A Note on a Generalized Elliptic Integral

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An expansion of

\[ \Omega_j(k) = \frac{\pi}{\Gamma(j - \frac{1}{2})} \int_0^\pi e^{-\frac{k_1}{2} e^{-\frac{1}{2} \int_0^\pi t \cdot e^{-\frac{t}{2}} \cdot I_0(k^2 t) dt} \]  

in the neighborhood of \( k^2 = 1 \) is obtained by a method based on an Abelian theorem.

We have therefore exposed \( \Omega_j(k) \) as a Laplace transform in which the coefficient in the first exponential, \( (1 - k^2)/k^2 \), approaches zero as \( k^2 \) approaches 1. We can now apply an Abelien theorem for Laplace transforms, [2], to determine the behavior of \( \Omega_j(k) \) in the neighborhood of \( k^2 = 1 \). To do this we note the asymptotic expansion

\[ e^{-\frac{1}{2} \int_0^\pi I_0(k^2 t) dt} \sim \frac{1}{\sqrt{2\pi x}} \sum_{n=0}^\infty \frac{(-1)^n}{2^n} \frac{1}{(2^n n!) \Gamma(\frac{1}{2} - n n!) \Gamma(\frac{1}{2} + n n!)} \]  

as \( x \) tends to infinity. Substituting eq (6) into eq (5) we find for the asymptotic expansion of \( \Omega_j(k) \):

\[ \Omega_j(k) \sim \sqrt{\frac{\pi}{2}} \frac{1}{k^{j+1/2}} \frac{1}{\Gamma(j + 1/2)} \left( \sum_{n=0}^{j-1} \frac{(-1)^n \Gamma(\frac{1}{2} + n)}{2^n \Gamma(\frac{1}{2} - n)} \right) \frac{(j-n-1)!}{n!} \left( \frac{k^2}{1-k^2} \right)^{-n} \]

\[ + \sum_{n=1}^{\infty} \frac{(-1)^n \Gamma(\frac{1}{2} + n)}{2^n \Gamma(\frac{1}{2} - n)} \frac{1}{(n-j)!} \left( \frac{k^2}{1-k^2} \right)^{n-j} \ln \left( \frac{1-k^2}{k^2} \right) \]  

Another useful representation for \( \Omega_j(k) \) can be obtained by noting that the Legendre function \( P_{n-1/2} (\cosh \eta) \) can be written, [3],

\[ P_{n-1/2} (\cosh \eta) = \frac{1}{\pi} \int_0^\pi \frac{d\theta}{(\cosh \eta + \sinh \eta \cos \theta)^{n+1/2}} \]
from which it follows that

\[
\Omega_j(k) = \frac{\pi}{2^{j+1}} P_{j+1/2} \left( \frac{1}{1 - k^2} \right)
\]

However, a derivation of asymptotic behavior starting with eq (9) is not as direct as the proof we have given.