Effect of Air Drag on the Motion of a Filament Struck Transversely by a High-Speed Projectile

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The effect of air drag on the motion of a filament struck transversely is treated theoretically. The air drag is shown to produce curvature in the transverse wave formed by the impact, to increase the speed of the transverse-wave front, and to increase the strain in the filament. The theory is applied to the case of a nylon yarn impacted transversely at 189 meters/second. The calculated radius of curvature of the yarn was 1.1 meter, which agreed well with 1.2-meter radius obtained by experiment. The calculated effects of air drag on the strain and distance traveled by the transverse-wave front 601 microseconds after impact were small.

When a long filament is subjected to a high-velocity transverse impact, strain waves and transverse waves are generated which travel outwards along the filament away from the point of impact. According to theory [1-6] the velocities and strain distributions in these waves can be calculated provided that the stress-strain curve of the filament under the impact conditions is known. Recently [7-9] the inverse problem has been considered for a textile yarn, i.e., the calculation of the stress-strain curve if the velocities or the strain distribution resulting from an impact are known. Unfortunately, the theory is incomplete for this practical application because it neglects the effect of air drag on the yarn. In this paper the theory of transverse impact is extended to include the effect of air drag, and the errors introduced by its neglect are estimated for a typical case.

1. Behavior of the Filament Neglecting Air Drag

Since the effect of air drag on the motion of the filament is treated as a perturbation on the motion without air drag, the theory neglecting air drag is first reviewed. The impact produces longitudinal-strain waves that propagate outward in each direction from the point of impact. These strain waves produce a strain $\varepsilon_0$ in the filament. In the region between the strain-wave fronts, material of the filament is set into motion longitudinally toward the point of impact with a velocity of $W_0$. This material is taken up by a tent-shaped wave of transverse motion, as shown in figure 1. The filament material between the projectile and the transverse-wave front moves vertically with the velocity $V$ of the projectile, and the wave front travels in a longitudinal direction with velocity $U_0$. The section of filament between the projectile and transverse-wave front is a straight line; the solid and dotted lines in figure 1 show the filament at two different times after impact.

The longitudinal velocity of the filament material, $W_0$, is given by

$$ W_0 = \frac{1}{m} \int_0^{e_0} \sqrt{\frac{dT}{de}} \, de $$

(1)

where $m$ is the mass per unit length of the unstrained filament, $e_0$ is the strain, and $T$ is the tension required to produce a strain $e$.

The transverse-wave front velocity is

$$ \lambda_0 = \sqrt{\frac{T_0}{m(1+\varepsilon_0)}} $$

(2)

with respect to the unstrained filament, i.e., with respect to a Lagrangian coordinate system. $T_0$ is the tension in the filament resulting from the impact. These velocities are related to the impact velocity $V$ by the equation

$$ V^2 = 2W_0(1+\varepsilon_0)\lambda_0 - W_0^2. $$

(3)

An expression for the transverse-wave front velocity $U_0$ in laboratory coordinates will now be

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1 Figures in brackets indicate the literature references at the end of this paper.
derived. Suppose ink marks 1 cm apart are placed on the unstrained filament before the filament is impacted. After impact, the transverse-wave front velocity in cm/sec is \( \lambda_0 \) with respect to the unstrained filament, so the transverse-wave front will pass \( \lambda_0 \) ink marks per second. However, these marks are now \((1+\epsilon_0) \) cm apart since the strain in advance of the transverse wave is \( \epsilon_0 \). The transverse-wave front, therefore, traverses a distance \( \lambda_0(1+\epsilon_0) \) cm on the filament per second. Since the filament in advance of the transverse wave has a velocity \( W_0 \) opposite to the direction of motion of the transverse-wave front, a stationary observer detects a transverse-wave front velocity of

\[
U_0 = \lambda_0(1+\epsilon_0) - W_0. \tag{4}
\]

The slope, \( \psi_0 \), of the filament behind the transverse-wave front is seen from figure 1 to be given by

\[
\tan \psi_0 = V/U_0.
\]

Equations (1) to (4) describe the motion of the impacted filament in the absence of air drag. These equations, however, must be modified slightly in certain cases. For instance, the above theory is based on the assumption that the stress-strain curve is always concave downwards, whereas many stress-strain curves for textile yarns have concave upward portions. The modification required in this case has been discussed by Smith et al. [10]. At very high impact velocities, part of the region of variable strain in the strain wave occurs in the wake of the transverse-wave front instead of in advance of it as assumed in the discussion above. The modifications required in this rarely occurring case [6] are not considered in this paper.

### 2. Effect of Air Drag

The force on the filament between the projectile and transverse-wave front due to air drag may be expressed as a component perpendicular and a component parallel to the filament. However, the force component parallel to the filament is small, so only the force component perpendicular to the filament will be considered.\(^2\)

This force produces curvature in the section of filament between the projectile and transverse-wave front, thereby increasing the strain and tension in the filament. This increase in strain produces strain waves of small amplitude that are propagated past the transverse-wave front and along the filament. Since the velocities of these strain waves are usually much greater than the velocity of the transverse wave, the strain, \( \epsilon \), in the filament between the projectile and transverse wave is nearly uniform at any time, but varies with time. Thus the Lagrangian velocity of the transverse-wave front is given by

\[
\lambda = \sqrt{\frac{T}{m(1+\epsilon)}} \tag{5}
\]

\(^2\) These force components are compared later in the paper.
With this approximation for the tension, eq (8) becomes

\[ W = \frac{1}{\sqrt{m}} \int_0^1 \sqrt{\frac{dT}{de}} de + \frac{1}{\sqrt{m}} \int_0^1 \sqrt{\frac{dT}{de}} de \]

\[ = W_0 + \sqrt{K/m} (e - e_0). \tag{12} \]

Equations (9) and (11) are now substituted in eqs (5), (6), (7), and (12) to express \( \lambda, S, X, \) and \( W \) as series in \( t \). Thus eq (5) becomes

\[ \lambda = \frac{1}{\sqrt{m}} \sqrt{\frac{T_0 + Ka_0 + K a_1 t + \ldots}{1 + e_0 + a_0 + a_1 t + \ldots}} \]

\[ = \sqrt{\frac{T_0}{m(1 + e_0)}} \sqrt{\frac{1 + ka_0 + ka_1 t + \ldots}{1 + a_0 + a_1 t + \ldots}} \]

\[ = \lambda_0 \sqrt{\frac{1 + ka_0}{1 + a_0}} \sqrt{\frac{1 + ka_1 t + \ldots}{1 + a_1 t + \ldots}} \tag{13} \]

where \( k = K/T_0 \).

Dividing and taking the square root of the series yields

\[ \lambda = \lambda_0 \sqrt{1 + ka_0} \sqrt{1 + e_0 + a_0 + a_1 t + 0t^2} \]

\[ = \lambda_0 \sqrt{1 + ka_0} \sqrt{1 + e_0 + a_0 + a_1 t + 0t^2} \tag{14} \]

where \( 0t^2 \) represents terms of the second and higher powers of \( t \). The following equations are similarly derived:

\[ S = \lambda_0 \sqrt{1 + ka_0} \sqrt{1 + e_0 + a_0 + a_1 t + 0t^2} + \frac{3 + (1 + e_0 + 4a_0)k}{4\sqrt{1 + e_0 + a_0 + a_1 t + 0t^2}} \lambda_0 \sqrt{1 + e_0 + a_0 + a_1 t + 0t^2} \tag{15} \]

\[ X = \lambda_0 \sqrt{1 + ka_0} \sqrt{1 + e_0 + a_0 + a_1 t + 0t^2 - W_0 - \sqrt{\frac{K}{m}} a_0} \]

\[ + \left[ \frac{1 + (1 + e_0 + 2a_0)k}{4\sqrt{1 + e_0 + a_0 + a_1 t + 0t^2}} \sqrt{1 + e_0 + a_0 + a_1 t + 0t^2} \right] a_1 t + 0t^2. \tag{16} \]

In order to solve for \( a_0, a_1, \) etc., a relationship must be found between \( S \) and \( X \). This relationship is derived by considering the configuration of the filament between the projectile and transverse-wave front.

The configuration of the filament exposed to air drag is shown in figure 2. Let \( s \) be the distance measured along the filament between the projectile and a segment \( ds \) of the filament.

The air drag on the segment \( ds \) is composed of a frictional component due to the viscosity of the air and a dynamic pressure component due to uneven air pressure around the diameter of the yarn. The frictional component of the air drag is only a few percent of the dynamic pressure component [11] and so will be neglected.\(^3\) By the “cross-flow principle” [11] the dynamic pressure component of the air drag is directed perpendicularly to every point of the filament. Figure 3 shows the forces on a segment \( ds \) of the filament. The equilibrium equations for the horizontal and vertical force components of the segment are

\[ \frac{d(T \cos \psi)}{ds} - F \sin \psi = 0 \]

and

\[ \frac{d(T \sin \psi)}{ds} + F \cos \psi = 0, \]

where \( F \) is the force per unit length on the filament due to air drag. These conditions simplify to

\[ \frac{dT}{ds} = 0, \tag{17} \]

and

\[ \frac{d\psi}{ds} = -\frac{F}{T}. \tag{18} \]

Equation (17) states that the tension is constant along the filament.

The force due to air drag is proportional to the square of the velocity of the segment, to \( \cos^2 \varphi \) where \( \varphi \) is the slope angle of the segment, and to the length of the segment. Since the case of small air drag is

\(^3\) If the slope of the filament is close to 90°, the pressure component of the drag becomes small.
being considered, the velocity may be approximated by the projectile velocity $V$ and the slope $\varphi$ by the slope $\varphi_0$ that the filament would have if no air drag were present. Therefore, the force on the element $ds$ is

$$FdS=qV^2 \cos^2 \varphi \, ds. \quad (19)$$

Since $F$ and $T$ are independent of $s$ and therefore constant along the yarn, eq (18) shows that the filament forms an arc of a circle of radius $T/F$. The angle subtended by the arc $S$ at the center of this circle is given by the length of the arc divided by the radius $SF/T$, where $S$ is the length of the filament between the projectile and transverse-wave front. The length of the corresponding chord of the circle is then $(2T/F) \sin (SF/2T)$ and is also $\sqrt{X^2+V^2t^2}$ by figure 2. Therefore,

$$(2T/F) \sin (SF/2T)= \sqrt{X^2+V^2t^2}. \quad (20)$$

Since $F$ is very small compared to $T$, the sine may be approximated by relatively few terms of its infinite series. Squaring both sides of the equation then gives

$$X^2+V^2t^2=S^2-\frac{F^2S^4}{12T^2} \quad (20)$$

(neglecting terms higher than $F^2$), which is the desired relationship between $X$ and $S$ to evaluate the coefficients $a_0, a, \text{etc.}$

Equations (15), (16), and (3) for $S$, $X$, and $V$ are substituted in eq (20) to yield the following identity in $t$.

$$\left[ \lambda_0 \sqrt{1+\epsilon_0} \sqrt{1+\epsilon_0+a_0 \sqrt{1+k a_0}} - W_0 - \sqrt{K/m} a_0 \right]$$

$$+ \left( \frac{1+(1+\epsilon_0+a_0 \sqrt{1+k a_0})}{\sqrt{1+\epsilon_0+a_0 \sqrt{1+k a_0}}} \right) a_1 t/4$$

$$+ 0t^2 \right]^2 + [2W_0(1+\epsilon_0)\lambda_0 - W_0^2] - \left[ \lambda_0 \sqrt{1+\epsilon_0} \right.$$

$$\sqrt{1+\epsilon_0+a_0 \sqrt{1+k a_0}} + \frac{3+(1+\epsilon_0+4a_0 \sqrt{1+k a_0})}{4 \sqrt{1+\epsilon_0+a_0 \sqrt{1+k a_0}}} a_0 \lambda_0$$

$$\left. \sqrt{1+\epsilon_0+a_0 \sqrt{1+k a_0}} \right] = 0^2=0. \quad (21)$$

Since this expression is zero for all values of $t$, the constant part and the coefficient of $t$ must each be zero. Simplification of the constant part gives

$$\left[ \frac{K a_0}{2m} - \frac{\sqrt{K/m} \lambda_0 \sqrt{1+\epsilon_0} \sqrt{1+\epsilon_0+a_0 \sqrt{1+k a_0}}} \right.$$

$$+ W_0 \sqrt{K/m} a_0 + W_0 \lambda_0 \sqrt{1+\epsilon_0} \left[ \sqrt{1+\epsilon_0} \right.$$

$$\left. \sqrt{1+\epsilon_0+a_0 \sqrt{1+k a_0}} \right] = 0. \quad (22)$$

By inspection $a_0=0$. Therefore, by eq (9), the strain just after impact, when $t$ is very small, is seen to be $\epsilon_0$, which is the strain for no air drag. This is reasonable since the length of filament in transverse motion just after impact is too short to be affected by air drag.

Substituting $a_0=0$ in eq (21), calculating the coefficient of $t$ and simplifying gives

$$[(\lambda_0(1+\epsilon_0) - W_0)\lambda_0 + \lambda_0(1+\epsilon_0)k - 2\sqrt{K/m}]$$

$$- \lambda_0^2(1+\epsilon_0)(3+k+ke_0) = 0 \quad (23)$$

This equation gives $a_1=0$ since the coefficient does not vanish.

Since $a_0$ and $a_1$ are zero, the terms in higher powers of $t$ must be evaluated in order to calculate the effect of air drag. Thus eqs (9) and (11) become

$$e = \epsilon_0 + a_2 t^2 + a_3 t^3 + \ldots$$

$$T = T_0 + K a_2 t^2 + K a_3 t^3 + \ldots \quad (24)$$

The quantities $X$ and $S$ in eqs (15) and (16) are recalculated to higher powers in $t$ and substituted in eq (20) to give

$$\frac{3}{4} \left[ (4\lambda_0 + 2\sqrt{K/m}+kw_0)(1+\epsilon_0) \right.$$ 

$$+ W_0(1-2\sqrt{K/m}/\lambda_0) a_2 t^2 - \frac{1}{12T_0} F^2 \lambda_0^2(1+\epsilon_0)^4 t^2$$

$$+ \frac{1}{4} [(6\lambda_0 + 2\sqrt{K/m}+kw_0)(1+\epsilon_0) \right.$$

$$\left. + W_0(1-2\sqrt{K/m}/\lambda_0)] a_3 t^3 + 0t^4 = 0. \right.$$ 

Putting the coefficient of $t^2$ equal to zero yields

$$a_2 = \frac{F^2(1+\epsilon_0)\lambda_0^2/(4T_0^3)}{(4\lambda_0 + 2\sqrt{K/m}+kw_0)(1+\epsilon_0)+W_0(1-2\sqrt{K/m}/\lambda_0)} \quad (25)$$

and putting the coefficient of $t^3$ equal to zero yields

$$a_3 = 0. \quad (26)$$

Thus the strain in the filament is given by

$$e = \epsilon_0 + a_2 t^2 + 0t^3. \quad (27)$$

Substituting this expression into eq (7) and integrating gives for the distance traveled by the transverse wave

$$X = U_0 t + [\lambda_0 k(1+\epsilon_0) + \lambda_0 - 2\sqrt{K/m}] a_2 t^3/6 + 0t^5 \quad (28)$$

where $a_2$ is given by eq (25).

4. Comparison With Experiment

This theory is compared with an impact study of a nylon yarn by Smith et al. [12]. Their multiframe

photograph of a high tenacity nylon yarn, impacted transversely by a projectile with a velocity of 189 m/sec is shown in figure 4. The yarn is photographed before impact, 201 μsec after impact, and every 50 μsec thereafter. Only half of the yarn was photographed in order to show greater detail. The projectile is seen along the right-hand edge of the photograph.

The curvature of the yarn in the transverse wave due to air drag can be seen in figure 4. The radius of curvature of the yarn for the first 10 exposures was measured and found to be approximately constant at 1.2 m.4

The linear density of this nylon yarn is 92.7 tex or $9.27 \times 10^{-5}$ kg/m. Its tension-strain curve, measured at a rate of strain of 4800 percent/sec by Smith et al. [9], is shown in figure 5. Using this curve, the strain $\varepsilon_0$ without air drag was computed from eqs (1), (2), and (3). For the impact velocity $V=189$ m/sec, $\varepsilon_0=2.031$ percent.

In order to calculate the effect of air drag, the drag coefficient, $F$, is calculated using eq (19). The value of $q$ in this equation is defined by Hoerner [11] as

$$q = \frac{1}{2} \rho d C_D$$

where $\rho$ is the density of air, $d$ is the diameter of the

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4 After the first 10 exposures, the reflected strain wave interacts with the transverse wave and produces curvature in the yarn in addition to the curvature produced by air drag [4].
yarn, 0.7 mm, and $C_D$ is the dimensionless drag coefficient. $C_D$ is a slowly varying function of the Reynolds number of the flow. For the case considered, the Reynolds number is $8.8 \times 10^5$ and the corresponding drag coefficient, $C_D$, for a cylinder is 1.10. However, the structure of a yarn affects the drag coefficient. Hoerner [11] results yield an average drag coefficient of 1.09 at a Reynolds number of $10^6$ for two metal cables that have a structure similar to a textile yarn. Since the drag coefficient of a cylinder is 1.19 for a Reynolds number of $10^5$, the drag coefficient of the yarn is given by

$$C_D = \frac{1.09}{1.19} \times 1.10 = 0.99.$$

By eqs (29) and (19) the force acting on the yarn is $F = 11.46$ newtons/m. The radius of curvature of the yarn, $T/F$, is calculated to be 1.1 m in good agreement with the experimental value of 1.2 m.

By eqs (25), (27), and (28) the strain, $e$, and distance, $X$, transversed by the transverse wave may be calculated. For 601 $\mu$sec after impact, the strain is 2.12 percent and the distance, $X$, traveled by the transverse-wave front is 19.07 cm, compared with a strain of 2.03 percent and an $X$ of 19.06 cm, calculated by neglecting air drag. Thus the effect of air drag on the strain in the yarn and the distance traveled by the transverse wave is small for these conditions, although it produces noticeable curvature in the yarn.

The effect of air drag increases slowly with the velocity of impact. For example, an impact of 300 m/sec on the same yarn produces a strain of 4.44 percent and a distance $X$ of 24.24 cm without air drag, and for air drag a strain of 4.54 percent and distance $X$ of 24.38 cm at 601 $\mu$sec after the impact. Thus the effect of air drag is small for the examples considered.

5. References