Graphical Determination of Radio Ray Bending in an Exponential Atmosphere

C. F. Pappas, L. E. Vogler, and P. L. Rice

(August 11, 1960)

This paper presents a simple engineering method for calculating the amount of bending undergone by a radio ray passing through an exponential model atmosphere. For any initial takeoff angle and for values of the surface refractivity ranging from 200 to 450, the bending angle $\tau$ may be determined as a function of height above the earth's surface, using a few graphs and a few calculations. Indications of the accuracy of the method are given at the end of the paper.

1. Introduction

The course of a radiofrequency electromagnetic wave traveling through the atmosphere is altered by variations of the atmospheric index of refraction, $n$. These variations, due to changes in vapor pressure, air pressure and temperature, are extremely complex in detail; however, mathematical models of refraction can be constructed which represent an average picture of the variations. This paper considers an "exponential" model (the CRPL Exponential Reference Atmosphere [1]), in which the refractivity $(n-1) \times 10^6$ decreases exponentially with height, causing the radio wave to be bent away from its initial direction. The amount of bending is measured by the refraction angle, $\tau$, and is important in such problems as the accurate determination by radar of the range and height of flying objects, the location of extra-terrestrial radio noise sources in radio astronomy, and the analysis of radio communication systems.

Mathematically, $\tau$ may be expressed in the following integral form [2,3]

$$\tau = - \cos \theta_0 \int_{s_0}^{r_0} (dn/n)(n_r/n_0)^2 - \cos^2 \theta_0^{1/2}$$

where $n$ is the atmospheric index of refraction, $n_0$ is the value of $n$ at the surface of the earth and $r$, $r_0$, and $\theta_0$ are defined in figure 1. The CRPL Exponential Reference Atmosphere is characterized by an index of refraction of the form

$$n = 1 + (n_0 - 1)e^{-c_0 h}$$

where $c_0$ is the decay constant and $h$ is the altitude above the surface of the earth. For this model atmosphere eq (1) is not integrable in closed form; it can be expanded in series, but the resulting expression is quite complicated for hand calculations. A numerical integration method has been used to compute values of $\tau$ by Bean and Thayer; these are listed in reference 1. This method is only practical through the use of a large scale computer.

It might be noted that when $\theta_0$ is large $\tau$ may be calculated by a formula which is quite simple and very accurate [4]:

$$\tau = \left( \frac{n_0 - 1}{n_0} \right) \cot \theta_0 \left( 1 - e^{-c_0 h} \right) \text{ (radians)}.$$ (2)

However, for small $\theta_0$ no simple expression is available for calculations; thus, an engineering method was developed to provide a quick and practical means to obtain $\tau$ in this case. This method has the added advantage over the tables in ref [1] in that the $N_s$ of $\tau$ is not limited to those listed.

![Figure 1. Geometry of radio-ray refraction.](image-url)
2. Calculation of $\tau_a$

The approximation of $\tau$ is denoted by $\tau_a$; the formula is given by

$$\tau_a = f(n_0) \cos \theta_0 q e^{B(n_0)} + pC$$

(3)

where the terms are explained as follows:

- $f(n_0) =$ the bending approximation in milliradians.
- $f(n_0) =$ read from figure 2 for a given $N_s$. If more accuracy is desired, this value can be computed by

$$f(n_0) = \left[\left(\frac{\pi}{2}\right)\left(\frac{n_0-1}{n_0}\right)(k-1)\right]^{1/2} \times 10^3.$$

- $n_0 =$ the index of refraction at the earth's surface.
- $k =$ the effective earth's radius factor.
- $N_s =$ the surface refractivity.
- $\theta_0 =$ the initial elevation angle expressed in milliradians.
- $h_m =$ the height above the surface of the earth in meters.
- $q =$ read from figure 3 for a given $h_m$ and $\theta_0$.
- $B =$ read from figure 4 or 5 for a given $h_m$ and $\theta_0$.
- $p =$ a correction factor for $N_s$, which is read from figure 6 for a given $N_s$.
- $C =$ a height correction factor which is obtained from figure 7 for a given $h_m$.

Two examples of the computation of $\tau_a$ are included to illustrate the use of the $\tau_a$ formula. One example is for an $N_s$ of 252.9 and the second is an example using an $N_s$ of 404.9 in which the $pC$ correction factor has an effect.

**Example 1.** (Calculation of $\tau_a$ with $N_s \leq 344.5$)

- Given: $N_s = 252.9$
- $\theta_0 = 40$ milliradians
- $h_m = 500$ meters

- Find: $\tau_a = f(n_0) \cos \theta_0 q e^{B(n_0)} + pC$

$$f(n_0) = 10.06$$
$$\cos \theta_0 = .99920$$
$$q = .0385$$
$$B = .00077$$
$$p = 0$$
$$pC = 0$$

$$e^{Bf(n_0)} + pC = 1.0077$$

$$\tau_a = .39 \text{ milliradians}$$

$$\tau = .38 \text{ milliradians}^2$$

**Example 2.** (Calculation of $\tau_a$ with $N_s > 344.5$)

- Given: $N_s = 404.9$
- $\theta_0 = 200$ milliradians
- $h_m = 30$ meters

- Find: $\tau_a = f(n_0) \cos \theta_0 q e^{B(n_0)} + pC$

$$f(n_0) = 24.70$$
$$\cos \theta_0 = .98007$$
$$q = .00472$$
$$B = .00117$$

Obtained by numerical integration.

---

**Figure 2.** $f(n_0)$ versus $N_s$.  
**Figure 3.** $q$ versus $h_m$.  

$^2$ Obtained by numerical integration.

176
3. Derivation of $\tau_a$ formula

By plotting $\tau$ versus $h$ for many different values of $\theta_0$ and $n_0$, it was decided that the simplest form that could be assumed for $\tau$ to obtain the accuracy desired was

$$\tau = f(n_0) \cos \theta_0 e^{A + Bf(n_0)}$$  \hspace{1cm} (4)

The range of $N_e$ considered lies between 200 and 450 since values outside this range rarely occur in actual practice.

Using two values of $N_e$, 200 and 344.5, and $\tau$'s obtained by the numerical integration procedure of ref [1], a least squares fit of

\[\begin{array}{ll}
Bf(n_0) &= .0289 \\
p &= .30 \\
C &= -.125 \\
pC &= -.0375 \\
F(n_0) + pC &= -.0086 \\
\tau_a &= .0113 \text{ milliradians} \\
\tau &= .0113 \text{ milliradians}^2
\end{array}\]
\[ \ln \left( \frac{\tau}{f(n_0) \cos \theta_0} \right) \]

was made to obtain values of \( A \) and \( B \) for a given \( h_m \) and \( \theta_0 \).

Values of \( e^A \), denoted by \( q \), were computed and graphed for \( h_m \) and given \( \theta_0 \) (see fig. 3). The \( B \) values were also plotted versus \( h_m \) for given \( \theta_0 \) (see figs. 4 and 5). It may be noted that \( q \) and \( B \) are not graphed for height values less than 10 m; at the upper range of height, a limit for \( q \) and \( B \) is approached and reached for a given \( \theta_0 \).

Using the \( A \) and \( B \) values it was found that for an \( N_s \) greater than 344.5 a correction factor was needed to modify \( q e^{B(n_0)} \) (or \( e^{A+B(n_0)} \)) so that the \( \ln (\tau/f(n_0) \cos \theta_0) \) and, consequently, the \( \tau \) error were within an acceptable range. The height correction \( C \) was obtained from the difference between \( \ln (\tau/f(n_0) \cos \theta_0) \) and \( q e^{B(n_0)} \) for an \( N_s \) of 450 and plotted for graphical use (see fig. 7). Through further calculations and comparisons the relationship of \( N_s \) to the height correction \( C \) was determined for \( N_s \) greater than 344.5, and less than 450. This relationship was the basis for the \( N_s \) correction \( p \) which was plotted versus \( N_s \) (see fig. 6). Inclusion of this additional correction element results in an expression \( q e^{B(n_0)+pC} \) for an \( N_s \) greater than 344.5. Since \( p \) equals zero when \( N_s \) is less than or equal to 344.5,

\[ \tau_a = f(n_0) \cos \theta_0 \quad q e^{B(n_0)+pC} \]

becomes the general form for the simplified calculation of \( \tau \).

### 4. Accuracy of \( \tau_a \) Method

In checking out the simplified calculation method various \( \tau_a \) were compared with values of the CRPL exponential reference atmosphere \( \tau \) for the same \( h_m \) and \( \theta_0 \). Of the values computed \( \tau_a \) showed the smallest absolute error at the lower heights and smaller \( N_s \). The largest errors calculated for \( N_s \) of 450, the maximum absolute error being 0.53 milliradians, with a maximum relative error of 6.8 percent.

Below is a table of computed values which gives indication of the range of absolute and percent error for several values of \( N_s \). It will be noted that error values for an \( N_s \) of 200 and 344.5 were not included in this list since these values of \( N_s \) were used for the least squares fit and were considered to have less error than the \( N_s \) listed.

Only error values for \( \tau_a \) greater than 1.0 milliradian were (arbitrarily) included in table 1. For \( \tau_a \) less than 1.0 milliradian the absolute error is quite low, but the percent error can be high since \( \tau_a \) is so small. This can give a somewhat distorted picture since a small \( \tau_a \) may have an error of only 0.0001 milliradian and still be in error by greater than 3 percent.

### Table 1: Range of Error for \( \tau_a > 1.0 \) milliradian

<table>
<thead>
<tr>
<th>( N_s )</th>
<th>Range of absolute error</th>
<th>Range of percent error</th>
</tr>
</thead>
<tbody>
<tr>
<td>252.9</td>
<td>0.0006 to 0.0739</td>
<td>0.01% to 2.11%</td>
</tr>
<tr>
<td>333.0</td>
<td>0 to 0.1918</td>
<td>0 to 1.91%</td>
</tr>
<tr>
<td>377.2</td>
<td>0.013 to 0.1567</td>
<td>0.01% to 1.56%</td>
</tr>
<tr>
<td>401.9</td>
<td>0 to 0.2539</td>
<td>0 to 1.86%</td>
</tr>
<tr>
<td>450.0</td>
<td>0.0024 to 5533</td>
<td>0.22% to 6.79%</td>
</tr>
</tbody>
</table>

### 5. Explanation of Symbols

\( B \) = figures 4 and 5.

\( C \) = height correction factor; figure 7,

\( c_e \) = decay constant; see ref [1],

\( f(n_0) \) = figure 2,

\( \frac{\pi}{2} \left( \frac{n_0-1}{n_0} \right) (k-1) \times 10^3 \)

\( h \) = altitude above the surface of the earth,

\( h_m \) = altitude above the surface of the earth in meters,

\( k \) = effective earth’s radius factor

\( n = \frac{n_0}{n_0 - r_0} \frac{1}{n_0 - 1} \)

\( n \) = atmospheric index of refraction

\( 1 + (n_0 - 1) e^{-e_{CRPL} \cdot \text{Exponential Reference Atmosphere}} \)

\( n_0 \) = index of refraction at the earth’s surface

\( n(h = 0) \)

\( N_s \) = surface refractivity

\( (n_0 - 1) \times 10^6 \)

\( p \) = \( N_s \) correction factor; figure 6,

\( \phi \) = angle at center of the earth (see fig. 1)

\( \phi = \theta + \tau - \theta_0 \)

\( q \) = figure 3,

\( r \) = radial distance from the center of the earth,

\( r_0 \) = distance from the center to the surface of the earth,

\( \tau = \text{bending} \)

\( = -\cos \theta_0 \int_{n_0}^{n_1} \frac{dn}{n} \left[ \frac{\left( \frac{nr}{n_r} \right)^2 - \cos^2 \theta_0}{n_r^2} \right]^{-1/2} \)

\( \tau_a = \text{bending approximation in milliradians,} \)

\( = f(n_0) \cos \theta_0 \quad q \quad e^{B(n_0)+pC} \)

\( \theta \) = local elevation angle

\( = \cos^{-1} \left( \frac{n_r \tau_a \cos \theta_0}{n_r} \right) \)

\( \theta_0 \) = initial elevation or takeoff angle.
6. References


(Paper 65D2–116)