Calculation of Certain Multiple Generating Functions

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This paper contains a discussion of the evaluation of generating functions of the form

\[ F({x}) = \sum_{n_1}^{\infty} \ldots \sum_{n_k}^{\infty} M_j(n_1, n_2, \ldots, n_k) x_1^{n_1} x_2^{n_2} \ldots x_k^{n_k} \]

where \( M_j(n_1, n_2, \ldots, n_k) \) is the \( j \)th largest of the integers \( (n_1, \ldots, n_k) \). An alternate technique to one proposed by Carlitz is used in the calculation.

In a recent paper Carlitz has considered the problem of evaluating the generating functions

\[ F_j({x}) = \sum_{n_1=0}^{\infty} \ldots \sum_{n_k=0}^{\infty} M_j(n_1, n_2, \ldots, n_k) x_1^{n_1} x_2^{n_2} \ldots x_k^{n_k} \]  \hspace{1cm} (1)

where \( M_j(n_1, n_2, \ldots, n_k) \) is the \( j \)th greatest of the set of integers \( \{n_j\} \), [1]. His method was essentially a combinatorial one. It is our purpose in the present note to reconsider this problem by evaluating, instead of \( F_j \), the generating function

\[ G_j({x}; s) = \sum_{n_1=0}^{\infty} \ldots \sum_{n_k=0}^{\infty} e^{-s M_j(n_1, n_2, \ldots, n_k)} x_1^{n_1} x_2^{n_2} \ldots x_k^{n_k} \]

from which it is possible to derive an expression for \( F_j \) by differentiation

\[ F_j({x}; s) = \left. \frac{\partial G_j} {\partial s} ({x}; s) \right|_{s=0} \]  \hspace{1cm} (3)

Expressions for related generating functions can also be obtained in this manner.

The principal tool in the following analysis is the identity

\[ e^{-s M_1(n_1, \ldots, n_k)} = s \int_{M_1(n_1, \ldots, n_k)}^{\infty} e^{-st} dt = s \int_{0}^{\infty} e^{-st} H(t-n_1) H(t-n_2) \ldots H(t-n_k) dt \]  \hspace{1cm} (4)

where \( H(x) \) is the Heaviside step function

\[ H(x) = 0, \quad x < 0 \]
\[ = 1, \quad x > 0. \]  \hspace{1cm} (5)

We can immediately derive the expression for \( G_1({x}; s) \) from eq (4) by multiplying the general term of this equation by \( x_1^{n_1} \ldots x_k^{n_k} \) and summing. An interchange of the orders of summation and integration is easily justified for \(|x_1|, |x_2|, \ldots |x_k| < 1\) and so we have only to evaluate the sum

\[ \sum_{n=0}^{\infty} H(t-n)x^n = \sum_{n=0}^{\left\lfloor \frac{|t|}{n} \right\rfloor} x^n = \frac{1-x^{|t|+1}}{1-x} \]  \hspace{1cm} (6)

\[ ^{1} \text{L. Carlitz, The generating function for max}(n_1, \ldots, n_k), \text{Portugaliae, Mathematica 21, 201 (1962).} \]
With this result we find
\begin{equation}
G_1(\{x\}; s) = \frac{s}{(1-x_1)(1-x_2)\ldots(1-x_k)} \int_0^\infty e^{-st}(1-x_1^{[t+1]})(1-x_2^{[t+1]})\ldots(1-x_k^{[t+1]})dt.
\end{equation}

In order to evaluate this expression we need the following Laplace transform
\begin{equation}
s \int_0^\infty e^{-st}a^{[t]}dt = \frac{a(1-e^{-s})}{1-ae^{-s}} = (e^s - 1)\left(\frac{1}{1-ae^{-s}} - 1\right)
\end{equation}
where \(a\) is a constant. Equations (7) and (8) together yield:
\begin{equation}
G_1(\{x\}; s) = \frac{1}{(1-x_1)\ldots(1-x_k)} \left\{ 1 - (e^s - 1) \sum_r \left( \frac{1}{1-x_r e^{-s}} - 1 \right) 
+ (e^s - 1) \sum_r \sum_{r_2} \left( \frac{1}{1-x_{r_1} x_{r_2} e^{-s}} - 1 \right) 
- (e^s - 1) \sum_r \sum_{r_2} \sum_{r_3} \left( \frac{1}{1-x_{r_1} x_{r_2} x_{r_3} e^{-s}} - 1 \right) + \ldots \right\}
\end{equation}

Now let us consider the evaluation of the other \(G_j\). We require, for these generating functions, identities similar to that of eq (4). The preceding analysis suggests that we try the sum of integrals
\begin{equation}
s \int_0^\infty e^{-st}[[1-H(t-n_1)]H(t-n_2)\ldots H(t-n_k)]H(t-n_1)[1-H(t-n_2)]H(t-n_3)\ldots H(t-n_k) 
+ \ldots + H(t-n_1)H(t-n_2)\ldots [1-H(t-n_k)] dt = U(n_1, n_2, \ldots n_k; s)
\end{equation}
for the evaluation of \(G_2\). By repeatedly using eq (4) we find
\begin{equation}
U(n_1, \ldots n_k; s) = [e^{-sM_1(n_2, \ldots n_k)} - e^{-sM_1(n_1, n_3, \ldots n_k)}]
+ [e^{-sM_1(n_1, n_3, \ldots n_k)} - e^{-sM_1(n_1, \ldots n_{k-1})} - e^{-sM_1(n_1, \ldots n_k)}].
\end{equation}
Assume now that the maximum of the \(\{n_j\}\) is \(n_1\). Then since \(n_1\) appears in each of the terms in all of the brackets except the first, these brackets must be equal to zero. Furthermore we must have
\begin{equation}
M_1(n_2, \ldots n_k) = M_2(n_1, \ldots n_k).
\end{equation}
Hence we find
\begin{equation}
G_2(\{x\}; s) - G_1(\{x\}; s) = \sum_{n_1} \ldots \sum_{n_k} U_1(n_1, \ldots n_k; s)x_1^{n_1}\ldots x_k^{n_k}.
\end{equation}
The function \(G_1(\{x\}; s)\) has already been calculated in eq (9) so that \(G_2\) can be determined to be
\begin{equation}
G_2(\{x\}; s) = \sum_{r=1}^k G_1(\{x\} - x_r; s) + (1-k)G_1(\{x\}; s)
\end{equation}
where \(\{x\} - x_r\) is the set \(\{x\}\) less the element \(x_r\). Similar expressions can be obtained for all of the \(G_j\) by starting from a sum of integrals similar to eq (4), each of which has \(j\) factors \(1-H(t-n_r)\) with the remaining factors being of the form \(H(t-n_r)\). In this way we can write the recurrence relation
\begin{equation}
G_{j+1}(\{x\}; s) = \sum_{r=1}^k G_j(\{x\} - x_r; s) + (1-k)G_j(\{x\}; s).
\end{equation}
The function $F_1(x)$ is now obtained from eqs (3) and (9) in a straightforward manner:

$$F_1(x) = \frac{1}{(1-x_1) \ldots (1-x_k)} \left\{ 1 - \sum_{r=1}^{k} \frac{x_r}{1-x_r} + \sum_{r_1 \geq r_2} \frac{x_{r_1} x_{r_2}}{1-x_1 x_{r_2}} - \ldots \right\}$$

$$+ (-1)^{k-1} \frac{x_1 x_2 \ldots x_k}{1-x_1 x_2 \ldots x_k} \right\}$$

(16)

and others of the $F_j$ can be derived by recurrence

$$F_{j+1}(x) = \sum_{r=1}^{k} F_j(x - x_r) + (1-k)F_1(x).$$

(17)

By these techniques we may derive Carlitz’s result

$$F_j(x) = \sum_{s=j}^{k} (-1)^{s+j} \left( \frac{s-1}{j-1} \right) U_{ks}$$

(18)

where

$$U_{ks} = \frac{1}{(1-x_1)(1-x_2) \ldots (1-x_k)} \int \frac{x_1 x_2 \ldots x_s}{1-x_1 x_2 \ldots x_s}$$

(19)

where $Jf(x_1, x_2, \ldots, x_s)$ is the symmetric function determined by $f(x_1, x_2, \ldots, x_s)$.

Similar techniques can also be used for the calculation of generating functions like

$$H_j(x) = \sum_{n_1=0}^{\infty} \sum_{n_k=0}^{\infty} M_j(n_1, n_2, \ldots, n_k) x_1^{n_1} \ldots x_k^{n_k} = \sum_{n_1=0}^{\infty} \sum_{n_k=0}^{\infty} [M_j(n_1, n_2, \ldots, n_k)] x_1^{n_1} \ldots x_k^{n_k}.$$  

(20)

When $\lambda$ is an integer $H_j$ can be expressed as a derivative of $G_j$. However, one can calculate $H_j$ for any $\lambda$ by the same technique as we have used for $\lambda=1$. Define a function $G_j^{(\lambda)}(x; s)$ analogous to that in eq (4) except that $M_j(n_1, \ldots, n_k)$ is replaced by $M_j(n_1^\lambda, \ldots, n_k^\lambda)$. Then eq (4) remains valid except that each $n_j$ is to be replaced by $n_j^\lambda$ and the succeeding steps lead, in the case $j=1$, to the expression

$$G_1^{(\lambda)}(x; s) = \frac{s}{(1-x_1) \ldots (1-x_k)} \int_0^\infty \frac{1}{e^{-u(1-x_1^{[\lambda]+1})} \ldots (1-x_k^{[\lambda]+1})} dt.$$  

(21)

The Laplace transform of $a^{[\lambda]+1}$ is

$$s \int_0^\infty e^{-st} a^{[\lambda]+1} dt = a \left[ 1 - (1-a) \sum_{n=0}^{\infty} a^n e^{-(n+1)\lambda} \right]$$

(22)

which leads to

$$G_1^{(\lambda)}(x; s) = \frac{1}{(1-x_1) \ldots (1-x_k)} \left\{ 1 - \sum_{j=1}^{k} x_j \left[ 1 - (1-x_j) \sum_{n=0}^{\infty} x^n e^{-(n+1)\lambda} \right] \right\}$$

$$+ \sum_{j>r} \sum_{x_r} \left[ 1 - (1-x_r) \sum_{n=0}^{\infty} (x_r^n e^{-(n+1)\lambda} - \ldots \right]$$

(23)

which reduces to eq (9) when $\lambda=1$.

The method suggested in this note can be generalized to deal with any functional of the form $M_j(\varphi(n_1), \varphi(n_2), \ldots, \varphi(n_k))$ providing that $\varphi(n)$ is a monotone increasing function which tends to infinity with $n$. It can also be used to calculate Laplace transforms rather than generating functions.

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