A Symmetric Continuous Poker Model

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Beginning with von Neumann, mathematicians concerned with the rational analysis of conflict situations have realized that investigation of accurate or simplified versions of common card games leads to techniques and insights applicable to situations of military or economic interest. In the present paper, a symmetric poker model one stage more complicated than the original von Neumann game is solved. There are two bet levels, a, b, and an ante of 1 unit $(a>b>1)$; no raises are permitted. The game has a unique optimal strategy, which forbids bluffing on a low hand. The limiting case $b=1$ is shown to yield the von Neumann model.

1. Introduction

The technical literature of the mathematical theory of games of strategy can be roughly subdivided into (i) *general* papers, which deal with concepts, existence theorems and computational methods relevant for broad classes of games, and (ii) *specific* papers, in which the solutions of particular games are determined. Although many of the studies in the second category have dealt with problems suggested by economic or military conflicts, e.g. [1, 2][2], a substantial fraction have dealt with accurate or simplified versions of common card games, the best known example being von Neumann's analysis of a simple poker model ([3], sec. 19.4 to 19.10). Bridge and baccarat have been treated [4, 5], but poker has remained the favorite topic for these studies [3, 6 to 10], perhaps because the poker models investigated have presented just the right degree of challenge ("not easy, but not impossibly hard") to the ingenuity of the analyst. Of course the techniques and insights developed in these investigations are then applicable to situations of greater practical importance.

The present paper deals with a symmetric poker game one stage more complicated than von Neumann's original model. The game is *continuous* in the sense that the difficulties [6] due to the enormous but finite number of possible hands have been avoided by assuming (as von Neumann did) an infinite continuum of possible hands. The (more difficult) asymmetric version has also been solved by the authors; the results will appear elsewhere [11].

2. Description of the Game

The rules of the game are as follows. The two players R and S first *ante* 1 unit. They then receive independent random numbers (*hands*) from the interval [0, 1]. Each player knows his own hand, but not that of his opponent. The players act simultaneously, each either *dropping*, betting an additional

$b-1$ units, or betting an additional $a-1$ units. If both drop, then play ends and no payment is made. If both bet $b-1$ units or both bet $a-1$ units, then play ends and the player with the higher hand [3] wins the ante and bet of his opponent. If one player bets $a-1$ units and the other bets $b-1$, then the latter can either *fold* (in which case he loses his ante and previous bet to his opponent) or *see* [4] by betting an additional $a-b$ units (in which case the player with the higher hand wins the ante and previous bet or bets of his opponent); in either case play ends. Naturally we assume $a>b>1$.

Each player has four courses of action for any given hand. These are (1) to drop, (2) to bet $b-1$ units with the intention of folding if the opponent bets $a-1$ units, (3) to bet $b-1$ units with the intention of seeing if the opponent bets $a-1$ units, and (4) to bet $a-1$ units. Figures 1 and 2 show the algebraic payoff from player S to player R for any choices of courses of action by the players.

A *strategy* (for either player) is a quadruple $r(z)=\{r_i(z)\}$ $(i=1,2,3,4)$ of functions defined for $0\leq z\leq 1$. Since $r_i(z)$ represents the probability [5] that the player will choose the ith course of action when his hand is z, the restrictions

$$r_i(z)\geq 0, \qquad r_1(z)+r_2(z)+r_3(z)+r_4(z)=1$$

are imposed. The "smoothness" conditions on these functions are (i) that they be integrable, (ii) that if $r_i(z')>0$ then, for any interval J of which z' is an interior [6] point,

$$\int_J r_i(t)dt>0,$$

and finally [7] (iii) that

$$\int_0^z [br_2(t)-ar_4(t)]dt=0 \qquad \text{on} \quad [(b-1)/b, (a-1)/a]$$

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[3] The possibility of equal hands has zero probability and so can be disregarded.
[4] This possibility distinguishes our model from the case $m=3$ of the version studied in part II of [6].
[5] More precisely, $r_i(z)$ is a probability density function.
[6] The words "interior" and "open" are to be understood relative to the interval [0,1].
[7] This very weak but *ad hoc* condition is precisely what is needed to prove (b) and (c) of lemma 8.

implies

$$br_2(z) - ar_4(z) = 0 \quad \text{on} \quad [(b-1)/b, (a-1)/a]$$

It is convenient to define, for any $r(z)$,

$$R_i = \int_0^1 r_i(t)dt.$$

For any $r(z)$, $G_i(z)$ will denote the expected payoff to player R, using $r(z)$, against a fixed hand z and a fixed course of action i of his opponent. Thus the expected payoff function of the game is given by

$$K(r,r^*) = \int_0^1 \sum r_i^*(z)G_i(z)dz \qquad (2.1)$$

The explicit formulas [8] for the $G_i(z)$ are readily found from figures 1 and 2:

R \ S	1	2	3	4
1	0	−1	−1	−1
2	1	b	b	−b
3	1	b	b	a
4	1	b	a	a

FIGURE 1

R has the higher hand.

R \ S	1	2	3	4
1	0	−1	−1	−1
2	1	−b	−b	−b
3	1	−b	−b	−a
4	1	b	−a	−a

FIGURE 2

S has the higher hand.

$$G_1(z) = \int_0^z [r_2(t) + r_3(t) + r_4(t)]dt$$
$$+ \int_z^1 [r_2(t) + r_3(t) + r_4(t)]dt, \quad (2.2)$$

$$G_2(z) = \int_0^z [-r_1(t) - br_2(t) - br_3(t) + br_4(t)]dt$$
$$+ \int_z^1 [-r_1(t) + br_2(t) + br_3(t) + br_4(t)]dt, \quad (2.3)$$

$$G_3(z) = \int_0^z [-r_1(t) - br_2(t) - br_3(t) - ar_4(t)]dt$$
$$+ \int_z^1 [-r_1(t) + br_2(t) + br_3(t) + ar_4(t)]dt, \quad (2.4)$$

$$G_4(z) = \int_0^z [-r_1(t) - br_2(t) - ar_3(t) - ar_4(t)]dt$$
$$+ \int_z^1 [-r_1(t) - br_2(t) + ar_3(t) + ar_4(t)]dt. \quad (2.5)$$

[8] Note that the $G_i(z)$'s are shown by these formulas to be *continuous* functions.

3. Plan of Attack

It would not be very instructive simply to state the solution of the game and demonstrate its correctness. Instead, a slightly polished version of the process actually used to solve the game will be presented. This is done in sections 4 and 5, in which it is proved (subject to the plausible assumption stated in sec. 4) that the only possible optimal strategy is the one obeying the conditions (a) to (d) of lemma 8, which deal with the behavior of the strategy on three intervals $[0,g)$, (g,g'), and $(g',1]$ which might be thought of as the sets of *low* hands, *intermediate* hands, and *high* hands. It is then relatively easy to verify that this particular strategy is indeed the unique optimal strategy; this is done in section 6, primarily by inspection of the expected payoff function (2.1). In section 7 the limiting case $b=1$ is dealt with, and this game is found to have a unique optimal strategy obtained [9] by "passing to the limit" in the optimal strategy for $b>1$. The von Neumann poker game of sections 19.4 to 19.10 of [3] is shown to arise as a "reduced game" from this limiting case.

In section 4 it is shown that $r_3(z)=0$ on $[0,1]$ for every optimal $r(z)$. The points g, g' are located, and the unique optimal $r(z)$ is isolated in section 5. The arguments of these sections are essentially all applications of the following *domination criterion*:

The strategy $r(z)$ is optimal only if the following is true: For each z, i for which $G_i(z)$ (as a function of i) does not assume its minimum, $r_i(z)=0$.

The correctness of this criterion follows from the smoothness condition *(ii)* and the fact that $r(z)$ is optimal only if

$$K(r,r) = \min_{r^*} K(r,r^*).$$

4. Elimination of One Course of Action

The elimination is effected as follows.

LEMMA 1. *For every optimal $r(z)$, $r_3(z)=0$ on $[0,1]$.*

Otherwise, for some optimal $r(z)$, $R_3>0$ and $r_3(z)>0$ for some $z>0$. If z' is the least upper bound of the set of all such z, then by (2.4) and (2.5)

$$G_4(z') - G_3(z') = -(a-b)R_3 - 2b\int_{z'}^1 r_2(t)dt < 0,$$

so that by continuity and the domination criterion $r_3(z)=0$ in a neighborhood of z', contradicting the definition of z'.

The following plausible assumption (which will be verified in sec. 6) is now made:

ASSUMPTION: *For every optimal $r(z)$, $r_4(1)>0$ (and thus $R_4>1$).*

[9] This is not quite the case for the asymmetric version ;see[11].

5. Derivation of the Necessary Conditions

Since application of the domination criterion requires examining the signs of the pairwise differences of the $G_i(z)'s$, the formulas for these differences (based on (2.2) to (2.5) and the condition $r_3(z)=0$ on $[0,1]$) will first be given:

$$G_2(z)-G_1(z)=(b-1)-bR_1-2b\int_0^z r_2(t)dt \quad (5.1)$$

$$G_3(z)-G_1(z)=bR_2+aR_4-1-2b\int_0^z r_2(t)dt$$
$$-2a\int_0^z r_4(t)dt \quad (5.2)$$

$$G_3(z)-G_2(z)=(a-b)R_4-2a\int_0^z r_4(t)dt \quad (5.3)$$

$$G_4(z)-G_1(z)=-1-bR_2+aR_4-2a\int_0^z r_4(t)dt \quad (5.4)$$

$$G_4(z)-G_2(z)=-2bR_2+(a-b)R_4-2b\int_0^z r_2(t)dt$$
$$-2a\int_0^z r_4(t)dt \quad (5.5)$$

$$G_4(z)-G_3(z)=-2bR_2+2b\int_0^z r_2(t)dt \quad (5.6)$$

The first four of these continuous functions are clearly monotone nonincreasing. The following table [10] defines the maximal open intervals in which these functions are positive $(+)$, zero (0), and negative $(-)$. For each interval in the $(+)$ and $(-)$ columns, the result of applying the domination criterion is given. Inspection of (5.1) to (5.4) reveals conditions on the intervals in the (0) column

[10] Of course lemma 1 renders superfluous the appearance of $r_3(z)=0$ in the table.

which are given below the interval. A question mark means that the interval may (so far as is known at the moment) be vacuous or degenerate to a single point; the condition written below such an interval is to apply only if the interval is non-degenerate.

The next three lemmas yield the values of R_1, R_2 and R_4 for optimal $r(z)$ (lemma 1 shows that $R_3=0$).

LEMMA 2. *For every optimal* $r(z)$:

    (a) $G_2(0)-G_1(0)=(b-1)-bR_1\geq 0$.

    (b) $G_4(0)-G_1(0)=-1-bR_2+aR_4\leq 0$.

If (a) were false, then the monotonicity of $G_2(z)-G_1(z)$ and the domination criterion would imply $R_1=0$, so that (a) would be true. If (b) were false, then since $G_4(1)-G_1(1)<0$ one would have

$$G_4(z')-G_1(z')=0$$

for some z' in $(0,1)$; by (5.10) $r_4(z)=0$ on $[0,z']$; so by (5.4)

$$G_4(0)-G_1(0)=G_4(z')-G_1(z')=0$$

and (b) would be true.

LEMMA 3. *For every optimal* $r(z)$:

    (a) $G_2(0)-G_1(0)=(b-1)-bR_1=0$.

    (b) $G_4(0)-G_1(0)=-1-bR_2+aR_2=0$.

If $G_2(1)-G_1(1)>0$. then the monotonicity of $G_2(z)-G_1(z)$ and the domination criterion would imply $R_2=0$, so that on the one hand the initial assumption would read

$$(b-1)-bR_1>0,$$

Difference	$(+)$	(0)	$(-)$	
$G_2(z)-G_1(z)$	$[0,g_-^{21})$? $r_2(z)=0$	(g_-^{21},g_+^{21})? $r_2(z)=0$	$(g_+^{21},1]$? $r_1(z)=0$	(5.7)
$G_3(z)-G_1(z)$	$[0,g_-^{31})$? $r_3(z)=0$	(g_-^{31},g_+^{31})? $r_1(z)=1$	$(g_+^{31},1]$ $r_1(z)=0$	(5.8)
$G_3(z)-G_2(z)$	$[0,g_-^{32})$ $r_3(z)=0$	(g_-^{32},g_+^{32})? $r_4(z)=0$	$(g_+^{32},1]$ $r_2(z)=0$	(5.9)
$G_4(z)-G_1(z)$	$[0,g_-^{41})$? $r_4(z)=0$	(g_-^{41},g_+^{41})? $r_4(z)=0$	$(g_+^{41},1]$ $r_1(z)=0$	(5.10)

and on the other hand $R_1=1-R_4$ (by lemma 1, using $R_2=0$). Combining these statements yields $R_4>1/b$, which together with (5.4) and $R_2=0$ yields

$$G_4(0)-G_1(0)=-1+aR_4>0,$$

contradicting (b) of lemma 2. Thus $G_2(1)-G_1(1)\leq 0$, which together with (a) of lemma 2 implies that

$$G_2(z')-G_1(z')=0$$

for some z' in $[0,1]$. By (5.7) $r_2(z)=0$ on $[0,z')$; so by (5.1),

$$G_2(0)-G_1(0)=G_2(z')-G_1(z')=0,$$

and (a) is proved. If (b) were false, then by (b) of lemma 2, the monotonicity of $G_4(z)-G_1(z)$, and the domination criterion, we would have $R_1=0$, contradicting (a).

LEMMA 4. *For every optimal $r(z)$:*

(a) $\quad R_1=(b-1)/b.$

(b) $\quad R_2=(a-b)/b(a+b).$

(c) $\quad R_4=2/(a+b).$

This follows from lemma 3 and the corollary $R_1+R_2+R_4=1$ of lemma 1.

Next the points g, g' will be located, and the behavior of an optimal $r(z)$ on $[0,g)$, (g,g') and $(g', 1]$ determined.

LEMMA 5. *For every optimal $r(z)$:*

(a) $\quad g_+^{21}=g_+^{41}=g=(b-1)/b.$

(b) $\quad r_1(z)=1 \qquad on\ [0,g).$

(c) $\quad r_1(z)=0 \qquad on\ (g,1].$

By (5.5) and lemma 4,

$$G_4(z)-G_2(z)=2b\int_0^z r_2(t)dt-2a\int_0^z r_4(t)dt \quad (5.11)$$

If $g_+^{21}<g_+^{41}$, then by (5.7) and (5.10) one has $r_2(z)=1$ on (g_+^{21}, g_+^{41}), $r_1(z)=1$ on $[0, g_+^{21})$, and $r_4(z)=0$ on $[0, g_+^{41})$, so that the last equation yields

$$G_4(g_+^{41})-G_2(g_+^{41})>0;$$

continuity and the domination criterion show that $r_4(z)=0$ in a neighborhood of g_+^{41}, but (5.4) and (b) of lemma 3 show that this contradicts the definition of g_+^{41}. An entirely similar argument refutes the possibility $g_+^{41}<g_+^{21}$, and so $g_+^{21}=g_+^{41}$. By (5.7) and (5.10) we have $r_1(z)=1$ on $[0, g_+^{21})$ and $r_1(z)=0$ on $(g_+^{21}, 1]$, so that $g_+^{21}=R_1$; an application of (a) of lemma 4 completes the proof.

LEMMA 6. *For every optimal $r(z)$:*

(a) $\quad g_+^{32}=g'=(a-1)/a.$

(b) $\quad r_4(z)=1\ on\ (g', 1].$

Note first that $r_4(z)=1$ on $(g_+^{32}, 1]$ since (i) $r_2(z)=0$ on $(g_+^{32}, 1]$ by (5.9), (ii) $r_1(z)=0$ on $(g, 1]$ by lemma 5, and (iii) $g<g_+^{32}$ because $G_3(g)-G_2(g)>0$ by (5.3) and the fact (see (5.10)) that $r_4(z)=0$ on $[0, g)$. It follows that

$$\int_0^{g_+^{32}} r_4(t)dt+(1-g_+^{32})=R_4=2/(a+b).$$

The definition of g_+^{32} together with (5.3) shows that

$$\int_0^{g_+^{32}} r_4(t)dt=(a-b)R_4/2a=(a-b)/a(a+b), \quad (5.12)$$

and combining the last equations yields the asserted value of g_+^{32}.

LEMMA 7. *For every optimal $r(z)$,*

$$G_4(z)-G_2(z)=0 \qquad on\ [g, g'].$$

Since $r_2(z)=0$ on $(g', 1]$, (5.11) and (5.12) imply

$$G_4(g')-G_2(g')=2bR_2-(a-b)R_4=0,$$

where the last equality follows from lemma 3. Let z^* be a point at which $G_4(z)-G_2(z)$ attains its maximum M on $[g,g']$; the last equation shows that $M\geq 0$. If $M>0$, then by continuity and the domination criterion one has $r_4(z)=0$ (and thus $r_2(z)=1$, since $r_1(z)=0$ on $(g,1]$) on some interval (z^*,z'); by (5.5) $G_4(z)-G_2(z)$ is strictly increasing on (z^*,z'), which contradicts the definition of z^* unless $z^*=g'$, in which case $M=0$ by the last equation. Thus $M=0$; an entirely similar argument shows that the minimum value of $G_4(z)-G_2(z)$ on $[g,g']$ is $m=0$, and so the lemma is proved.

The completion and summary of the preceding material is as follows.

LEMMA 8. *Let $g=(b-1)/b$ and $g'=(a-1)/a$. If the assumption at the end of section 4 is correct, then the only possible optimal strategy is the strategy defined by*

(a) $\quad r_1(z)=1 \qquad on\ [0,g).$

(b) $\quad r_2(z)=a/(a+b)\ on\ (g,g').$

(c) $\quad r_4(z)=b/(a+b)\ on\ (g,g').$

(d) $\quad r_4(z)=1 \qquad on\ (g',1].$

From lemma 1, lemma 5, and lemma 6 one has (a), (d), and also

$$r_2(z)+r_4(z)=1 \qquad on\ (g,g'). \quad (5.13)$$

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From lemma 7, (5.5), and lemma 4 one has

$$\int_0^z [br_2(t) - ar_4(t)]dt = 0 \qquad \text{on} \quad (g, g'),$$

which by smoothness condition (iii) (see section 2) implies

$$br_2(z) - ar_4(z) = 0 \qquad \text{on } (g, g');$$

this and (5.13) imply (b) and (c).

6. Proof That the Conditions Are Necessary and Sufficient for Optimality

In this section the notation $r(z)$ is reserved for the particular strategy described in lemma 8, while arbitrary strategies are denoted $r^*(z)$.

LEMMA 9. $K(r, r_0^*) = \min\limits_{r^*} K(r, r^*)$ *if and only if* $r_0^*(z)$ *obeys*

(a) $\quad r_3^*(z) = 0 \qquad\qquad$ on $[0, g)$.

(b) $\quad r_1^*(z) = r_3^*(z) = 0 \qquad$ on (g, g').

(c) $\quad r_1^*(z) = r_2^*(z) = 0 \qquad$ on $(g', 1]$.

The $K(r, r^*)$ of (2.1) is first written as the sum of these integrals:

$$\int_0^g [G_1(z) + r_2^*(z)(G_2(z) - G_1(z)) + r_3^*(z)(G_3(z) - G_1(z)) + r_4^*(z)(G_4(z) - G_1(z))]dz.$$

$$\int_g^{g'} [G_2(z) + r_1^*(z)(G_1(z) - G_2(z)) + r_3^*(z)(G_3(z) - G_2(z)) + r_4^*(z)(G_4(z) - G_2(z))]dz.$$

$$\int_{g'}^1 [G_4(z) + r_1^*(z)(G_1(z) - G_4(z)) + r_2^*(z)(G_2(z) - G_4(z)) + r_3^*(z)(G_3(z) - G_4(z))]dz.$$

It is easily verified that $r(z)$ has the properties described in lemma 4, and since $r_3(z) = 0$ on $[0, 1]$, $r(z)$ obeys (5.1)—(5.6). Using these facts, one finds that on $[0, g)$ the coefficients of $r_2^*(z)$ and $r_4^*(z)$ in the first integral vanish, while that of $r_3^*(z)$ is positive. Hence $r_0^*(z)$ minimizes the first integral if and only if it satisfies (a). Similarly, $r_0^*(z)$ minimizes the second integral if and only if it satisfies (b), and minimizes the third integral if and only if it satisfies (c).

THEOREM. $r(z)$ *is the unique optimal strategy.*

Since $r(z)$ obeys (a) to (c) of lemma 9,

$$K(r, r) = \min\limits_{r^*} K(r, r^*)$$

which (since the game is symmetric) implies that $r(z)$ is optimal. Thus *any* optimal $r_0^*(z)$ must satisfy

$$K(r, r_0^*) = \min\limits_{r^*} K(r, r^*)$$

and thus, by lemma 1 and (c) of lemma 9, must satisfy $r_4^*(1) = 1$. Hence the assumption at the end of section 4 is correct, and so, by lemma 8, $r(z)$ is the *only* optimal strategy.

7. Discussion of a Limiting Case

The limiting case $b = 1$ will now be briefly discussed. (Recall that $b > 1$ previously.) It is easily verified that the results of the previous sections remain valid (many of the proofs can be considerably simplified), so that the game with $b = 1$ has a unique optimal strategy which is obtained from that for $b > 1$ (i.e., from (a) to (d) of lemma 8) by setting $b = 1$. Since $g = 0$ for $b = 1$, the unique optimal strategy is given by

(a) $\quad r_2(z) = a/(a+1) \qquad$ on $[0, (a-1)/a)$.

(b) $\quad r_4(z) = 1/(a+1) \qquad$ on $[0, (a-1)/a)$

(c) $\quad r_4(z) = 1 \qquad\qquad$ on $((a-1)/a, 1]$.

Upon changing

$$\text{player } R, \text{ player } S, r_2, r_3, r_4, a$$

to $\qquad$ player 1, player 2, ρ_3, ρ_2, ρ_1, a/b

respectively, one finds that this strategy is precisely the unique optimal strategy of the continuous von Neumann poker game discussed in sections 19.4 to 19.10 of [3]. To explain this, note that (2.2) and (2.3), with $b = 1$, yield

$$G_2(z) - G_1(z) = -R_1 - 2\int_0^z [r_2(t) + r_3(t)]dt \leq 0$$

for all strategies $r(z)$, which on the one hand shows that the first course of action is dominated, and on the other hand yields a quick proof[11] that $r_1(z) = 0$ on $[0, 1]$ for all optimal $r(z)$. Thus, without loss of optimal strategies, the game can be "reduced" by "striking out" the first course of action for both players. This reduced game is easily seen to be identical with the von Neumann game, except for some changes of language and a multiplicative factor b in the expected payoff function. Thus the von Neumann game appears as a reduced form of the limiting case $b = 1$ of the game solved in this paper.

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[11] If $r_1(z') > 0$ for some z', then $R_1 > 0$; the formula and the domination criterion yield $r_1(z) = 0$ for all z, a contradiction.

8. References

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