822 | 1.9893 | 1.9929 | 1.9950 | 1.9957 |
| .070 | 1.9448 | 1.9678 | 1.9757 | 1.9797 | 1.9821 | 1.9829 |
| .080 | 1.9287 | 1.9540 | 1.9627 | 1.9671 | 1.9697 | 1.9706 |
| .090 | 1.9132 | 1.9406 | 1.9501 | 1.9548 | 1.9577 | 1.9587 |
| .100 | 1.8983 | 1.9277 | 1.9379 | 1.9430 | 1.9461 | 1.9472 |

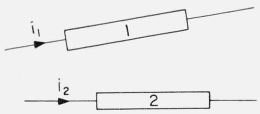


FIGURE 7. *Two arbitrary circuit elements carrying currents* $i_1$ *and* $i_2$.

## 4. Inductance of Complicated Geometries

The inductance of complicated geometries or of conductors having nonuniform current densities often can be calculated from a consideration of the magnetic energy of the conductor. Consider the two circuit elements shown in figure 7. The instantaneous energy, $w$, stored in the magnetic field created by the sinusodial currents $i_1$ and $i_2$ may be written

$$w = \tfrac{1}{2}L_1 i_1^2 + \tfrac{1}{2}L_2 i_2^2 + M_{12} i_1 i_2, \tag{22}$$

where $L_1$ and $L_2$ are the self-inductances of elements "1" and "2", respectively, and $M_{12}$ is the mutual inductance between these two elements. In (22) the currents are the instantaneous values

$$i_k = i_{k0} \sin (\omega t + \phi_k), \qquad k = 1, 2,$$

where $i_{k0}$ is the maximum value of $i_k$ and $\phi_k$ is the phase angle.

It is often more convenient to use the complex notation

$$I_1 = a_1 + jb_1, \ I_2 = a_2 + jb_2.$$

Using this notation, the average magnetic energy $W$ stored in the system of figure 7 may be written

$$W = \tfrac{1}{2}L_1 I_1 \cdot I_1 + \tfrac{1}{2}L_2 I_2 \cdot I_2 + M_{12} I_1 \cdot I_2. \tag{23}$$

It is assumed that the currents $I_1$ and $I_2$ have the same frequency but not necessarily the same phase angle. The pseudo vector notation $I_1 \cdot I_2$ has been used to take care of the difference in phase angle.

$$I_1 \cdot I_2 = a_1 a_2 + b_1 b_2 = \frac{i_{10} i_{20}}{2} \cos (\phi_1 - \phi_2).$$

Equation (23) for the average magnetic energy of two circuit elements may also be written in summation form as

$$W = \frac{1}{2} \sum_{i=1}^{2} \sum_{j=1}^{2} M_{ij} I_i \cdot I_j \tag{24}$$

where $M_{ii}$ = self-inductance of element $i$.

For a circuit having $n$ elements, the average magnetic energy of the total circuit is

$$W_n = \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} M_{ij} I_i \cdot I_j. \tag{25}$$

The equivalent inductance, $L_n$, of a closed circuit having $n$ elements is that inductance which will give the same average energy, $W_n$, when the same total current, $I_T$, is flowing through the circuit.

$$W_n = \tfrac{1}{2} L_n I_T \cdot I_T. \tag{26}$$

Solving (25) and (26) for the equivalent self-inductance gives

$$L_n = \frac{1}{I_T \cdot I_T} \sum_{i=1}^{n} \sum_{j=1}^{n} M_{ij} I_i \cdot I_j. \tag{27}$$

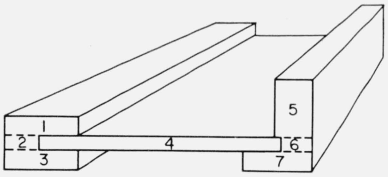


FIGURE 8. *A conductor of complicated cross-sectional area divided into elements whose inductances can be calculated.*

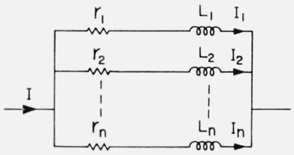


FIGURE 9.—*An equivalent circuit of the conductor shown in figure 8.*

This equation can be used to calculate not only the equivalent self-inductance of a closed circuit but also the equivalent self-inductance of part of a circuit if there is negligible interaction between the part under consideration and the rest of the circuit. The self-inductance of conductors having complicated geometries, therefore, may be calculated from (27). The procedure is to subdivide the conductors into simple geometries whose inductances can be calculated, and then sum these values with (27).

For example, let us calculate the equivalent self-inductance of the conductor shown in figure 8, assuming that there is negligible interaction between it and the rest of the circuit. There are no formulas for the self-inductance of such a geometry, so divide the conductor into elements whose inductances can be calculated. Since the self- and mutual inductance of parallel rectangular bars in any position can be calculated, divide the conductor into $n=7$ rectangular bars. The conductor now looks like the circuit in figure 9 with $n=7$ elements in parallel. The voltage drop across each element can be written

$$e=I_k r_k+j\omega \sum_{i=1}^{n} M_{ki} I_i, \tag{28}$$

where $r_k$ and $M_{kk}$ are the resistance and self-inductance of element $k$, and $M_{ki}$ is the mutual between elements $k$ and $i$. Since the desired self-inductance is independent of the value of $e$, set $e=1+j0$ for simplicity. Equation (28) gives $n$ equations which may be solved for the $n$ unknown currents $I_1$ through $I_n$. With the currents known, the equivalent inductance can be calculated from (27);

$$L=\frac{1}{I\cdot I}\sum_{i=1}^{n}\sum_{j=1}^{n} M_{ij} I_i \cdot I_j, \qquad n=7, \tag{29}$$

where, for this example,

$$I=\sum_{i=1}^{n} I_i, \qquad n=7.$$

At low frequencies (28) becomes

$$e\approx I_k r_k,$$

in which case (29) reduces to

$$L=R^2 \sum_{i=1}^{n}\sum_{j=1}^{n}\frac{M_{ij}}{r_i r_j}, \tag{30}$$

where $R$ is the total resistance of the conductor. Since the conductor is assumed to have a constant cross-sectional area along its length, each element will have a common length, $l$. The total resistance, $R$, may be calculated from

$$\frac{1}{R}=\sum_{k=1}^{n}\frac{1}{r_k}=\frac{1}{l}\sum_{k=1}^{n}\frac{A_k}{\rho_k}, \tag{31}$$

where $A_k$ is the area of element $k$ which has a resistivity $\rho_k$. Equation (30) shows that at low frequency the inductance of a conductor is not only a function of its dimensions but also of the resistivities of its different elements. When the resistivities of all elements are the same, however, (30) reduces to

$$L=\frac{1}{A^2}\sum_{i=1}^{n}\sum_{j=1}^{n}M_{ij}A_iA_j, \tag{32}$$

which is a function of only the dimensions. If the 3 sections of the conductor in figure 8 all have the same resistivity, the low frequency inductance may be calculated directly from (32) where the total area $A$ is

$$A=\sum_{i=1}^{n}A_i, \qquad n=7.$$

The procedure outlined in this example is valid for a conductor of any arbitrary constant cross-sectional area, provided of course that the conductor can be subdivided into $n$ elements whose inductances can be calculated, and that there is negligible interaction between this conductor and the rest of the circuit.

If the interaction is not negligible between the part being calculated and the rest of the circuit, the whole circuit must be considered. If there is interaction between the conductor of the previous example and, say, $m$ other conductors in the circuit, the sum from $i=1$ to $n$ in (28) becomes $i=1$ to $n+m$. There would then be $n+m$ currents to be evaluated, after which the equivalent inductance of the conductor could be calculated from (29), $i$ and $j$ going from 1 to $n$ as before.

If exact values of inductance are not required, the equivalent self-inductance often may be calculated easier using geometric-mean-distances [1–7] (abbreviated g.m.d.). It should be remembered, however, that even exact equations for the g.m.d. gives approximate values of inductance because all g.m.d. formulas assume that lengths are infinite.

## 5. Conclusions

Exact equations have been given for the mutual inductance between the following parallel conductors: a filament and a thin tape, a filament and a rectangular bar, two thin tapes, and two rectangular bars. There is no restriction on the size or spacing of the conductors. Exact equations have also been given for the self-inductance of a thin tape and of a rectangular bar. Since these equations are quite long, they have been put in a form most suited for programming on high speed computers.

These equations for simple rectangular conductors may be used to calculate the self-inductance of any other more complicated geometry if that geometry can be divided into simple rectangular elements. A general procedure for calculating the self-inductance of complicated geometries has been given. The mutual inductance between complicated geometries may also be calculated using a procedure similar to that used for self-inductance.

In deriving the equations given in this paper, the assumption was made that the current density was uniform throughout the conductor (or throughout each element in the case of complicated geometries). This restriction is not a severe one however, since in many cases the conductor may be divided into elements small enough so that the current density in each element may be considered uniform. Applications of these equations to the calculation of high frequency inductances where the current density is not uniform are given in a forthcoming paper by Brooke, Hoer, and Love [8].

## 6. References


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