Report of U.S. Commission 6, URSI RADIO WAVES AND CIRCUITS

Subcommission 6.1—INFORMATION THEORY

Part 1. Information Theory and Coding

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Since 1957, there has been considerable progress in the theory of coding messages for transmission over noisy channels. There have been three main directions of advance. First, there has been work on the foundations of the theory. During this time American mathematicians interested in probability have shown a serious interest in information theory, since Feinstein's work (now available in book form) [Feinstein, 1958a] and since the interest shown by Kolmogorov and Khinchin. Second, a great deal of work has been done on error-correcting block codes for noisy binary channels. This work has involved a good deal of modern algebra, and some mathematical algebraists have been joining the communications research workers in attacking these problems. Third, there has been continuing investigation of procedures in which input messages are coded and decoded sequentially rather than in long blocks. This work and the work on binary block codes both have significant practical implications for electrical communications.

1. Foundations

Shannon's original demonstration of the noisy channel coding theorem was an existence proof [Shannon, 1949]. Given a channel of capacity C bits per second and a rate of transmission R bits per second, the transmitter sends sequences of N channel input symbols. The receiver receives sequences of N channel output symbols and decides which input sequence was transmitted, making this decision incorrectly with probability P. What Shannon showed was that for $R < C$, P could be made arbitrarily small by increasing N. The proof was not constructive, and nothing quantitative was said about how rapidly P decreased as a function of N for given R and C. Feinstein [1954; 1958a] showed that P could be bounded by a decaying exponential in N. His proof covered channels with a simple kind of finite memory. While constructive in principle it could not be used in practice to construct a code with large N. In 1957, Shannon [1957] gave a remarkably concise proof based on his original random coding argument but more detailed and precise, which also gave an exponential bound to P as a function of N, and extended the proof to channels with considerably more complex memory. Blackwell, Breimann and Thomasian [1958] proved the existence theorem for channels with a finite-state memory of a still more general kind. Wolfowitz [1960] and Feinstein [1959] have also proved converse theorems—the weak converse being that for $R > C$, P cannot approach zero, and the strong converse being that for $R > C$, P must approach 1.

The kind of technique used by Shannon [1957] can be extended to obtain upper and lower bounds to the rate of exponential decay of P with N. Earlier work on binary channels had shown that for a considerable range of R less than C the upper and lower bounds essentially agreed, and best possible behavior could be uniquely specified. Similar results have been obtained by Shannon for more general channels. This work is not yet published, but the case of a continuous channel with additive Gaussian noise has been treated in detail [Shannon, C. E., 1959].

The increasing interest of mathematicians in this field is evidenced by an article by Wolfowitz [1958]. In general the results which the mathematicians have obtained are firmer proofs under more general circumstances of theorems whose general character was not surprising to communications researchers. However a recent paper [Blackwell, Breimann, and Thomasian, 1959], has presented an interesting new problem, defining capacity and proving a coding theorem for a channel whose parameters are not known precisely, but are constrained to lie in known ranges. This work might be relevant to incompletely measured and time-varying radio channels. So might a paper by Shannon [1958] on channels in which the transmitter has side information available about the state of a channel with memory: an example would be the information obtained by measurements of the propagation medium obtained while communicating.

2. Binary Channels

Starting with the earlier work of Hamming [1950] and Slepian [1956a, 1956b], error-correcting block codes for binary channels have been investigated extensively. Peterson and Fontaine [1959] have searched for best possible error-correcting codes of short block length (up to 29), using a computer. The number of codes grows so rapidly with block length that it was necessary to use many equivalence relations and shortcut tests to eliminate codes from consideration early. A number of counterexamples were found to common conjectures about optimum codes.

The use of error-correcting codes in practice has been limited by the difficulty of implementation, and by the fact that in many applications of interest the errors in the channel are not independent, but occur in runs or bursts. In earlier work Huffman [1956] had shown a coding and decoding procedure

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for the Hamming code which was simple to implement, and Green and SanSoucie [1958] have shown an easy implementation for a short multiple-error-correcting code. Hagelbarger [1959] has described codes which correct errors occurring in bursts whose implementation is not too difficult, and Abramson [1959] has described a highly efficient and easily implemented set of codes with similar properties.

Work on codes of longer block length, which can correct multiple errors, started with a decoding procedure given by Reed [1954] some time ago for the Reed-Muller family of codes. For really large block lengths these codes are not efficient, but Perry [1958] has built a coder and decoder for a Reed-Muller code which has block length of 128 digits, 64 of which are information digits and 64 check digits. This code can correct any set of 7 or fewer errors among the group of 128 and the efficiency is quite good. Using microsecond switching devices, the units can keep up with millisecond binary digits.

Calabi and Haefeli [1959] have investigated in detail the burst correcting properties of a family of codes which has been introduced earlier for correction of independent errors [Elias, P., 1954]. They also discuss the implementation of these codes.

A new family of codes discovered by Bose and Ray-Chaudhuri [1959, 1960] is much more efficient than the Reed-Muller codes for larger block lengths. Although in the limit of infinite block length these codes may also have zero efficiency, at lengths of a few thousands digits they are still quite good. Peterson [1960] has discovered an economical way to decode these codes. There is a great deal of current work on finding more properties of these codes, finding similar codes for channels which are symmetric but not binary, and so forth.

There has been a good deal of recent work on cyclic codes, including some encouraging results on step-by-step decoding due to Prange [1959]. Cyclic codes are closely related to the sequences which can be generated by shift registers with feedback connections. Recent discussions of these sequences have been given by Elspas [1959] and by Zierler [1959]. A review of the recent algebraic work on coding theory, including the Galois field theory which enters in the Bose-Chaudhuri codes, will be given by Peterson in a monograph to be published shortly [Peterson, 1960]. Most of the results in this area extend to channels which have an input alphabet of symbols whoses number is not 2 but any prime to any power, the channel still being completely symmetric in the way it makes its errors. Nonbinary channels have been investigated in their own right by Lee [1958] and by Ulrich [1957].

The introduction of two thresholds rather than one in a continuous channel introduces a null zone. The transmitter sends a binary signal, but the receiver makes a ternary decision, not attempting to guess the value of signals received in the null zone. Introducing the null zone may increase channel capacity, as shown by Bloom et al. [1957]. It also has the valuable effect of reducing the amount of computation required in decoding, since it is easier to replace missing digits than to correct incorrect ones. This is especially relevant for application to channels with Rayleigh fading.

3. Sequential Decoding

Earlier work had shown that the block coding procedure could be modified (in the binary case) by constructing codes in a convolutional fashion, so that the coding and decoding of each digit was of the same character and involved the same delay [Elias, 1955]. The parameter which replaces block length in such an argument is the delay between the receipt of a digit and the attempt to decode it reliably. This simplified the coding but left the decoding procedure as complicated as ever. However Wozencraft [1957] has shown that a suitable sequential coding procedure may be followed by a sequential decoding procedure which reduces the average amount of decoding computation immensely. Like the best of the long block codes now in prospect, this procedure promises millisecond communication with microsecond switching circuitry in the decoder at very high reliability. Unlike the block codes, however, Wozencraft's procedure is statistical and not highly algebraic, and it may be expected to generalize to other discrete channels with no special symmetry properties. On the other hand the computation remains reasonable only for a range of R well below C. Epstein [1958] has studied a sequential decoding procedure for the erasure channel, and work on more general channels is under way.

4. Conclusions on Coding

The general conclusions of interest for applications of error-correcting codes are two. First, there are now several good small codes which correct bursts of errors, which could be instrumented fairly easily for use in situations in which a rate well below capacity can be tolerated so that short codes may be used. These may find early application in sending digital data over telephone lines. Second, there are now available several kinds of large block codes and sequential codes which will permit very reliable transmission over long distance scatter channels, which can also be implemented. The cost of implementation is appreciable in these cases, but current computer circuitry is fast enough to permit decoding at transmission rates of the order of 1,000 binary digits per second, coded in blocks or with sequential constraints hundreds of digits in length, and the alternative of more large antennas or greater transmitter power are also expensive. It seems likely that such systems will be in experimental use by the next international URSI meeting in 1963.

5. Other Topics

Less progress has been made in the economical coding of information sources. In part this is because such progress becomes work in speech analysis of television systems and not information theory as such. However it might be worth noting that a scheme for coding runs of constant intensity in television has been demonstrated at full televison speed by Schreiber [1958].

A relation between the bandwidth and the duration of a signal is imposed by the Heisenberg uncertainty principle, whose applicability to time functions was pointed out by Gabor many years ago. Kay and Silverman [1959] have examined this relationship more carefully, and a form of the uncertainty principle which places a lower bound on the sums of entropies rather than on the products of second moments is discussed by Leipnik [1960]. Stam [1959] also discusses this entropic inequality and closely related results.

The sampling theorem is closely related to these questions. Linden and Abramson [1960] have given a generalization which permits the closed form expression of a bandlimited function in terms of samples of the function and its first k derivatives, taken at time intervals $(k+1)$ times as far apart as is required for samples of the function value alone. This extends earlier work by Jagerman and Fogel [1956]. Results bearing both on the uncertainty principle and on approximate sampling theorems—i.e., theorems concerning functions which include all but a fraction δ_1 of their energy in bandwidth W and all but a fraction δ_2 of their energy in a time interval of duration T—are the subject of active current work.

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Part 2. Random Processes

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Research on random processes in the period under consideration may be conveniently summarized under three main headings: statistical properties of the output of nonlinear devices; estimation theory for random processes; and representation theory for random processes.

Under the first heading, the investigations concern the statistical properties of the output of a nonlinear device, or of a linear filter following a nonlinear device, when the input is a random process having prescribed statistics. These problems are of great interest since this is a model for many types of receivers. The period 1957 to 1960, continuing earlier work, has seen the buildup of a large inventory of results and of methods for attacking this class of problems.

One of the most comprehensive approaches is reported on in papers by Darling and Siegert [1957], and by Siegert [1957, 1958]. These papers report on work actually done earlier. The problem considered is that of finding the (first order) probability distribution function of the quantity

$$\int \phi[x(\tau),\tau]d\tau,$$

where ϕ is a prescribed function and $x(\tau)$ is a component of a stationary n-dimensional Markoff process. Many problems in the category under consideration are special cases of this. The approach is via the characteristic function of the required probability distribution; it is shown that this characteristic function must satisfy two integral equations. Under certain conditions, it can also be shown that the characteristic function must satisfy two partial differential equations.

Another type of problem in this category is the investigation of the second or higher order probability distributions of the output, and particularly of the autocorrelation function of the output or the cross-correlation between two or more such outputs. For example, Price [1958] gives a theorem which is useful in deriving such auto- and cross-correlations when the inputs are Gaussian. The theorem stated can be used in many cases to calculate the quantity

$$R = \text{Expected Value of} \left\{ \prod_{i=1}^{n} f_i(x_i) \right\},$$

where $(x_1, \ldots, x_n)$ is a Gaussian vector and f_i are prescribed functions.

Many other papers, for example Leipnik [1958], Pierce [1958], Kielson et al., [1959], Helstrom and Isley [1959], McFadden [1959], Campbell [1957], and Leipnik [1959], have been written giving special results and using a number of different approaches.

Work has also continued on the problem of the distribution of zero crossings of Gaussian processes [Helstrom, 1957, and Brown, 1959].

Under the heading of estimation theory for random processes one might first mention the subject of estimating the spectral density of stationary Gaussian processes. Two references [Grenander and Rosenblatt, 1957, and Blackman and Tukey, 1959] summarize much work on this problem, a great deal of which had been done previously (but not all of which had been published previously). Blackman and Tukey discuss two types of estimates of the power spectrum, viz: estimation of the autocorrelation function, multiplication by a prescribed function of time called a "lag window," followed by Fourier transformation; or, passing the observed process through a filter of specified transfer function and calculating the average power of the output. They derive expressions for the first and second moments of such estimates, as well as of the cross-moments of estimates of the spectral density at two different frequencies. Grenander and Rosenblatt discuss similar types of spectral estimates, emphasizing and utilizing the fact that these as well as most other useful estimates of spectral density are quadratic forms in the observed data. They derive first and second order moments, as well as asymptotic probability distributions for large observed samples, of such estimates.

A recent paper of Grenander, Pollak, and Slepian [1959] discusses the small sample case, relying heavily on the fact that spectral density estimates are usually quadratic forms in the observed data.

In an interesting paper Slepian [1958] has discussed the following hypothesis-testing problem: given an observed sample of a Gaussian random process, known to be characterized by either one of two prescribed power spectra, which power spectrum does the process actually have? It turns out that in problems of this type, the measures induced by the two alternative hypotheses may be singular with respect to each other; in which case, it is possible to decide between the alternatives with arbitrarily small error probability, and with an arbitrarily small sample. Slepian gives various sufficient conditions for this. The power spectra satisfying his conditions are, moreover, standard types very frequently postulated. This emphasizes that the mathematical model one chooses must be carefully chosen to be appropriate to the problem one is trying to solve.

Another type of estimation problem for random processes is considered by Swerling [1959]. Suppose a prescribed waveform, depending on one or more unknown parameters, is observed in additive Gaussian noise having prescribed autocovariance function

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and zero mean. Expressions are derived for the greatest lower bound for the variance of estimates of the unknown parameters having prescribed bias. These greatest lower bounds are found to coincide in certain special cases with the variance, obtained by Woodward, of maximum likelihood estimates of the unknown parameters. Similar problems are investigated in Middleton [1959].

In the field of representation theory for random processes, work has continued on the subject of representation of nonlinear operations on random processes—especially for Gaussian processes. Papers by Zadeh [1957] and Bose [1959] and a book by Wiener [1958] deal with this problem. The approach followed is, first, to express the initial random process $\{x(t)\}$ as a series

$$x(t)=\sum_{n=1}^{\infty}u_n\alpha_n(t),$$

where $\{\alpha_n(t)\}$ is a set of orthonormal functions over the interval of definition of $\{x(t)\}$. If $\{x(t)\}$ is Gaussian, the u_n are Gaussian and, if $\alpha_m(t)$ are properly chosen, can be made independent. Any linear or nonlinear functional of $\{x(t)\}$ can then be regarded as a function of $u_1, \ldots, u_n, \ldots$. Second, one may choose a set of functions of the variables u_n which are orthonormal in the stochastic sense as explained, for example, in Zadeh [1957] with respect to the process $\{x(t)\}$. Then, nonlinear functionals of $\{x(t)\}$ may be expanded in a series of the orthogonal functions of the variables u_n.

Other research in the field of representation theory has treated such subjects as:

Use of bi-orthonormal expansions [Leipnik, 1959], envelopes of waveforms [Arens, 1957, and Dugundji, 1958], the sampling theorem and related topics [Balakrishnan, 1957, and Lerner, 1959], and harmonic analysis of multidimensional processes [Weiner and Masani, 1957 and 1958].

Much of this work in representation theory provides useful tools for attacking the problems discussed under the first two headings above.

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Part 3. Pattern Recognition

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Pattern recognition, in its widest sense, cuts across many fields of engineering interest—from character sensing to learning theory, and from machine translation to decision-making techniques. Inasmuch as the problem of recognizing patterns is that of simulating human thinking processes, it is also closely related to nonengineering fields such as physiology, psychology, cryptology and linguistics. No attempt is made in this report to summarize the developments in all these areas. Rather, pattern recognition developments are reported only to the extent that they represent a direct contribution to the theory of information. The enclosed bibliography is compiled primarily from engineering journals; consequently, it will be found that the emphasis in this report is placed on the recognition of visual patterns, rather than vocal, linguistic or other patterns, which are mainly covered in nonengineering publications.

The reason for the acute engineering interest in visual patterns is the recent emergence of the following two urgent problems: (a) How can redundancy be removed from television pictures, so that video signals could be transmitted at a greatly reduced waveband; (b) How can printed documents be read automatically, so that the most serious bottleneck—the human typist or card puncher—could be eliminated from digital data-processing systems. Although these two topics are treated separately in the literature, both represent different aspects of the same general problem of pattern recognition. This problem may be divided, somewhat artificially, into three phases: (1) Redundancy removal, (2) Recognition programs, (3) Recognition system design. This division will be adopted in the following summary. Since the boundaries between three phases are not well defined, the corresponding bibliography classification should not be regarded as too rigorous.

3.1. Redundancy Removal

Both the compression of television bandwidth and the design of character recognizers, require the determination of the source redundancies, and the establishment of scanning-coding schemes which would minimize these redundancies. Considerable work has been done in the past three years on the "run-length" scheme, where lengths of pattern runs, rather than values of individual cells, constitute the transmitted information. [Capon, 1959; Michel, 1958; 1957]. The redundancies which may be eliminated under this scheme were measured for some sources of practical interest, and bounds were found for the potential bandwidth saving [Deutsch, 1957; Powers and Staras, 1957; Schreiber and Knapp, 1958]. Another scheme that was explored is one in which scanning is confined to the minimal set of cells necessary for recognition under noiseless and noisy conditions [Gill, 1959]. Progress has also been made in the techniques of measuring the autocorrelation function of two-dimensional patterns [Kovasznay and Arman, 1957].

3.2. Recognition Programs

Although the removal of redundancies from the given patterns simplifies and accelerates their recognition, the recognition itself is a result of a predetermined series of decision rules—applied sequentially or simultaneously—which is called "a recognition program." The program invariably involves a set of transformations performed on the unknown patterns, followed by a comparison of the transformed pattern with a precompiled library of reference patterns. The size of the library and the length of the comparison process depend on the chosen set of transformations. Thus far, no universal procedure has been formulated for selecting a necessary or sufficient transformation set for a given pattern source; rather, each investigator uses intuitive or heuristic arguments to propose such a set for the specific source under investigation [Bledsoe and Browning, 1959; Dimond, 1957]. The approach which seems to be the most popular in the case of character recognition, is the association of each pattern with a distinct set of two-dimensional features ("corner," "intersection," "arc," etc.) which can be abstracted from each pattern with the aid of digital computers [Bomba, 1959; Kamentsky, 1959; Unger, 1959]. The necessary set of concepts is, again, presented heuristically. Similar situation exists in recognition programs proposed for other classes of patterns [Gold, 1959].

3.3. Recognition System Design

Once a set of transformations is selected for the recognition program, a system has to be constructed for executing the program. The intuitive basis on which the program is constructed, forces most investigators to plan a flexible system, in which transformations can be readily varied either manually or automatically as more experience is gained on the nature of the pattern source and the performance of the program (the automatic method is closely related to problems concerning "adaptive systems," which are not reviewed in this report). The majority of all recognition systems built to date are found to be still in the "learning" stage, serving as testing grounds for the various schemes devised by the respective investigators [Grenias et al., 1957; Kirsch et al., 1957; Tersoff, 1957]. A byproduct of these

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circumstances are the so-called "pattern synthesis" techniques, developed for simulating various pattern sources for test purposes [Flores and Ragonese, 1958; Grenias and Hill, 1957]. These techniques are also applied to the design of optimal-style patterns, where a limited degree of freedom may be exercised over the construction of the source itself.

It seems that although a considerable progress has been made in various areas of pattern recognition, it is still minute in comparison with the problems that still remain unresolved. The scanning-coding techniques devised for transmitting visual patterns compress the currently employed bandwidth by at most a factor of 10, while a factor of a million is required in order to approach the recognition capacity of the human eye. Automatic reading of relatively standardized characters is in a relatively high development stage, but the mechanical recognition of handwriting or speech are still practically unfeasible. Further progress in this field seems to lie in three directions: (a) Deeper analysis of the redundancies inherent in the various classes of pattern sources, (b) Formulation of procedures for determining optimal sets of transformations required for recognizing given sets of patterns, (c) Simulation of learning processes with digital computers. It is hoped that the next three years will witness significant contributions to these basic problems.

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Gold, B., Machine recognition of hand-sent Morse code, IRE Trans. on Inform. Theory **IT–5,** 17 (March 1959). (The code is recognized by applying a set of fixed language rules, and statistically determined threshold tests.)

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Kirsch, R. A., L. Cahn, C. Ray and G. H. Urban, Experiments in processing pictorial information with a digital computer, 1957 Proc. EJCC, 221. (Laboratory apparatus is described for performing pattern-recognition research.)

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Michel, W. S., Statistical encoding for text and picture communication, Commun. and Electronics **77,** 33 (March 1958). (Various coding schemes proposed for different classes of patterns are discussed and compared.)

Michel, W. S., W. O. Fleckenstein and E. R. Kretzner, A coded facisimile system, 1957 Wescon Conv. Record, pt. 2, 84. (Variable-length coding of binary patterns scanned through a variable-sweep scanning system may result in 3–1 saving; the transmission rate may be made proportional to the complexity of the scanned material.)

Powers, K. H., and H. Staras, Some relations between television picture redundancy and bandwidth requirements, Commun. and Electronics **32,** 492 (Sept. 1957). (Picture redundancy is separated into nonpredictive redundancy, resulting from nonoptimal first-order probability distribution, and predictive redundancy, resulting from statistical correlation between successive signals. Nonpredictive redundancy in television is essentially zero. Predictive redundancy allows at least 2 to 1 saving.)

Schreiber, W. F., and C. F. Knapp, TV bandwidth reduction by digital coding, 1958 IRE Nat. Conv. Record, pt. **4,** 88. (Picture statistics and characteristics of human vision are utilized in coding video signals and transmitting the code at a uniform rate.)

Tersoff, A. I., Automatic registration in high-speed character-sensing equipment, 1957 Proc. EJCC, 232. (A device is described which minimizes the effects of tilt, poor registration and extraneous marks on the recognition process.)

Unger, S. H., Pattern detection and recognition, Proc. IRE **47,** 1737 (1959). (Characters are recognized through a spatial computer, capable of detecting a specified set of geometrical properties uniquely associated with each pattern.)

Part 4. Detection Theory

Robert Price*

4.1. Remarks

The period since the XII General Assembly has seen a consolidation of the closely related concepts of Wald, Woodward, Middleton, and Van Meter, and Peterson, Birdsall, and Fox into a fairly unified theory of detection, together with the successful application of the theory to a variety of problems. Through this approach, 'optimal' detector structures for electronic systems can be synthesized provided that the designer has *a priori* knowledge of the governing statistics and error costs. At the same time, older and more standard detection techniques have continued to receive attention, the theoretical results generally being stated in terms of probability-of-error or signal-to-noise ratio at the detector output. If one must attribute the discovery of any new, guiding principles to the preceding three-year period, the most likely candidates would seem to be found in those few studies which have sought theories which can cope effectively with situations in which *a priori* knowledge is seriously lacking.

It appears that roughly half the effort of the past three years has been devoted to specific detection problems in radar and communications. In contemporary communications studies considerable heed is paid to 'optimum' detection procedures, there being less inclination to examine conventional, suboptimum detectors than in the radar analyses. The reason for this may be that the radar designer faces considerably greater *a priori* uncertainty, both with regard to the signal and the channel through which it comes. By contrast, relatively simpler channels have usually been assumed without loss of realism in communications problems, while the communications system designer also has more direct control of the signal. The appropriate optimum detectors for communications then turn out to be rather elementary, and can at present be constructed with hardly more effort than suboptimum devices require. In fact, the communications environment is generally 'clean' enough that much recent work has been concerned with determining good sets of transmitted signal waveforms, the use of an optimum receiver being taken for granted.

The bulk of the remaining effort has dealt with special topics in detection of quite general application. Further study in sequential decision has been made both theoretically and through Monte Carlo computer experimentation, in the hope of achieving significant speedup in detection over fixed-sample operation. Greater understanding of the detection of stochastic signals in noise has been sought for applications in such fields as radio astronomy and in systems where rapidly fading channels are encountered. There has been some work on parameter estimation for a finite number of parameters, a subject which is virtually inseparable from detection theory. Detection losses in nonlinear devices have also received further examination.

Attempts to circumvent the *a priori* difficulty represent only a small fraction of the output of the past three years in detection analysis, but have perhaps the most significance for future work. Original attacks have been made through game theory, comparison of experiments, nonparametric techniques, dynamic programing, and inductive probability. It is hoped that one or more of these tools will prove effective in breaking new ground.

4.2. Papers

The following list of references has been drawn largely from the American journals concerned with statistical communication theory and information theory, but also contains a few laboratory technical reports. This selection omits papers on multiple parameter estimation, and the estimation of signal waveforms and impulse responses, since these topics verge on filtering theory. Other closely related subjects which are not covered are classical studies in hypothesis testing that do not refer to electronic systems, investigations into ambiguity functions of radar waveforms, and information-feedback systems.[1] The future pursuit of feedback studies may well lead to wider interchanges in detection notions between radar and communications.

4.3. Bibliography

a. Applications to Radar

Bussgang, J. J., P. Nesbeda and Safran, A unified analysis of range performance of CW, pulse and pulse doppler radar, Proc. IRE **47,** 1753 (1959). (Includes a simplified analysis for sweep integrator systems containing square-law detectors.)

Cohn, G. I., and L. C. Peach, Detection of radar signals by direct measurement of their effects on noise statistics, Proc. Nat. Electronics Conf. **14,** 821 (1959). (Describes equipment for measuring waveform probabilities.)

Dilworth, R. P., and E. Ackerlind, The analysis of post detection integration systems by Monte Carlo methods, 1957 IRE Nat. Conv. Record, Pt. 2, 40. (Measurement of output probability distributions for filter-linear detector-integrator and filter-squarer-integrator combinations.)

Galejs, J., and W. Cowan, Interchannel correlation in a bank of parallel filters, IRE Trans. on Inform. Theory **IT–5,** 106 (1959). (Study of first-order effects on false alarm and detection probabilities.)

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McCord, H. L., An estimation of the degradation in signal detection resulting from the addition of the video voltages from two radar receivers, IRE Nat. Conv. Record, Pt. 2, 83 (1957). (A loss of about 2 db for linear addition, only 0.2 db for peak selection.)
Miller, K. S., and R. I. Bernstein, An analysis of coherent integration and its application to signal detection, IRE Trans. on Inform. Theory **IT–3,** 237 (1957). (In covering a region of doppler uncertainty, the more filters the better.)
Pachares, J., A table of bias levels useful in radar detection problems, IRE Trans. on Inform. Theory **IT–4,** 38 (1958). (Values of the incomplete Gamma function.)
Reed, I. S., E. J. Kelly, and W. L. Root, The detection of radar echoes in noise. Pt. I: Statistical preliminaries and detector design. Pt. II: The accuracy of radar measurements. Tech. Repts. 158 and 159, Lincoln Lab., M.I.T., (1957). (Derivation of optimal detectors or estimators using orthogonal expansion coordinates, followed by evaluation of how well such devices perform.)
Siebert, W. McC., Some applications of detection theory to radar, IRE Nat. Conv. Record, Pt. 4, 5 (1958). (Discusses a constant-false-alarm-rate detector for use when the noise is of variable strength.)
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b. Applications to Communications

Cahn, C. R., Performance of digital phase-modulation communication systems, IRE Trans. on Comm. Systems **CS–7,** 3 (1959).
Lawton, J. G., Theoretical error rates of 'differentially coherent' binary and 'Kineplex' data transmission systems, Proc. IRE **47,** 333 (1959). Also C. R. Cahn and J. G. Lawton, Comparison of coherent and phase-comparison detection of a four-phase digital signal, Proc. IRE, 1662 (1959). (Binary phase-comparison shown to be asymptotically as good as a coherent phase-reversal system, but in four-phase modulation there is a relative asymptotic loss of 3 db.)
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Pierce, J. N., Theoretical diversity improvement in frequency-shift keying, Proc. IRE **46,** 903 (1958). (Derivation and performance evaluation of optimum detectors for diversity reception.)
Reiger, S., Error rates in data transmission, Proc. IRE **46,** 919 (1958). (Effectiveness of simple codes shown.)
Shannon, C. E., Probability of error for optimal codes in a Gaussian channel, Bell System Tech. J. **38,** 611 (1959). (Upper and lower bounds to error probability obtained for best set of waveforms for signalling through independent Gaussian noise.)
Sunde, E. D., Ideal binary pulse transmission by AM and FM, Bell System Tech. J. **38,**, 1357 (1959). (With a bandwidth constraint, concludes that FM is better than AM for signalling through white noise.)
Turin, G. L., Error probabilities for binary symmetric ideal reception through nonselective slow fading and noise, Proc. IRE **46,** 1603 (1958). (Considers pulse signals of familiar shapes.)
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Turin, G. L., Some computations of error rates for selectively fading multipath channels, Proc. Nat. Electronics Conf. **15** (in press). (Even if one of two independently fading paths is relatively weak, error probability is considerably lower than if only the stronger path is present.)

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Middleton, D., On the detection of stochastic signals in additive normal noise, Pt. I, IRE Trans. on Inform. Theory **IT–3,** 86 (1957). (A filter-squarer-integrator shown to be an optimum structure.)
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f. Loss in Nonlinear Devices

Galejs, J., Signal-to-noise ratios in smooth limiters, IRE Trans. on Inform. Theory, **IT–5,** 79 (1959). (Behavior of loss as a function of the saturation of an error-function limiter characteristics, for various noise spectral shapes.)

Green, B. A., Jr. (see under Radar).
Manasse, R., R. Price and R. M. Lerner, Loss of signal detectability in band-pass limiters, IRE Trans. on Inform. Theory **IT-4,** 34 (1958). (Analysis at low-signal-to-noise ratios shows little detectability loss through a hard limiter.)

g. Attacks on the *A Priori* Problem

Abramson, N., The application of 'Comparison of Experiments' to detection problems, IRE Nat. Conv. Record, Pt. 4, 22 (1958). (A simple but realistic example demonstrates the possibility of showing that one system is superior to another regardless of cost assignments and *a priori* message probabilities.)
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Stewart, J. L., and E. C. Westerfield, A theory of active sonar detection, Proc. IRE **47,** 872 (1959). (Considers reverberation, using a pseudo-random transmission and a cross-correlator detector.)

i. Books

Recent books, available or in press at the end of 1959, in which detection analysis receives major treatment:

Middleton, D., An introduction to statistical communication theory (McGraw-Hill Book Co., Inc., New York, N.Y., 1960).
Davenport, W. B. Jr., and W. L. Root, An introduction to the theory of random signals and noise (McGraw-Hill Book Co., Inc., New York, N.Y., 1958).
Helstrom, C. W., Statistical theory of signal detection, Pergamon Press (in press).

Part 5. Prediction and Filtering

L. A. Zadeh*

Much on the research on prediction and filtering conducted in the United States during the period 1957–1960 was concerned essentially with various extensions of Wiener's theory. In particular, extensions involving nonstationary continuous time processes, vector-valued processes, stationary and nonstationary discrete-time processes, nonGaussian processes, incompletely specified processes, and nonlinear filters and predictors have received attention.

A new and very promising direction in prediction theory has been opened by the application of Bellman's dynamic programing to the determination of optimal adaptive filters and predictors. Actually, the basic work of Bellman and Kalaba [1958, 1959, 1960] and its extensions and applications by Freimer [1959], Aoki, Kalman, and Koepcke [1958], and Merriam [1959] are not concerned with prediction and filtering as such. However, the recent work of Kalman shows that, mathematically, there is a duality between the filtering problem and the control problems considered by Bellman and Kalaba, and others. Thus, these contributions are likely to have a considerable impact on the course of development of the theory of filtering and prediction in the years ahead, and point toward an increasing utilization of digital computers and the concepts and techniques of discrete-state systems both in the design of predicting and filtering schemes and in their implementation.

During the past two years four books containing in aggregate a substantial amount of material on prediction and filtering have been published. Davenport and Root [1958] present a clear exposition of Wiener's theory and some of its extensions. Wiener [1958] discusses orthogonal expansions of nonlinear functionals but stops short of applying them to prediction problems. Bendat [1958] presents a general survey of linear prediction and treats some special problems in considerable detail. Middleton [1960] contains a thorough exposition of the classical prediction theory together with a theory of reception in which the problems of prediction and filtering are formulated in the framework of decision theory. The appendix of Middleton's book includes an informative section on the solution of the Wiener-Hopf equation and some of its variants.

A more detailed discussion of the contributions to filtering and prediction theory is presented in the following pages. For convenience, the subjects of nonlinear filtering, nonstationary and discrete-time filtering, and miscellaneous contributions are dealt with separately.

5.1. Nonlinear Filtering

The contributions to nonlinear filtering and prediction have centered largely on the fundamental work of Wiener [1953] and its earlier extensions by Bose [1956] and Barrett [1955]. A discernible trend in research in this area is to consider special types of processes for which optimal nonlinear filters assume a simple form. A key work in this connection is that of Barrett and Lampard [1955], in which the class, Λ,[1] of all second order density functions admitting a diagonal representation of the form.

$$p(x_1,x_2;\tau)=p(x_1)p(x_2)\sum_{n=0}^{\infty}A_n(\tau)\theta_n(x_1)\theta_n(x_2) \qquad (1)$$

is introduced. Here $p(x_1,x_2;\tau)$ denotes the second order density of a stationary process $\{x(t)\}$, $x_1=x(t)$, $x_2=x(t+\tau)$, $p(x)$ is the first order density, and $\{\theta_n(x)\}$ is a family of polynomials with the orthogonality property

$$\int p(x)\theta_m(x)\theta_n(x)dx=\delta_{mn}. \qquad (2)$$

In particular, Barrett and Lampard have shown that Gaussian and Rayleigh processes are of this type, with the θ_n being Hermite and Laguerre polynomials, respectively. Convergence and other aspects of the Barrett-Lampard expansion were investigated by Leipnik [1959], while necessary and sufficient conditions under which $P(x_1,x_2;\tau)$ can be expressed in the form (1) have been given by J. L. Brown [1958]. Brown also studied [1957] a more general class of densities for which the expansion (1) is nondiagonal and the coefficients $A_{mn}(\tau)$ are restricted by the relation $A_{m1}(\tau)=d_m a_{11}(\tau)$, $m=1, 2, \ldots$, the d_m being real constants. As shown by Brown, processes with densities of this type exhibit a number of interesting properties.

One way in which the Barrett-Lampard expansion can be used in nonlinear filtering was pointed out by Zadeh [1957]. Specifically, assume that the second order density of a process with zero mean can be represented by (1), with the $\theta_n(x)$ not necessarily having the form of polynomials. Then, if an optimal (minimum variance) filter is sought in the class of filters admitting the representation

$$F(x)=\sum_{n=0}^{\infty}\int_0^{\infty}K_n(\tau)\theta_n[x(t-\tau)]d\tau, \qquad (3)$$

where the $K_n(\tau)$ are undetermined kernels, and the desired output is written as

$$F^*(x)=\sum_{m\epsilon M}\int_{-\infty}^{\infty}K_n^*(\tau)\theta_n[x(t-\tau)]d\tau, \qquad (4)$$

where M is a finite index set and the $K_n^*(\tau)$ are given kernels, the determination of the $K_n(\tau)$ reduces to the solution of a finite number of Wiener-Hopf

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[1] In Barrett and Lampard's definition of Λ, $p(x_1, x_2; \tau)$ is not assumed to be symmetrical.

integral equations

$$\int_0^\infty K_m(\tau)A_m(t-\tau)d\tau=\int_{-\infty}^\infty K^*(\tau)A_m(t-\tau)d\tau,\ m\epsilon M \tag{5}$$

with $K_n\equiv 0$ if $n\neq M$.

Another type of process—for which the problem of determining an optimal nonlinear predictor is greatly simplified—was introduced by Nuttall [1958]. Specifically, Nuttall calls a process separable [2] if the conditional mean of x_2 given x_1 can be represented as

$$E\{x_2|x_1\}=\int(x_2-\mu)p(x_2;\tau|x_1)dx_2=(x_1-\mu)\rho(\tau) \tag{6}$$

where μ is the mean value of the process and $\rho(\tau)$ is its normalized autocorrelation function. Separable processes form a slightly broader class than that defined by Brown [1957].

Among the many interesting properties of separable processes is the following prediction property. Let $s(t)$ be a signal mixed with additive noise. Then, if $\{s(t)\}$ is a separable process, the best estimate of $s(t+\tau)$ in terms of the best estimate of $s(t)$ is given by

$$s^*(t+\tau)=s^*(t)\rho_s(\tau)+\mu_s[1-\rho_s(\tau)], \tag{7}$$

where $\rho_s(\tau)$ and μ_s are the normalized autocorrelation and the mean value of the signal process, and starred quantities, represent optimal (minimum variance) estimates. In the absence of noise, the explicit formula for the best predictor in terms of $s(t)$ becomes

$$s^*(t+\tau)=s(t)\rho_s(\tau)+\mu_s[1-\rho_s(\tau)]. \tag{8}$$

Still another type of process for which the prediction problem is manageable was considered by D. A. George [1958]. Here the observed signal $f(t)$ is assumed to be the output of an invertible nonlinear system N preceded by an invertible linear system L to which a white Gaussian signal $x(t)$ is applied. Thus, symbolically, $f=NLx$ and $x=L^{-1}N^{-1}f$. Then, if an optimal estimate of $f(t+\alpha)$ is denoted by $\tilde{f}(t+\alpha)$, it is not difficult to find an operator H_α acting on the present and past values of $x(t)$ such that $\tilde{f}(t+\alpha)=H_\alpha[x(t)]$. Once H_α has been found, $\tilde{f}(t+\alpha)$ can be expressed in terms of the present and past values of $f(t)$ by the relation $\tilde{f}(t+\alpha)=H_\alpha L^{-1}N^{-1}f$.

While some authors have sought to simplify the prediction problem by considering processes with special properties, others have turned to special types of nonlinear operators. In particular, the work of Bose [1956, 1959] was extended by D. A. Chesler [1958] to operators of the form $F(\sum_{n=1}^{N}C_n\phi_n)$, where F denotes either a linear operator with memory, or a nonlinear memoryless operator, or a more general nonlinear operator possessing an inverse; the C_n are adjustable constants, and the ϕ_n are nonlinear operators such that the expectation $E\{\phi_n(x)\phi_m(x)\}=0$ for $m\neq n$, x being the input to the filter. As was shown by Bose in the absence of F the optimal value of each C can be determined by measuring the mean-square error as a function of, say, C_i and assigning to C_i the value which minimizes the mean-square error. This method is shown by Chesler to be applicable also when F is a linear operator or a nonlinear operator with no memory. The extension is less straightforward when the only assumption on F is that it possesses a realizable inverse.

In all the foregoing analyses the signal process is assumed to be stationary. However, there are many situations of practical interest in which an appropriate representation for the signal is a series of the form

$$s(t)=\sum_{i=1}^{n}\alpha_i\phi_i(t), \tag{9}$$

in which the $\phi_i(t)$ are known functions of time and the α_i are unknown constants or random variables. In such cases, the problem of filtering or predicting $s(t)$ reduces to the estimation of the coefficients α_i.

It was shown some time ago by Laning [1951] that when (a) the noise if additive, stationary and Gaussian, (b) the joint distribution of the α_i is known, and (c) the loss function $L(\epsilon)$ is nonnegative and vanishes for $\epsilon=0$, optimal estimators for the α_i are memoryless nonlinear functions of linear combinations of values of the input over the interval of observation. In a recent paper, similar results were obtained by a different and more rigorous method by Kallianpur [1959]. More specifically, for the case where the interval of observation is $[0, T]$, and the loss function is quadratic, Kallianpur derived explicit expressions for the best estimate of $s(t)$ at time $T+T_1$ in terms of n linear functionals of the form

$$\int_0^T x(t)p_i(t)dt,\ i=1, 2, \ldots, n,$$

where $x(t)$ is the sum of signal and noise, and the $p_i(t)$ are square integrable solutions of integral equations

$$\int_0^T R(t-\tau)p_i(\tau)d\tau=\phi_i(t), i=1, 2, \ldots, n, \tag{10}$$

in which $R(\tau)$ is the correlation function of the process.

More concrete results for the same general problem were obtained by Middleton [1959] and Glaser and Park [1958]. In particular, Middleton found explicit expressions for minimum variance estimators of the α_i for the cases where (a) the α_i are jointly normally distributed, (b) the α_i are independent and Rayleigh distributed, (c) the α_i are independent and their distributions are not symmetrical, (d) the α_i are independent and their distributions are symmetrical. Of these cases, only (a) and (d) yield linear estimators for the α_i.

[2] It should be noted that the term "separable process" is used in the theory of stochastic processes in an altogether different sense.

The relation between maximum likelihood, minimum variance and least squares estimates of the α_i was studied in earlier papers by Mann [1954] and Mann and Moranda [1954]. A number of interesting properties of minimum variance estimates of $s(t)$ and its derivatives for the case where the $\phi_i(t)$ are polynomials in t were found by I. Kanter [1958, 1959]. A central result of Kanter is that an optimal weighting function for predicting the j^{th} derivative of n^{th} degree polynomial can be expressed uniquely and simply in terms of optimal estimators of k^{th} derivatives of k^{th} degree polynomials, with k ranging between j and n.

5.2. Filtering and Prediction of Nonstationary, Discrete-Time, and Mixed Processes

As is well known [Miller, Zadeh, 1956], extensions of Wiener's theory to nonstationary processes lead to integral equations of the general form

$$\int_a^b R(t, \tau)x(\tau)d\tau = g(t),\ a \leq t \leq b, \tag{11}$$

in which $R(t, \tau)$ is the covariance function of the observed process. Little can be done toward the solution of this equation when $R(t, \tau)$ is an arbitrary covariance function. Thus, contributions to the theory of prediction of nonstationary continuous time processes consist essentially of methods of solving (11) in special cases.

Along these lines, Shinbrot [1957] discussed the solution of (11) for the case where $R(t, \tau)$ can be expressed in the form

$$R(t, \tau) = \sum_{n=1}^{N} \alpha_n(\tau) b_n(t). \tag{12}$$

Using Shinbrot's methods, the solution of (11) reduces to the solution of a system of differential equations with time-varying coefficients. There is some advantage in such a reduction when one has available a differential analyzer or an equivalent machine. Similar results are yielded by a theory due to Darlington [1958, 1959], in which many of the concepts and techniques of time-invariant networks are extended to time-varying networks. As in the paper of Miller and Zadeh [1956], a key assumption in these approaches is that the observed process may be generated by acting on white noise with a product of differential and inverse-differential operators, or equivalently, with a lumped-parameter linear time-varying network. Darlington's paper [1958] contains also a simplified technique for finding a finite memory Wiener filter for stationary signal and noise.

A special case for which explicit solution can be found has been studied by Bendat [1957]. Here the basic assumption is that the signal is of the form $s(t) = 0$ for $t < 0$, $s(t) = \sum_1^N (\alpha_n \cos nwt + b_n \sin nwt)$ for $t \geq 0$, where the α_n and b_n are random variables with known covariance matrices, while the covariance function of the noise is of the form

$$R(t_1,t_2) = Ae^{-\beta|t_1-t_2|} \cos\gamma(t_1-t_2) \text{ for } t_1,\ t_2 \geq 0$$
$$= 0 \text{ for } t_1 < 0 \text{ or } t_2 < 0. \tag{13}$$

Closely related cases in which the prediction problem can be solved completely are those in which the nonstationarity of signal and noise processes is due to a truncation (e.g., multiplying the signal and noise by a step function) of stationary processes. This is true also in the case of discrete-time processes, as is demonstrated by several examples in Friedland's [1958] extension of Wiener's theory to nonstationary sampled-data processes.

Several interesting results concerning the linear prediction of filtering of stationary discrete-time processes were described by Blum [1957a, 1958, 1957b]. In particular, Blum has developed recursive formulas which express the estimate at time n in terms of a finite number of past estimates and past values of the observed process. This type of representation is especially useful in connection with so-called growing memory filters, i.e., filters which act on the entire past of the input. Thus, if the input sequence (starting at $t=0$) is denoted by $x_0, x_1, \ldots, x_n$, and the filter output at time n is denoted by Z_n, then Z_n is expressible as $Z_n = \sum_{r=1}^{n} C_r X_r$, in which the C_r depend on n. A shortcoming of this representation is that as time advances the C_r have to be recomputed at each step and their number grows with n. On the other hand, a recursive relation (if it exists) is of the form

$$Z_n = \alpha_1 Z_{n-1} + \ldots + \alpha_k Z_{n-k} + b_0 x_n + b_1 x_{n-1} + \ldots + b_e x_{n-e}, \tag{14}$$

where α's, b's, k and e are constants independent of n and hence need not be recomputed. One complication in this approach to the problem is that in order to start the recursion one must know initially $Z_0, Z_1, \ldots, Z_k$.

A somewhat related but more general approach has been formulated recently by Kalman. Specifically, Kalman assumes that the observed process is an n-dimensional vector process $\{\underline{y}(t)\}$ which is generated by acting with a linear discrete-time system on a white noise $\{u(t)\}$: thus,

$$\underline{y}(t) = P(t)\,\underline{x}(t)$$
$$\underline{x}(t+1) = G(t)\underline{x}(t) + \underline{u}(t), \tag{15}$$

where the bars denote vectors and $P(t)$ and $G(t)$ are given time-varying matrices. (This assumption is analogous to the usual one in the case of nonstationary continuous-time prediction, viz, that the observed process can be generated by acting on white noise with a time-varying network.) Kalman shows that an optimal (minimum variance) estimate of $x(t)$ is given by the recursive relation

$$\underset{\sim}{x}^*(t+1)=[G(t)-A(t)p(t)]\underset{\sim}{x}^*(t)+A(t)\underset{\sim}{y}(t) \quad (16)$$

where

$$A(t)=G(t)M(t)P'(t)[P(t)M(t)P'(t)]^{-1} \quad (17)$$

and $M(t)$ is given by

$$M(t+1)=[G(t)-A(t)P(t)]M(t)G'(t)+Q(t) \quad (18)$$

where G' is the transpose of G and $Q(t)$ is the covariance matrix $Q(t)=E\{\underset{\sim}{u}(t)u'(t)\}$. The matrix $M(t)$ is the expectation of the matrix $\underset{\sim}{\epsilon}(t)\epsilon'(t)$, where $\epsilon(t)$ is the error at time t. In this formulation, to start the recursion one must know $x^*(0)$ and $M(0)$. However, in most cases the effect of the initial choices of $x^*(0)$ and $M(0)$ will be insignificant by the time the system reaches its steady state.

An interesting observation made by Kalman is that the prediction problem as formulated by him is dual to a problem in control theory in which the objective is to find an input which minimizes a quadratic loss function.

In additions to extensions of Wiener's theory to nonstationary continuous and discrete-time processes, extension to processes of mixed type were also reported. In particular, Robbins [1959] solved the mean-square optimization problem for the case where the filter consists of a linear time-invariant system followed by a sampler which is followed in turn by another linear time-invariant system. Janos [1959] gave a complete analysis of the case where a stationary signal is multiplied by a train of rectangular pulses, yielding a periodic pulse-modulated time series. The filter is assumed to be a time-invariant linear network. The integral equation satisfied by the impulsive response of the optimum filter is of the Wiener-Hopf type, but a multiplying factor involving trains of rectangular pulses complicates its solution. A method of solution of this equation is given by Janos for the infinite memory as well as the finite memory case.

5.3. Miscellaneous Contributions

There are several not necessarily unimportant problems in filtering and prediction which have received relatively little attention during the period under review. Contributions concerned with such problems are discussed in this section.

It has long been recognized that the use of a quadratic loss function imposes a serious limitation on the applicability of Wiener's theory. Under certain conditions, however, optimality under the mean-square-error criterion implies optimality under a wide class of criteria. Such conditions have been found by Benedict and Sondhi [1957], and, independently, by Sherman [1958]. Thus, Benedict and Sondhi have shown that in the case of a Gaussian process optimality with respect to a loss function of the form $L=\epsilon^2$, where ϵ denotes the error, implies optimality with respect to any loss function of the form $L=\sum_n|\epsilon|^n$, where $n>0$ but is not restricted to integral values. In Sherman's result, $L=f(\epsilon)$ is an even function and $\epsilon_2\geq\epsilon_1\geq 0$ implies $f(\epsilon_2)\geq f(\epsilon_1)$. More special cases involving the design of optimal filters under nonmean-square-error criteria have been considered by Bergen [1957] and Wernikoff [1958]. A time-weighted mean-square-error criterion which can be used to reduce the settling time of an optimal linear filter was employed by Ule [1957].

An extension of Wiener's theory to random parameter systems was described by Beutler [1958]. In Beutler's formulation, the signal and noise are assumed to have passed through a time-invariant random linear system before being available for application to a filter or predictor. The linear system is assumed to be characterized by a transfer function $H(w, \gamma)$, in which γ is a random parameter with a known distribution. In effect, this amounts to modifying the statistical characteristics of the original signal and noise processes.

The multiple series prediction problem for the infinite memory case was considered by Hsieh and Leondes [1959]. In their paper, Hsieh and Leondes describe a simplified method of solving the simultaneous integral equations for the weighting functions. Their technique is not applicable, however, to the finite memory case.

The optimization of continuous-time filters and predictors is frequently carried out by discretizing time and then letting the interval between successive samples approach zero. There are many published papers in which limiting processes of this type are used without adequate justification. A careful and rigorous analysis of the problems involved in obtaining optimum continuous-time linear estimates as limits of discrete-time estimates was given by Swerling [1958].

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Circuit Theory

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In this paper a report is presented on the research in circuit theory in the United States during the period 1957–1960. The paper was prepared as a progress report for submission to the XIII Triennial General Assembly of URSI, held in London in September 1960. The following subdivisions of circuit theory are treated:
1. Introduction.
2. Topology or linear graphs, including associated matrix formulations.
3. Synthesis by pole-zero techniques.
4. Realizability conditions and positive real matrices.
5. Systems with time-varying and nonlinear reactances.
6. Active systems.
7. Concluding remarks.

The discussion considers problems that have been solved in these areas as well as a number of important problems for which answers are still not available.

1. Introduction

In the past decade the boundaries of circuit theory [1] have expanded explosively; as a result the present range of circuit-theory research is enormous. It is thus manifestly impossible to give a short account of this research in the United States for the past three years. This would be true even if the "old" or more conventional definition [2] of circuit theory were used; use of a "new" or more encompassing definition [3] makes it hold *a fortiori*. The best one can do is to offer a few examples to suggest the vigor, pertinence, and extent of the present research in circuit theory. For this purpose we have chosen to concentrate on the following subdivisions of circuit theory; (2) Topology or linear graphs, including associated matrix formulations, (3) Synthesis by pole-zero techniques, (4) Realizability conditions and positive real matrices, (5) Systems with time-varying and nonlinear reactances, and (6) Active systems.

The above divisions are obviously overlapping. We subdivide them in this way merely for convenience of discussion and we shall not hesitate to point out interrelations.

In addition, we omit from detailed consideration a number of research areas that fall within the field of circuit theory and also overlap other fields. Among them are: (a) contact networks and digital computers; (b) data processing; (c) noise theory; (d) sequential circuits; (e) synthesis of distributed-parameter systems; and (f) matched filters. However, we will not completely neglect these areas, but will briefly mention some of the outstanding work in a few of them, though without a precise formulation of the problems. It is clear that these subdivisions of the circuit theory field, e.g., the research in data processing, have great relevance to the problems of interest to URSI, and it is recommended that some provision be made for their detailed discussion in the next triennial report.

It is difficult if not impossible to discuss the research accomplishments of the past three years in the United States without reference to much antecedent work and to work done in other countries; we see farther than our predecessors only by standing on their shoulders, and it is thus essential to refer to some of the accomplishments of the giants of former days. The presentation given here should be considered more in the nature of a portrait rather than a photograph.[4] We shall have to invoke the artist's privilege of emphasizing certain aspects of the subject to the exclusion of other aspects. To mix a metaphor, in some respects, as is true for any attempted summary of a vast subject, this report takes on the character of a personal odyssey through the present circuit-theory research in the United States.

Finally we hasten to point out that the references are intended to be merely representative, not exhaustive. Because of the fact that parallel lines of endeavor are going on at many research centers, almost an entirely different set of references could be given to illustrate the identical discussion. If we succeed in indicating the problems that have been agitating research workers and in elucidating some of those that have been solved and others that remain unsolved, we will have accomplished our purpose.

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[1] We use this term synonymously with *network theory*.

[2] Such a definition was proposed by Professor B. D. H. Tellegen at the 1957 URSI General Assembly held at Boulder, Colo. He suggested that the following definition be used to guide the deliberations of Subcommission 6.2:

Circuit theory is the theory of networks composed of black boxes characterized by relations between the currents and voltages at the terminals, which relations contain only time as an independent variable, and contain neither space nor temperature coordinates.

[3] In the ensuing discussion of Professor Tellegen's definition it appeared that many of the delegates of Subcommission 6.2 considered the definition too restricted. An *ad hoc* group, of which the writer was a member, proposed the following definition of circuit theory in the wide sense:

Circuit theory is the theory of networks of black boxes which are characterized by relations between the voltages, currents, or other variables at their terminals, and which are in general abstractions of physical components of electrical systems.

There appears to be slight difference between the definitions as stated. However, the discussion made clear that the proponents of the second definition wished to include areas like sequential circuits and networks with probabilistic elements (and in general such areas that overlapped the interests of Subcommission 6.1 on Communication and Information Theory), whereas those holding to the first definition would exclude these areas.

[4] The writer borrows this useful metaphor from his friend, Professor R. M. Foster.

2. Combinatorial Topology or Linear Graphs

The past decade has witnessed advances in circuit theory that are expressed in different ways. Much of what is being said about these advances becomes merely a babel unless the circuit theorist is multilingual. This should be interpreted in each of two ways. First, the same problems are being considered by competent scientists in many countries of the world. Second, different mathematical languages are being used to attack and gain insight into these problems. The use in circuit theory of the languages of function theory and some elementary aspects of matrix theory is fairly well established; new languages that have been introduced in the recent past are the language of linear graphs, the language of lattice theory, the language of vector spaces, and the language of sophisticated matrix theory [Trans. IRE 1959b]. We discuss below the field of linear graphs and associated matrix formulations of circuit-theory problems.

Though the basic concepts of linear graphs and their applications to network theory were introduced by Kirchhoff himself [1847], it is only recently that their great power for both analysis and synthesis has been widely recognized. A large part of this recognition stems from attempts to solve the synthesis problem for networks without transformers. One evidence of the intense and widespread interest in this field is the issue of the IRE Transactions of the PGCT that was devoted to this field [1958b]; another is the number of letters and industrial publications that treated this subject [Nerode and Shank, 1957; Nakagawa, 1958; Weinberg, 1958c; Kim, 1958; Calabi, 1956; and Hatcher, 1958].

A good proportion of the publications on graph theory are devoted almost exclusively to a reformulation of Kirchhoff's "Third and Fourth Laws," by which laws we mean his rules for writing down a network function almost by inspection. Some, however, do give basically new material. Mason [1956; 1957], for example showed how to determine system functions of *active* networks by topological rules. This represents an important extension, since Kirchhoff's techniques were restricted to the solution of passive networks without transformers. There were also a number of others who formulated topological rules for solving active networks [Boisvert, 1958; Coates, 1957; Mayeda, 1958b].

Mason's graphs, it should be pointed out, differ from Kirchhoff's; Mason calls them *signal-flow graphs*. These graphs are similar to the block diagrams used in system analysis; thus one difference from Kirchhoff graphs is that the algebraic sum of the signals at a node of a signal-flow graph is not zero, and a second difference is that signals flow along a branch in only one direction.

Two other problems that were solved are the realization of a loop matrix or cut-set matrix by a graph and the realization of a homogeneous polynomial as the discriminant of a network. The first problem is related to the still unsolved problem of realizing a real matrix as the resistance or conductance matrix of an n-port network containing only resistances and no ideal transformers [Slepian and Weinberg, 1958b]. Indeed, it may also be said to be a problem in any field where linear graphs are applicable, e.g., information theory and linear programing [Elias et al., 1956; Dennis, 1958, 1959; Jewell, 1958]. For a long time this problem was unsolved [5] and then as so often happens a number of solutions appeared almost simultaneously. Two solutions were presented at the 1959 International Symposium on Circuit and Information Theory. One paper by Guillemin is motivated by problems in network theory [Guillemin, 1959]; the second by Löfgren (of Sweden) is stated in terms of contact networks and appears to be fairly simple to apply [Löfgren, 1959]. If we exclude the work of the Russians, it is probably true that Gould was the first to solve this problem [Gould, 1957, 1958]; [6] his solution appears to be complicated in its application. Subsequently Auslander and Trent gave an alternative solution (1959).[7]

We have thus gone from poverty to an embarrassment of riches with regard to this problem; we now have what could be considered a plethora of solutions.[8] It is critically necessary at this point to consolidate our advances. All these procedures should be compared for their generality and ease of application; their merits and advantages for solving different types of problems should be illustrated. It would also be desirable that they be stated in a common simple language so that their differences and similarities become evident. Finally, if it is possible, an everyday design procedure should be formulated. Perhaps part of this task will be accomplished at the *Fifth Midwest Symposium on Circuit Theory: Topology in Circuit Theory* to be held on May 8 and 9, 1961 at the University of Illinois.

The problem of realizing a specified homogeneous polynomial that was mentioned above and the story of one of its solutions illustrate the fact that the pace at which we are finding solutions to problems of long standing is an accelerating one. An exceedingly difficult problem in the past [Foster, 1952] was the determination of the necessary and sufficient conditions for a homogeneous polynomial of n variables to be the discriminant of a realizable network—that is, the determinant of the system matrix of the loop or node equations. Some only partially successful attacks [9] on this problem were previously made by Cohn [1950], Shannon and Hagelbarger [1956], and Melvin [1956]. Dr. Campbell of BTL had also been interested in this problem around 1917, but he is

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[5] Perhaps it would be more accurate to state that the problem was not even formulated, since an awareness of the problem became explicit only in the last few years.

[6] The Russians have written many papers on contact networks; the writer believes there is a high probability that solutions to this and other "unsolved" problems are waiting to be exhumed from the Russian literature.

[7] In their paper Auslander and Trent [1959] give what could be considered an abstract solution. They have subsequently written a paper (as yet unpublished) that gives a constructive procedure for realizing the graph.

[8] The reader should not assume that we have mentioned all the solutions. There are, for examples, a solution by Harry Lee in his MIT master's thesis done under Professor Guillemin's supervision, and a solution by W. Mayeda, which he has submitted for publication to the Transactions PGCT.

[9] The writer is indebted to Professor R. M. Foster for this discussion of the earlier attacks on the problem.

not known to have reached any significant conclusions. Only one necessary condition was put forth in the three cited papers. If we let D be the homogeneous polynomial in the n variables R_k, and further let D_k be the partial derivative of D with respect to R_k, then the expression $(D_kD_1-DD_{k1})$ is the square of a homogeneous function of the R_k of degree one less than D and with coefficients that may be -1 as well as $+1$. Cohn's paper attempted to show that this condition was also sufficient, but a counterexample can demonstrate this to be impossible.

The problem was then mentioned by the writer in a talk he gave at Princeton. Dr. F. Harary, who was present at the talk, casually passed the problem on (during the 1959 International Symposium on Circuit and Information Theory) to Tom Crowley of BTL, who was commentator for the session on Switching Theory. Using the techniques of Löfgren's paper, Crowley announced he had solved the problem.[10] Subsequently the writer discovered that Mayeda had previously solved the problem [1958a]. It also appears that another solution has now been given by Duffin [1959].

This is not the only instance of a problem's being solved at the Symposium. A different problem that arises in linear programing [Heller and Tompkins, 1956; Hoffman and Kruskal, 1956] is the specification of a set of necessary and sufficient conditions on a real matrix for it to be a unimodular matrix, where a unimodular matrix is defined as a rectangular matrix all of whose subdeterminants (including each element considered as a subdeterminant of order one and also the determinant itself, if the matrix is square) are equal to ± 1 or 0. This problem also arises in network theory and in the theory of contact networks, and generally in any discipline that can be described in graph-theoretic terms; for example, the incidence matrix introduced by Kirchhoff is a unimodular matrix and so is the loop matrix based on a fundamental set of loops. The writer mentioned that this problem was unsolved in chairing the session on Graph and Matrix Theories; the following day D. Anderson of the Hughes Aircraft Company (who, it should be mentioned, had also been introduced to this problem previously) indicated he had a solution.[11]

The research mentioned above—that is, the realization of a loop or cut-set matrix, the complete characterization of the unimodular matrix, and the realization of a homogeneous polynomial—are all important in what the writer considers to be the crucial network problem at the present time, namely, the realization of networks containing no ideal transformers. Distinguishing classes of networks with regard to the inclusion or exclusion of ideal transformers is a fundamental method of differentiation. For example, it can be shown that the exclusion of transformers makes the realization of the n-terminal network a problem distinct from that of the n-port network, whereas when transformers are allowed a solution to one class of problem also solves the other. It is felt, furthermore, that solution of the problem of realizing transformerless networks will throw light on the problem of equivalent networks, and on how to obtain them by linear transformations.

As Cederbaum [1958] points out, most of the synthesis procedures for n-port networks use the artifice of the ideal transformer to solve the realization problem; to use his apt simile, the ideal transformer has been used like the *deus ex machina* of classical drama. By a suitable arrangement of transformers one can get combinations of voltages and currents which otherwise would be impossible. It is clear that new types of synthesis procedures are required; instead of assuming the network configuration in advance, as is done when we use one of the presently known procedures, the configuration will be derived from the mathematical characterization of the network. The resulting structure will probably be a complex interconnection of elements, a network in the true sense of the word, rather than one of the known canonical configurations. All of this indicates to the writer that the concepts of linear graphs will become increasingly important.

To mention one result for which no derivation is known other than a graph-theoretic one, we have the necessary condition that an impedance matrix or an admittance matrix of a pure resistance n-port must be a paramount matrix, where by a paramount matrix we mean a real symmetric matrix each of whose principal minors of order $p(p=1, 2, \ldots, n)$ is not less than the absolute value of any pth-order minor built from the same rows. Tellegen [1952] derived this result for a three-port by use of the fact that the voltage ratio of a resistance network cannot exceed unity; he also showed the condition to be sufficient for a three-port. However, for an n port with $n>3$, this method does not suffice and Cederbaum [1958] was forced to use linear-graph concepts for his derivation. These results and others on dominant matrices for resistance networks are summarized by Slepian and Weinberg [1958a]. The latter authors also derive a sufficiency condition on dominant *residue* matrices for two-element kind networks; this result was subsequently useful in the realization of active RC networks [Kinariwala, 1959]. In the above we use the term *dominant* matrix to mean a real symmetric matrix each of whose main-diagonal elements is not less than the sum of the absolute values of the elements in the same row.

2.1. Future Research Activity and Evaluation

The writer feels that the problem of realizing an n-port resistance network will be solved before the next General Assembly; implicit in this solution there will probably be a method for realizing RLC networks without transformers. This may appear to be a rash prediction since it was way back in 1952 that Foster wrote [Foster, 1952], "There is room for much further progress in the investigation of general n-terminal pair networks, especially the delineation

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[10] Though the writer has a copy of the paper that Crowley wrote, he does not believe it has yet been published.

[11] Again this solution has not yet been published but has been studied by the writer.

of just what can be done without ideal transformers, without mutual inductance, or with only two kinds of elements. Furthermore, even the theory of the true 3-terminal network (without pairing of terminals) for two kinds of elements (without mutual inductance or ideal transformers) is almost wholly unknown." Today each of these problems is still unsolved. However, we should recall that the problem of the discriminant that Foster also mentions is now solved. Furthermore, such men of the calibre of Guillemin and Darlington are now looking at problems of this nature. Guillemin is using linear transformations of matrices [1960a; b] as his method of attack, whereas Darlington has informed the writer in informal conversation that he was using vector spaces in his analysis of the problem.

The topological approach (as presented mainly in Cederbaum's papers) is also a promising one and should not be neglected. It has led to the brink of a major breakthrough on this problem—e.g., the statement of the paramountcy condition on impedance or admittance matrices of n ports containing no transformers—and provides a formulation of the problem in matrix terms that is elegant. Cederbaum [1958] has shown that a necessary and sufficient condition for a given symmetric nth-order matrix Z to be the impedance matrix of an RLC n port containing no real or ideal transformers is that it is a principal submatrix of the inverse of the triple matrix product $BY_{br}B'$, where Y_{br} is a diagonal matrix whose main-diagonal elements are $\underline{a}$, bs, c/s with $\underline{a}, b, c>0$, s is the complex variable $s=\sigma+j\omega$, B' is the transpose of B, and B satisfies the conditions for a cut-set matrix corresponding to an adequate system of node-pair voltages—that is, B can be realized as the cut-set matrix of a graph by one of the procedures mentioned previously. A necessary condition on B is that it be a unimodular matrix. A complete statement of the necessary and sufficient condition on B for it to be such a matrix is that there exists a decomposition of B of the form

$$B=K^{-1}Q$$

where Q is a reduced incidence matrix of the desired connected network and K is a reduced incidence matrix of the tree of node-pair voltages (that is, of the tree that is formed by drawing a branch for each voltage variable). An analogous condition can of course be stated for the admittance matrix of an n port.

This necessary and sufficient condition differs from those ordinarily given in synthesis, where sufficiency is demonstrated by a synthesis procedure.[12] Here no synthesis procedure exists because no method is known for decomposing Z into a principal submatrix of the desired congruent transformation of a diagonal matrix. Solution of this matrix problem would be a contribution of the first magnitude.

[12] For this reason it has been objected that such a form of necessary and sufficient condition is not of great value, that it in effect merely restates the problem. The writer does not agree since the restatement of the problem allows other mathematical artillery to be used in the solution. As an illustration of the value we should note that it has led to a solution of the problem of realizing a resistance n-port network that has only $(n+1)$ terminals, Cederbaum [1957].

To make this statement apply to two-element kind networks—e.g., to the RC case—we merely require that the elements of Y_{br} be of the form $\underline{a}$ and bs. To convert it to the problem of realizing a pure resistance n port, we stipulate that the diagonal elements of Y_{br} be positive numbers. In this case the elements of Z are of course no longer rational functions of s but are real numbers.

Some necessary conditions on Z are known: Z must be a symmetric positive real matrix; in addition, Z must be a paramount matrix for each value of s in the range $0<s<\infty$. However, a set of necessary and sufficient conditions is not known even for $n=2$, that is, the two-port network without transformers; it is also not known for the RC or LC case with $n=2$.

For the case of the resistance network when the n port is formed from the links pertaining to a tree, Cederbaum [1959] has furnished a solution. For the admittance case this represents a solution for the resistance network when the n port network has only $(n+1)$ terminals. The solution consists of an algorithm whereby the decomposition

$$Z=BY_{br}B'$$

if it is possible at all, can be carried out. Here we note the problem is simpler in that Z is not required to be a principal submatrix of the triple product but is equal to it. However, Cederbaum's techniques may be suggestive in solving the more general problem.

It should also be mentioned that a similar formulation as a triple matrix product is given by Bryant [1959a]. He shows that the necessary and sufficient condition for a real symmetric matrix Z to be the impedance matrix of a resistive n port is that Z be of the form

$$Z=S'G^{-1}S$$

where S' is the transpose of S, S is a submatrix of a reduced incidence matrix of a tree, and G is a dominant matrix with nonpositive off-diagonal elements. Again this should not be looked upon as a mere restatement of the problem. It may have an advantage over the Cederbaum formulation in that G can be recognized by inspection; however, the problem to be mentioned below of recognizing G^{-1} (the inverse of a dominant matrix with nonpositive off-diagonal elements) still remains. The transformation matrix S can also be recognized by inspection since the necessary and sufficient conditions for a matrix to be the reduced incidence matrix of a tree is that it be nonsingular, have elements ± 1 or 0, and in each column have at most two nonzero elements, specifically, one $+1$ and one -1. Cederbaum's transformation matrix, it should be recalled, must be unimodular, a test for which is laborious; and even if it is unimodular it may still not be realizable by a graph. Of course, the unimodular test may be omitted when this is convenient and the procedure for realizability as a graph may be applied directly. Bryant [1959b] considers additional formulations for resistance networks in his doctorate thesis.

We mention, finally, one more approach that may yield useful insights for solving the problem of realizing a resistive n-port network. We might prefer to assume that the network possesses accessible *terminals* rather than n ports—i.e., terminals paired into ports or terminal pairs—or we might find it convenient to switch between the two representations. There is a simple formula relating system functions in one representation to system functions in the other. This formula, which is given below, is not so widely known as it should be; its first appearance and proof in the literature are somewhat in doubt, and it is continually being rediscovered. One of the conceptual advantages of the $2n$-terminal network representation is that only driving-point measurements need be made; these characterize the n port uniquely. Thus an obvious necessary condition on each measurement is that it is a nonnegative number.

Consider a resistance n port with an open-circuit resistance matrix $R=[R_{ik}]$. Of course, since the n port obeys reciprocity, of the n^2 driving-point and transfer resistances only $n(n+1)/2$ are independent—that is, the matrix is symmetrical. Now consider this network as a $2n$-terminal network with the terminals numbered from 1 to $2n$, and with the ports so numbered that port 1 comprises terminals 1 and $(n+1)$, the assigned positive direction being from terminal 1 to terminal $(n+1)$. In general port k will run from terminal k to terminal $(n+k)$.

For the representation of the $2n$-terminal network let $S_{i,k}$ denote the measured driving-point impedance between *terminals* i and k, all other terminals being left free. Then we define $S_{k,k}=0$, since this measurement corresponds to both of the measuring leads connected to the same terminal. It is clear that

$$R_{kk}=S_{k,n+k}$$

The general formula for the elements of matrix R is

$$R_{ik}=\frac{1}{2}\,[S_{i,n+k}+S_{k,n+i}-S_{i,k}-S_{n+i,n+k}]$$

which reduces to the previous formula when $i=k$. A simple proof of this formula that uses Kirchhoff's topological rules has been given by Professor Foster in a private letter to the writer.

There are some other unsolved problems raised by graph-theoretic considerations. For example, can a simple test for a paramount matrix be devised? A direct test that follows from the definition is to check the required conditions on each principal minor of order $p\leq n-1$ and each of its corresponding nonprincipal minors. The evaluation of all possible minors, however, can be laborious, and the question naturally arises whether all minors must be tested. In other words can a simplification be effected as, for example, in the test for a positive definite matrix? We recall that for an nth-order matrix to be positive definite, it is necessary that all the principal minors are positive; it is sufficient, however, to test only a subset of n principal minors. It has been shown by Slepian and Weinberg [1958] that we must test all minors of order two. It can be shown, furthermore, that not much can be done to shorten the work of testing a matrix for its paramount character.[13]

Another matrix problem is the formulation of a simple method for determining whether the inverse of a nonsingular paramount matrix is a dominant matrix with nonpositive off-diagonal terms. This would then give a set of necessary and sufficient conditions for the realization of a real matrix as the impedance matrix of a $(n+1)$-terminal network containing only pure resistances. Also, with regard to a dominant matrix, though we know that the condition of dominance is sufficient for realization of a given matrix as the *admittance* matrix of a resistive n port, we still don't know whether this is true for realization as the *impedance* matrix.

A final problem may be mentioned for the paramount matrix. As indicated previously, it is known that paramountcy is sufficient for the realization of an n port for $n\leq 3$; though the writer conjectures that it is not sufficient for $n>3$, this has never been demonstrated. A method that has been suggested [14] for proving or disproving the sufficiency for $n=4$ is to consider the dominant admittance matrix

$$Y=\begin{bmatrix}7 & 1 & 2 & 3\\ 1 & 12 & 4 & 5\\ 2 & 4 & 15 & 6\\ 3 & 5 & 6 & 18\end{bmatrix}$$

This is realizable by a general procedure [Slepian and Weinberg, 1958a], but (as has been shown by Cederbaum [1959]) not by a four-port with only five terminals. Now suppose that Y is reduced to the irreducible [15] paramount matrix

$$Y_1=\begin{bmatrix}3 & 1 & 2 & 3\\ 1 & 5 & 4 & 5\\ 2 & 4 & 6 & 6\\ 3 & 5 & 6 & 53/7\end{bmatrix}$$

Now the question is whether there exists *any* four-port with Y_1 as its admittance matrix. It may be worth while to investigate this particular case and perhaps by the use of the possible geometrical configurations [Foster, 1932] for a four-port, the fact that a nonplanar network has no dual, and by the

[13] An example to illustrate that we cannot eliminate testing minors of order $n-1$ in an nth-order matrix was furnished the author in a private letter—etc. (See letter).

[14] This suggestion was made to the writer in a private letter from Professor R. M. Foster.

[15] By reducing a paramount matrix we remove main-diagonal elements without destroying the paramountcy condition. Then the reduced matrix is inverted and this reduction, if it is possible, is repeated. This yields shunt and series elements in the corresponding network. When a matrix is reached for which this is no longer possible since the paramountcy condition will be violated by such a step, this matrix is called irreducible. A detailed discussion of the reduction of a third-order matrix is given in chapter 8 of the author's book, "Network Analysis and Synthesis," to be published by the McGraw-Hill Book Co.

process of complete induction, it can be demonstrated that no such network exists and consequently that paramountcy is not sufficient.

Another irreducible paramount matrix suggested by Foster is

$$Y_2 = \begin{bmatrix} 3 & 2 & 1 & 3 \\ 2 & 3 & 2 & 3 \\ 1 & 2 & 3 & 3 \\ 3 & 3 & 3 & 5 \end{bmatrix}.$$

Does any four-port exist with Y_2 as its admittance matrix? Not only can this example throw light on the question of paramountcy, but it may also furnish an answer to an unresolved aspect of equivalent networks—specifically, do there exist matrices which are admittance matrices of networks without transformers but not impedance matrices, and vice versa? We recall that any matrix realizable as an admittance matrix can also be realized as an impedance matrix, *if ideal transformers are allowed;* however, this is an unanswered question for transformerless networks. For the matrix given above as Y_2 there is a simple network if this matrix of numbers is considered as an impedance matrix, namely, a chain of five one-ohm resistances, with the ports chosen as indicated in figure 1.

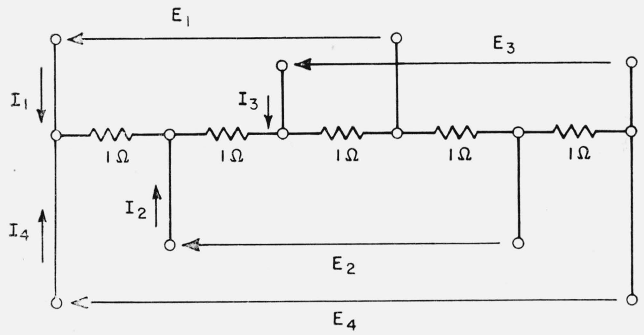

FIGURE 1. *Chain of five 1-ohm resistances realizing the given impedance matrix.*

Some important *analysis* problems still remain with regard to graph theory. We know that the driving-point and transfer functions of an n port may be expressed in terms of the independent driving-point functions of the network considered as a $2n$-terminal network. Thus a simple method for determining these driving-point functions is required. This is known [Weinberg, 1958a] for those driving-point functions measured across a branch of a symmetrical graph; however, the problem of specifying the driving-point function across two nodes not connected by a branch is still unsolved. Solution of this problem is important since it would permit simple calculation of the currents and voltages of large graphs used to simulate other physical systems. For an indication of the extensive computations that are presently required the reader is referred to Branin's paper [1959].

Another aspect of graph theory that should see wider use in the future is Wang Algebra [Duffin, 1959]. This appears to be ideally suited for the application of digital computers to network investigations. One of the troublesome problems that previously held up digital-computer research on networks by use of graph-theory concepts is the direct determination of all the trees of a network without duplication [Hobbs, 1959; Mayeda, 1959]; a method has now been given by Fujisawa [1959].

It is felt that the applications of graph theory to physical systems will increase rapidly in the next few years. The rate of increase will depend on the size of the cultural lag—that is, the length of time before linear graphs is taught in the schools as a routine tool of the engineer. In the past, network theorists have assimilated mathematical disciplines like function theory, matrix theory, and Laplace transform theory, and have developed general methods for analyzing exceedingly complex networks without becoming lost in a maze of detail. Then these concepts and techniques that were developed in network theory were recognized to be of great value to the applied mathematician and physicist [Mathews, 1959], irrespective of his field of specialization. Network concepts such as input-output and others have even found their way into a recent book on pure mathematics [Kaplan, 1958]. In addition to their use in exact sciences, the input-output concept and the ubiquitous black box have yielded rich rewards in such fields as biology and economics [Leontief, 1959]. Thus network theorists have insisted that their subject had a great deal to offer other fields and hence that engineers and physicists should learn the language of network theory [Weinberg, 1960].

This situation may now be changing; the change is illustrated by the field of graph theory. Graph-theory applications are being made with great speed in other engineering fields such as linear programing, information theory, and switching circuits. It may now become necessary for engineers and physicists to use the common language of graph theory rather than redefine the concepts in a manner appropriate for their own specialty. Perhaps future teachers of electrical engineering instead of stating Kirchhoff's current law in the old form that the algebraic sum of the currents at a node is zero will teach the equivalent linear-graph statement that the 1-chain I is orthogonal to the coboundary of each point of a graph G. In any case, however it is taught, we can be fairly certain that graph theory will eventually become as established as function theory or matrix theory in the educational background of the engineer and physicist.

3. Synthesis by Pole-Zero Techniques

Though Darlington's [1939] work was done about 20 years ago his techniques are still not being used by the average engineer. Following the lead of Grossman's paper in early [1957], which attempted to make Darlington's results on elliptic function filters more readily available, Henderson published nomographs [1958], Weinberg published tables of element values for Butterworth, Chebyshev and

Bessel-polynomial networks [Weinberg, 1957a; b; c], and Henderson and Kautz presented a whole series of graphs [1958] of the transient response of such networks. The large demand for reprints of these papers with the letters of comments attest to the cultural lag between what is known about filter theory by workers in this field and the methods used to design filters by the engineers in the laboratory.

The research activity in this area for the past three years has been devoted largely to an application of the classical RLC synthesis techniques to new types of systems and secondly to the extension of synthesis procedures to include rational functions with nonreal coefficients. We shall discuss these two trends, mention a new synthesis procedure, and then briefly consider the approximation problem. Finally, we shall give a fairly thorough discussion of the problem of finding explicit formulas for the element values of ladder networks.

The design of crystal filters has generally been treated by image-parameter techniques. Kosowsky [1958] has extended these techniques in his treatment of methods for realizing such filters. O'Meara [1958a; b; c] in a series of papers has attempted to show the value of modern synthesis techniques by applying them to particular crystal-filter configurations. With the increasing stress on transformation techniques for achieving desired network configurations [Saal and Ulbrich, 1958]—e.g., the so-called *zig-zag* filter—it is felt that general synthesis procedures for crystal filters may yet be formulated.

A new RLC synthesis procedure is that of Macnee [1958]; this may be useful in frequency-multiplexing problems. The network yielded by Macnee's procedure has open-circuited inputs and paralleled outputs. He thus realizes a set of transfer impedances in contrast to Guillemin's related procedure of realizing a transfer admittance by means of ladder networks paralleled at both ends.

Lewis applied RLC synthesis techniques to the realization of pulsed networks [1958], whereas Levenstein [1958] showed that the realization of networks with linearly varying resistances—i.e., potentiometer networks—was analogous to the RC synthesis problem. This correspondence will probably be extended in the future and has already led to the analysis of positive real functions of two variables [Ozaki and Kasami, 1959].

Baum has made a significant contribution to the design of narrow-band filters [1957; 1958a]. He has extended the techniques of synthesis to apply to rational functions whose polynomials have nonreal coefficients; this requires that he consider as additional types of elements in the low-pass domain fictitious frequency-independent positive and negative reactances; when the transformation to the band-pass domain is made, the networks become physically realizable. Baum [1958b] has also shown how to use fewer elements than in the Brune procedure in the realization of driving-point functions with geometric symmetry, such as are obtained in the low-pass to band-pass reactance transformation.

The application of RLC synthesis techniques to transmission-line networks by means of Richards' transformation was considered by Grayzel [1958]. A useful summary and extension of methods for handling this problem are given by Welsh and Kuh [1958].

In considering the approximation problem we find that Kuh has presented an additional solution for approximating the ideal delay function [1957]. The solution, which is found by means of the potential analogy, yields a tandem connection of a low-pass ladder network and an all-pass bridged network. It is more efficient than the maximally flat time delay yielded by Bessel polynomials in the sense that a wider bandwidth is achieved for a prescribed number of singularities and time delay. However, the Bessel-polynomial approximation is of course much simpler.

Papoulis [1958] considered the approximation of a magnitude characteristic and found the class of polynomials that has the maximum cutoff rate under the constraint of a monotonic response. Thus his polynomials give a magnitude function that combines the monotonic property of the Butterworth polynomials and the optimum cutoff property of the Chebyshev polynomials. Again, as in the case of Kuh's approximation, some measure of simplicity is lost: the Butterworth polynomials are much simpler than the set of new polynomials.

We now come to the discussion of ladder networks and explicit formulas for their element values. This problem has tantalized research workers ever since Norton [1931], who was the first to contribute to this problem, derived the formulas for the element values of ladder networks with a Butterworth characteristic and with a resistance termination at only one end. Bennett [1932] extended Norton's work by giving the formulas for the element values for the maximally flat ladder that is terminated in resistance at both ends.[16] However, Bennett's formulas are restricted to the case of equal resistance terminations. About 20 years later Belevitch [1952] derived the formulas for the Chebyshev-polynomial or equal-ripple ladder. Again, the formulas are not general: Belevitch's apply only to the matched ladder network. Orchard [1953] then extended Belevitch's formulas to the open-circuited or short-circuited Chebyshev ladder.

In 1954 a major breakthrough came when Green [1954] provided a generalization of all the preceding work; he discovered the formulas for ladders with a Butterworth or Chebyshev characteristic and with *any ratio of resistance terminations*. These formulas did not solve the complete problem since they apply only when the zeros of the reflection coefficient are chosen to lie in one half-plane. Depending on the choice of the zeros of the reflection coefficient, a number of other networks is possible. For a transfer function whose denominator is of odd degree, Wein-

[16] Bosse [1951] who appeared to be unaware of Bennett's work independently solved the same problem. In addition, Bosse was the first to give complete proof of the formulas. Bennett had his proof practically complete and Norton's analysis gave general formulas for 1st, 2nd, 3rd, etc., element of ladder for any total number of elements without proving the general formula for the mth element in n-element structure.

berg [1957c] solved the case of a symmetrical distribution of the zeros of the reflection coefficient—that is, for zeros chosen to alternate in the left and right half-planes.

These formulas have led the writer and others to conclude that we are somehow "missing the boat" on the ladder network. Though in a mathematical sense the Darlington method is an elegant solution to the general problem of realizing a lossless network terminated in resistances, the computations seem unusually complicated when applied to a simple configuration like the ladder. The writer has felt for a long time that the simplest methods of analyzing and synthesizing a ladder are still to be found. The discovery (and proof) by Indjoudjian [1954] of formulas for the element values of an n-stage RC amplifier have bolstered this feeling.

One of the disconcerting aspects of most of the above results on the Chebyshev and Butterworth ladders is that they were never rigorously proved, although their correctness was universally accepted. The formulas were derived by carrying out the calculations in detail for cases of low degree and then guessing the general result. An attempt to prove the general case, in the hope that such a proof would show how to solve related problems, resulted only in a proof for the Butterworth case [Doyle, 1958]. As remarked by Doyle, his proof is a "hammer-and-tongs" one in that it gives no clue to the reason for the amazing simplicity of the final formulas. It is thus not possible to extend the proof to formulas for the Chebyshev case or for other zero distributions of the reflection coefficient or finally, to formulas for the elliptic-function filter.

The above was the state of knowledge on the ladder network at the time of the last URSI General Assembly in 1957. What followed reads almost like a detective story. At a meeting of Sub-Commission VI–2, the writer mentioned the significant problems of finding formulas for the inverse Chebyshev and the elliptic-function filters and in passing commented that the formulas given by Green for the Chebyshev-polynomial case had never been proved. One of the participants in the discussion was Dr. H. Takahasi. After the meeting the writer and Dr. Takahasi had supper at which the latter casually mentioned that he had derived and proved the formulas in 1951. It must be admitted that the reaction of the writer was disbelief; it was so hard to imagine this to be true that he felt he hadn't explained the problem clearly to Dr. Takahasi. However, the ensuing conversation showed that Dr. Takahasi was aware of all facets of the problem. He promised to send a copy of his paper [Takahasi, 1951], and some time later he did.

Thus one of the unexpected effects of this URSI meeting of scientists from different countries is the uncovering of Dr. Takahasi's paper. This gives another illustration (if any are needed) of the desirability of more such international conferences.

In this paper Takahasi derives the formulas that were later independently given by Green. The wealth of new results, the elegance of the proof, and the implications for future work are adequately covered in the paper by Weinberg and Slepian [1960a] based on Takahasi's paper. Suffice it to say here that it is literally incredible that these results could have remained unknown to workers outside of Japan (and, it may be added, to many Japanese also) for so long a time. Perhaps the history of this problem as presented here can make some small contribution to eliminating a repetition of similar occurrences. The amount of duplication in research and calculating effort that could have been eliminated and the additional progress that could have been made in this field by widespread knowledge of Takahasi's paper are incalculable.

3.1. Future Research Activity

Though some work has been done on the realization of true RLC networks—that is, where the coupling network contains resistance elements inserted in a controlled manner—we are still in need of a general synthesis procedure. The state of knowledge even on the problem of incidental dissipation is not complete. We still don't know how to realize a network where each inductor is not restricted to the same dissipation factor. Perhaps graph theory may be useful here; some work has already been done on showing how equivalent ladder networks can be derived by the use of graph theory [Simone, 1959]. The procedures of Darlington [1939] and Bader [1942] apply to the case of nonuniform dissipation where d_C is the dissipation factor of the capacitors and d_L) is the dissipation factor of the inductors. Darlington did not present his procedure in detail; as a result a generation of readers has probably had difficulty in applying it. Desoer thus performs a useful service in giving a clear interpretation of Darlington's procedure [1959]. When the network is terminated at only one end, the problem is greatly simplified. Geffe [1959] considered such a singly loaded network whose reciprocal voltage ratio is a polynomial and gave formulas for the coefficients of the polynomial after predistortion; thus the need for the Darlington or Bader procedure is eliminated in this case.[16a] As is remarked by Bennett, in his proof appended to Geffe's letter, the fact that this closed-form solution for the coefficients is obtained should not be taken to imply that the general case can be treated similarly. However, further investigation of a possible simplification of the doubly loaded case would be worth while.

The writer also feels that *formulas for the element values* should exist for the uniformly predistorted Butterworth and Chebyshev cases. We know these formulas when no dissipation is introduced. One's sense of propriety is outraged when he finds that making a simple translation of the frequency variably forces him to carry out the computationally awkward continued-fraction expansion. Nature is generally not so perverse. After all, the poles of the Butterworth and Chebyshev functions still lie

[16a] It might be added that Orchard maintains (in a letter to the Editor, Trans. **PGCT,** June 1960) that direct calculation of the element values is simpler than using the closed-form expression for the coefficients of the polynomial.

on a circle and ellipse, respectively, except that the figures are shifted to the right. Perhaps the deep insights of Takahasi on the properties of the continued-fraction expansion for Butterworth and Chebyshev functions should help in this problem as well as in some of the other problems mentioned below.

It appears that there is no end to the closed-form formulas that can be found for the Butterworth and Chebyshev functions. For the Butterworth transfer function given by

$$|K(j\omega)|^2=\frac{1}{1+\omega^{2n}}$$

the time delay $T_d=-d\beta/d\omega$ (where $K(j\omega)=|K(j\omega)| e^{j\beta}$) has been found to be

$$T_d=\frac{\sum_{m=0}^{n-1}\frac{\omega^{2m}}{\sin (2m+1)\pi/(2n)}}{1+\omega^{2n}}$$

For the Chebyshev function given by

$$|K(j\omega)|^2=\frac{1}{1+\epsilon^2T_n^2(\omega)}$$

where $T_n(\omega)$ is the Chebyshev polynomial of the first kind (i.e., $T_n(\omega)=\cos (n \cos^{-1} \omega)$), the time delay in closed form[17] is

$$T_d=\frac{\sum_{m=0}^{n-1}\frac{U_{2m}(\omega)\sinh (2n-2m-1)\phi_2}{\epsilon^2 \sin (2m+1)\pi/(2n)}}{1+\epsilon^2T_n^2(\omega)}$$

In the above $U_n(\omega)$ is the Chebyshev polynomial of the second kind—i.e., $U_n(\omega)=\sin (n+1)\phi/\sin \phi$, where $\omega=\cos \phi$—and ϕ_2 is the imaginary part of ϕ given by $\phi_2=1/n \sinh^{-1} 1/\epsilon$. It should now not be difficult to go further; perhaps the time delay for the inverse Chebyshev may be found in closed form, and even that of the elliptic-function filter. One of the other problems where the insights of Takahasi should be helpful is the determination of formulas for the element values of an inverse Chebyshev transfer function—that is, the function obtained by a simple transformation of a Chebyshev-polynomial function which yields a low-pass filter with a maximally flat pass band and an equal-ripple stop band. Again it should be possible to determine the effect of the transformation on the formulas for the element values of the Chebyshev transfer function. Of course, the inverse Chebyshev has finite transmission zeros so that each arm of the ladder network no longer consists of a single inductance or capacitance, but the finite zeros are known in closed form so that it should be a simple matter to add proper resonating elements to an element given by a formula. The difficulty that is introduced by the steps of the continued-fraction expansion may now be removed by the properties derived by Takahasi. One should perhaps start with the simplest ladder network, that is a ladder with a resistance at only one end.

The problem of greatest moment with regard to the derivation of formulas for the element values is the case of the elliptic-function filter—that is, the filter with an optimum cutoff characteristic and equal ripples in both the pass and stop bands. This problem is exceedingly difficult,[18] but well repays long study. Discovery of the formulas could well bring about a revolution in the applications of modern filter theory. It is suggested that some of the relationships presented by Helman [1955] may be useful here since they relate the elliptic-function filter to the Chebyshev filter (for which formulas are known) without the introduction of elliptic functions.

Most of the above could be looked upon as an attempt to achieve a general understanding of the ladder network, one aspect of which is to answer the question whether formulas for the element values can be found when the reciprocal transfer function is a polynomial many of whose properties are known analytically. For example, can such formulas be found for the transfer function with a maximally flat time delay, that is, the function yielded by use of the Bessel polynomials? The continued-fraction expansion about the origin of the ratio of the even and odd parts of the polynomial representing the reciprocal transfer function is simple, the rth coefficient of $1/s$ being given by $2r-1$; perhaps a related functional form also exists for the coefficient of s in the expansion about infinity. The transfer function corresponding to the so-called synchronously tuned amplifier, treated by Indjoudjian [1954], should be investigated for any insights it may offer. Indjoudjian derived and proved formulas for the singly loaded case; the doubly loaded case is thus still unsolved.

Finally, formulas for the element values of the network with a distribution of zeros of the reflection coefficient other than all in one half-plane are known in only one case [Weinberg, 1957c]; some effort will probably be expended in determining the formulas for other zero distributions.

Perhaps future research in this area will demonstrate that each type of function must be investigated individually, that fortuitous circumstances permitted the determination of the known formulas. At any rate, it would be desirable to establish some conclusion; such investigations will surely yield insights valuable for further research in network theory.

It appears that research on network functions expressed as functions of *two* complex variables may be accelerated in the next few years. Such functions arise in many different investigations. An analysis of the positive real functions of two variables that arise in potentiometer circuits has already been mentioned [Levenstein, 1958]. Reference to Takahasi's work will show that he makes elegant use of the properties of symmetrical polynomials in two

[17] Both of these formulas for the time delay were sent to the writer in a private letter from H. J. Orchard.

[18] One should be optimistic, however. In a private communication H. J. Orchard writes that he believes he has found the formula for the first reactance of the elliptic-function filter.

variables. Furthermore, the functions describing networks containing resistances, inductances, capacitances, and transmission lines of commensurable length are functions of two complex variables after a substitution has been made to remove the exponential terms. Such functions also arise in control theory when systems containing a transportation lag (i.e., a pure time delay) are treated.[19] It has also been suggested [20] that the realization of networks containing inductors with unequal dissipation factors might be attacked by the use of rational functions of two variables.

The positive function introduced by Baum in his theory of narrow-band filters, as contrasted with the positive real function, will get further study and application in network theory. Already Belevitch [1959a] has used it to obtain what he feels to be a more natural derivation of the Brune cycle.

Finally, the analysis and synthesis of nonlinear networks will come in for increasing attention. A start on this problem is represented by the treatment of the piecewise linear case obtained by using networks of diodes and resistances; some work in this area is that by Stern [1956] and Dennis [1959].

4. Realizability Conditions and Positive Real Matrices

In the section on graph theory we discussed the still unsolved problem of characterizing the second-order impedance or admittance matrix of a grounded RC quadripole. Some necessary conditions have been derived using function theory rather than graph theory. It has been shown by Slepian and Weinberg [1958b] that the order relationships that hold for the numerator coefficients of the z_{ik} before cancellation of possible common factors—namely, that the coefficients of z_{21} must be positive and not greater than the corresponding coefficients of z_{11} and z_{22}—must hold even after cancellation for the z_{ik} or the y_{ik} in the case of a network with less than six nodes; in other words, for such a network it is impossible for both sets to violate the conditions. These results have been extended in a doctorate thesis by Olivares [1959]. Some additional results have been obtained in other countries [Bryant, 1959; Adams, 1958], but the general problem still remains unsolved.

For the case when only a transfer function of the RC three-terminal network is specified, some additional work has been done. Kuh [1958] has given an alternative synthesis procedure, and Kuh and Paige [1959] have determined the maximum possible multiplier for the voltage ratio of an RC ladder network.

Some recent work has been done on characterizing networks containing negative elements in addition to positive resistances, inductances, and capacitances. A general theory for the synthesis of networks containing negative elements becomes more urgently needed with the widespread use of the negative-impedance converter and especially with the discovery of the tunnel diode. A basic attempt at the formulation of realizability conditions for such networks is given in Bello's doctorate thesis [Bello, 1959]. Some additional papers are scheduled for presentation at the Polytechnic Institute of Brooklyn Symposium on Active Networks and Feedback Systems to be held in April 1960. Further discussion of the synthesis aspect of this problem is given in the section on Active Systems.

The problem of characterizing the matrices of passive n ports has received much attention. The paper of Youla, Castriota, and Carlin [1959] uses the scattering matrix and attempts to derive a rigorous theory of linear, passive, time-invariant networks on an axiomatic basis. Their discussion is heavily mathematical, being replete with the concepts of Hilbert space. The papers of Weinberg and Slepian [1958; 1960b] use the impedance and admittance characterizations and thus discuss the positive real matrix. They give new realizability conditions on different types of networks; in addition, they attempt to establish simple tests for checking the realizability of specified matrices whose elements are rational functions. One of the properties of their tests is the elimination of the usually required step of solving for the roots of polynomials. Some other work on the positive real function was done by Seshu and Balabanian [1957]; they consider transformations of a positive real rational function that keep the positive real property invariant.

4.1. Future Research

Enough has been said in the graph-theory section on the unsolved problems of finding necessary and sufficient conditions for the realizability of transformerless networks, where these conditions include a synthesis procedure. We have only to add here what should be obvious, namely, that the techniques of function theory can be used to supplement those of graph theory for answering these knotty problems, and we mention one problem whose solution may give significant insight. Then we discuss positive real functions.

The problem on the RC grounded quadripole is the proving or disproving of a conjecture made by Darlington [1955]. He has stated his belief that the series-parallel network constitutes a canonical subclass for the general three-terminal RC network—that is, every second-order matrix realizable by an RC grounded quadripole may be realized in series-parallel form. Darlington has proposed the realization of the following impedance matrix to test his conjecture:

$$Z=\frac{1}{2s(s+1)}\begin{bmatrix} s^2+6s+1 & s^2-s+1 \\ s^2-s+1 & s^2+6s+1 \end{bmatrix}.$$

Though this matrix satisfies all the known necessary conditions on the impedance matrix of an RC grounded network, no realization as a series-parallel

[19] It should perhaps be pointed out that stability problems in this case can be treated precisely by Pontryagin's [1955] theorem. It appears that this is not known to workers in feedback control theory since a search of many of the leading books in this field reveals that complicated approximate techniques are used for determining stability.

[20] This suggestion was made to the writer by Prof. N. DeClaris.

structure is possible and no other type of grounded quadripole has yet been found [Olivares, 1959]. Olivares has stated his belief that the matrix is unrealizable as a three-terminal network; however the matrix is readily realizable as a symmetrical lattice.

It is inevitable that much research effort will be devoted in the future to the positive real function and the positive real matrix. The definition that is usually given [Weinberg and Slepian, 1960b] is satisfactory provided we are dealing with matrices whose elements are rational functions. However, the concept of a positive real function is important enough to be defined in general; such functions arise whenever passive systems are considered. For example, the matrices treated by Wigner and von Neumann [1954] are positive real matrices except for a trivial rotation of the axes of the complex plane. The authors show the necessary and sufficient conditions on their matrices, but these are given in a nonconstructive form; in other words, no simple way of testing these matrices is given. The question arises whether tests can be devised for these matrices as was previously done for matrices of lumped-parameter electrical networks [Weinberg and Slepian, 1958; 1960b].

A general definition of a positive real function (PRF) is even needed [21] in electrical theory, for use with distributed systems or with the limits of finite lumped systems as the number of elements increases without limit. Under such circumstances for example, the simple function $\sqrt{s}$ should obviously be PRF; here we see that the function is real only for s real and nonnegative. Furthermore, with many transcendental functions, the definition of the function differs for s real and positive and of *zero* angle from s real and positive and of angle 2π. Accordingly, Professor Foster proposes that the fundamental definition of a positive real function be expressed in terms of the argument of the complex variable rather than in terms of a half-plane. The proposed definition of a PRF $F(z)$ is as follows:

(i) $F(z)$ is an analytic function of z for $-\pi/2<\arg z<\pi/2$
(ii) $F(z)$ is real for $\arg z=0$
(iii) $\text{Re}[F(z)]\geq 0$ for $-\pi/2<\arg z<\pi/2$
(iv) $F(z)$ to be defined by analytic continuation when possible beyond the sector in which it is defined by (i)—(iii).

Remark 1. The restriction in (i) may possibly be too strong, but at the present moment, the restriction to analyticity seems to be the only feasible assumption to make. Further study may be needed on this point.

Remark 2. If the restriction in (iii) were made stronger, that is, $\text{Re}[F(z)]>0$, the only effect would be to exclude the special case of $F(z)$ being identically zero. From a certain point of view, this might be an advantage, since then, in quite a number of theorems, this particular case would not have to be excluded by a special statement. On the other hand, this would exclude the simple case of a short circuit (if we were talking of impedances). On the other hand, in normal mathematical procedure, we would have automatically excluded the case corresponding to $F(z)$ identically infinite. This particular impasse might perhaps be completely avoided if we defined an entirely new sort of function $W(z)$ corresponding to

$$W(z)=\frac{F(z)-1}{F(z)+1}.$$

This corresponds to what is accomplished by using the scattering matrix as opposed to either the short-circuit admittance matrix or open-circuit impedance matrix. More study with respect to this particular point also seems desirable.

Remark 3. Note that nothing is postulated explicitly concerning the behavior of the function on the imaginary axis. For a study of this behavior we depend entirely on analytic continuation from the sector $-\pi/2<\arg z<\pi/2$. Also note that we do *not* include the suggestion of Richards that essential singularities on the imaginary axis be specifically excluded.

Remark 4. An example of a PRF:

$$F_1(z)=z^r$$

where $-1\leq r\leq 1$. Interestingly enough, the physiologists have investigated the electrical properties of many different organic materials where the impedance over a very wide frequency range is of the form $F_1(z)$ with a constant value of r characterizing each particular kind of material, the actual numerical values somewhere in the neighborhood of -0.7 for the most part.

Remark 5. An example of a PRF with an essential singularity at the origin:

$$F_2(z)=1-\exp(-1/z).$$

That this function is PRF may readily be seen by noting that

$$\text{Re}\,[F_2(x+iy)]=1-\exp\left(\frac{-x}{x^2+y^2}\right)\cos\frac{y}{x^2+y^2}.$$

Hence the positive character when x is positive.

Remark 6. Another example of PRF with an essential singularity at the origin which is not isolated, being a cluster point of poles:

$$F_3(z)=\frac{1}{1+z}+\frac{1}{2(1+2z)}+\frac{1}{4(1+4z)}+\frac{1}{8(1+8z)}+\cdots$$

Here each component in the infinite series is itself a PRF, and the series is seen to be absolutely and

[21] For the remaining discussion on positive real functions the writer is indebted to Professor R. M. Foster, who communicated these ideas in a private letter. He writes that his ideas are tentative, but the whole note is so suggestive for future research that it is used (with permission) in its entirety.

uniformly convergent in the sector $-\pi/2<\arg z<\pi/2$ since in that sector

$$\frac{1}{N(1+Nz)}<\frac{1}{N}.$$

Remark 7. An example of a PRF with all poles and zeros on the imaginary axis (i.e., i PRF to use Richards' notation [1947]:

$$F_4(z)=\tanh\ z.$$

That this function is PRF may readily be seen by noting that

$$\mathrm{Re}\,[F_4(x+iy)]=\frac{e^{4x}-1}{(e^{2x}\cos 2y+1)^2+(e^{2x}\sin 2y)^2}$$

which is positive when x is positive.

Remark 8. An example of a PRF with a natural barrier on the imaginary axis:

$$F_5(z)=\tanh\ z+\frac{\tanh 2z}{2}+\frac{\tanh 4z}{4}+\frac{\tanh 8z}{8}+\cdots$$

Here each component in the infinite series is itself a PRF. That the series is absolutely convergent and represents an analytic function in the sector $-\pi/2<\arg z<\pi/2$ is seen by the following consideration:

$$|\tanh(x+iy)|^2=\frac{e^{4x}-2e^{2x}\cos 2y+1}{e^{4x}+2e^{2x}\cos 2y+1}$$

$$\leq\frac{e^{4x}+2e^{2x}+1}{e^{4x}-2e^{2x}+1}$$

$$=\mathrm{ctnh}^2\mathrm{x}$$

and now assume that x is in the domain

$$0<r\leq x$$

where r is some fixed positive number. Then

$$\left|\frac{\tanh N(x+iy)}{N}\right|\leq\frac{\mathrm{ctnh}\ Nx}{N}\leq\frac{\mathrm{ctnh}\ Nr}{N}\leq\frac{\mathrm{ctnh}\ r}{N}.$$

Thus the function is well defined.

To show the existence of a barrier, consider any value of y of the form

$$y=\frac{\pi(2m+1)}{2^{n+1}}$$

where m is any integer, positive, negative, or zero, and n is a positive integer. And then consider the term in $F_5(z)$ which is

$$\frac{\tanh\ Nz}{N}$$

where $N=2^n$ and for this particular value of y. The real part of this term is equal to $\dfrac{\mathrm{ctnh}\ Nx}{N}$ and becomes positively infinite as x approaches zero keeping y at this same fixed value.

It is believed that these thoughts of Professor Foster on positive real functions will stimulate future research on this important problem.

5. Systems With Time-Varying and Nonlinear Reactances [22]

During the past three years networks containing nonlinear reactances have been used as amplifiers; these amplifiers have been called *parametric amplifiers*. The concept of sustained oscillations in nonlinear systems is an old one, Lord Rayleigh having described in 1877 the stability conditions for a system excited at twice the frequency of the unstable vibrations [Valdes, 1958]. Similar behavior was studied in connection with electromechanical systems. In addition, nonlinear reactive modulators were used in radiotelephony before 1914. It wasn't until 1957, however, that the present-day flurry of activity on parametric amplifiers started. At that time Suhl [1957] suggested using the anomalous dispersion effect in ferrites to make a variable-inductance parametric amplifier. In the succeeding three years much literature and many devices have appeared under the title of parametric amplifier. Most of the theoretical works during this period, with a few notable exceptions, are reviews, rediscoveries, and adaptations of theories already known. The device technology, however, started from zero and made startling advances. We do not discuss devices here since this will be reported on by Commission VII.

The theoretical background for parametric amplifiers can be divided into two fairly distinct categories. The first is the energy-conversion properties of nonlinear reactances, and the second is the circuit theory of linear networks with periodic time-variant parameters.

The fundamental energy-conversion property of nonlinear reactances is characterized by a set of energy relations known as the Manley-Rowe equations [1956].[23]

Manley and Rowe showed that when a nonlinear capacitance imbedded in a linear fixed-parameter network is excited by sources at two frequencies ω_0 and ω_1, the power flow $P_{m,n}$ into the capacitance at the various combination frequencies $m\omega_0+n\omega_1$ is characterized by the equations

$$\sum_{m=1}^{\infty}\sum_{n=-\infty}^{\infty}\frac{mP_{m,n}}{m\omega_0+n\omega_1}=0$$

$$\sum_{m=-\infty}^{\infty}\sum_{n=1}^{\infty}\frac{nP_{m,n}}{m\omega_0+n\omega_1}=0.$$

[22] The writer thanks Dr. B. J. Leon for his help in the preparation of this section.

[23] The so-called Manley-Rowe equations go back further than this paper. They were contained in a paper by J. M. Manley, "Some General Properties of Magnetic Amplifiers," Proc. IRE, 39, 259 (1951); this paper was based on unpublished work done by Manley much earlier, at least going back to the late 1930's.

Many subsequent papers have appeared deriving the Manley-Rowe equations in various ways. Manley and Rowe used a method of Fourier analysis. Penfield [1959] used an energy-function approach. From this he was able to extend the equations to reactive n ports with k noncommensurable exciting frequencies. Haus [1958] showed that the Manley-Rowe equations apply to the power carried by an electromagnetic field in a nonlinear lossless medium.

To see how the Manley-Rowe equations are applied to a parametric amplifier problem, let us consider the circuit of figure 2. For this circuit the powers $P_{m,n}$ are zero for $(m,n) \neq (1,0), (0,1), (1,-1)$ because the capacitance faces an open circuit at these frequencies. Thus the Manley-Rowe equations become

$$\frac{P_{1,0}}{\omega_0}+\frac{P_{1,-1}}{\omega_0-\omega_1}=0$$

$$-\frac{P_{1,-1}}{\omega_0-\omega_1}+\frac{P_{0,1}}{\omega_1}=0.$$

Since there is no source at frequency $\omega_0-\omega_1$, $P_{1,-1}$ must be negative (flowing out of the capacitance) if at least one of the P's is nonzero. If $\omega_0<\omega_1$, $P_{0,1}$ is also negative and energy is converted from the frequency ω_0 to both ω_1 and $\omega_0-\omega_1$. With this condition the "signal" at frequency ω_1 is "amplified."

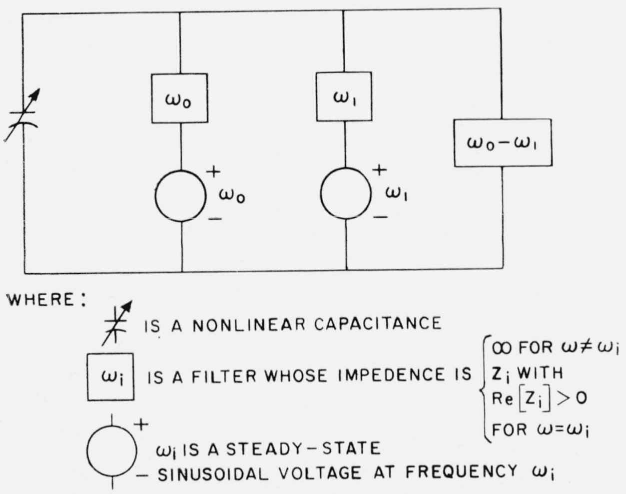

FIGURE 2. *Idealized parametric amplifier.*

In general the Manley-Rowe equations give an indication of feasibility for a particular amplifier with sharply tuned filters, and, in addition, they give quantitative information on the conversion efficiency. They do not give any information on the possible gain and bandwidth for a particular circuit model.

In order to get quantitative information about the performance of a parameteric amplifier one must analyze a complete circuit model. No general theory exists for analyzing nonlinear circuits of the type used in parametric amplifiers. However, if the excitation at one frequency, known as the pump, is much larger than the other excitations called the signal, a linear model characterize this small-signal performance. The general class of circuit models which characterize the small-signal performance of parametric amplifiers are linear circuits with a few periodic time-variant parameters imbedded in a network of time-invariant parameters. The most complete proof of this statement is given by Duinker [1958]. We shall refer to networks of this type as *linear parametric networks* (LPN).

The case of linear parametric networks with limped elements (LLPN) has been studied in great detail. The time-domain equations that characterize these networks are linear differential equations with periodic coefficients. Homogeneous equations of this type have been discussed quite extensively in the mathematical literature (Starzinskii, 1955; McLachlan, 1947]. The techniques used on these equations are so involved that they cannot conveniently be used to obtain the general transient and steady-state solution to inhomogeneous equations. It is the latter type of equation that is of interest for the design of an amplifier. Bolle [1955] pointed out that when the excitation to an LLPN is a steady-state sinusoid, one can write down the frequencies of all the resulting currents and voltages. He discussed the case of one variable element and showed that if the element were described by a finite number of sinusoids and if the network were such that all but a finite number of the voltage and current terms were zero, then the amplitudes of the nonzero voltages and currents could be computed. Duinker [1958] extended Bolle's method to include more variable elements, but he did not eliminate the two qualifying conditions. Virtually all of the recent papers on LPN use Bolle's method [Rowe, 1958; Heffner and Wade, 1958; Seidel and Hermann, 1959].

For the case of an LLPN with a single variable element described by a finite number of sinusoids, Desoer [1959] presented a method of steady-state analysis. His method, which is exact for these circuits, consists of an algorithm for computing the amplitudes of the voltages and currents in the same manner as Bolle, but Desoer proved, in addition, that the neglected terms do not have to be zero. Desoer gave a bound for the error introduced by neglecting the higher-frequency terms, and he showed that this bound tends to zero as we increase the number of terms used.

A more general method of analysis has been presented by Leon [1959; 1960 (in press)]. He showed that the frequency-domain equations that characterize both the transient and steady-state behavior of LLPN's are linear difference equations with rational-function coefficients. The difference-equation approach yields exact computational techniques for analyzing specific circuits. It also gives formal solutions that can be discussed in general terms. Although the two papers of Leon have answered a lot of questions about the analysis of LLPN's, there are many more to be solved before a synthesis procedure for these networks can be formulated. The second paper states many of these problems in detail [Leon, in press].

For distributed LPN's Tien and Suhl [1958] showed that the approximations of Bolle's method lead to a pair of coupled equations similar to the equations for traveling-wave tubes. This has led to a number of devices of both the forward and backward traveling-wave type. For iterative LLPN's consisting of a cascade of single variable-element circuits Currie and Weglein [24] of the Hughes Aircraft Company have shown that Bolle's method also leads to a pair of coupled equations similar to traveling-wave tube equations. A number of other analyses of distributed LPN's has appeared [Roe and Boyd, 1959; Bell and Wade, 1959; Kurowawa and Hamasaki, 1959; Pierce, 1959; and Shafer, 1959].

The theory of LPN's is far from complete and many interesting problems exist. A bigger problem is that of obtaining a quantitative LPN approximation to a pumped nonlinear circuit. To find the model, one must analyze the nonlinear circuit with a single-frequency (pump) excitation. All solutions to date have been very approximate and apply only to very special cases.

6. Active Systems

It is well known that a passive finite lumped-parameter network can achieve all the characteristics of any stable active finite lumped-parameter network except possibly for the gain; in other words, the transfer function of the active network cannot be more complicated than a rational function. Thus figure 3 represents a possible realization of any active transfer function, where the purpose of the amplifier is merely to supply gain.

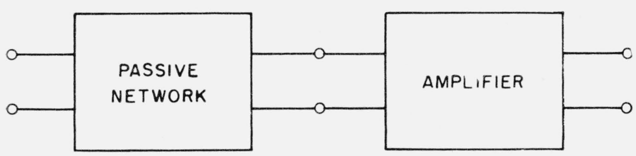

FIGURE 3. *Possible form for realization of any active transfer function*

It is useful to remember this fact; it saves our chasing rainbows for the proverbial pot of gold. We can add feedback loops within feedback loops almost *ad infinitum* (and often *ad nauseam*); alas, we still cannot get more than a quotient of two polynomials as the transfer function. Recognition of this fact makes us determine precisely why we are using feedback in a configuration—e.g., in the adaptive systems to be discussed below; surely not for achieving a desired transfer function that has, for example, a fast response. An open-loop configuration would do as well.

A similar simple characterization applies to an active stable driving-point function. It can always be realized by a passive driving-point function plus a negative resistance, either in series or in parallel. This is schematically illustrated in figure 4. Though

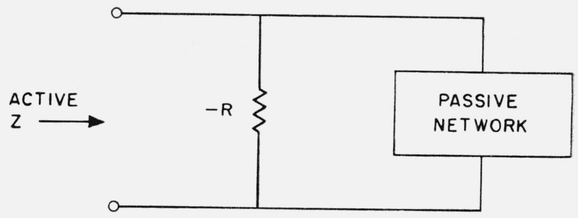

FIGURE 4. *Representation of an active driving-point function.*

this is not suggested as a practical means for realizing such a function, it is a feasible one.

We will not discuss systems such as those shown in figure 3, where the active element furnishes only gain (and perhaps isolation when a number of such networks are cascaded as, for example, in a flat staggered n-tuple amplifier). We will discuss two classes of active networks. In the first class the active element is used in a feedback configuration to achieve some desired result that cannot be achieved by a passive system upon which *some restriction has been placed*. Such a restriction may be the requirement of using no inductances in a network; here feedback is used as a tool in the synthesis of an RC network to achieve RLC characteristics. Another restriction can be the specification of a fixed network (called the *plant*) whose parameters vary in some manner; this network is required to yield a specified transfer function that is insensitive to the parameter variations of the fixed system. This sensitivity requirement necessitates the use of feedback; we shall discuss this in the context of the research on adaptive systems.

The second class of systems that we discuss involves negative elements that have not been achieved by a feedback circuit; more specifically, we consider networks containing tunnel diodes.

6.1. Active RC Synthesis

The use of inductances at low frequencies introduces many difficulties. Thus to achieve RLC characteristics—e.g., complex poles close to the imaginary axis—attempts have been made to use only resistances and capacitances with an active element to achieve a desired pole-zero pattern. The first active RC synthesis in the literature is due apparently to Fritzinger [1938] and Scott [1938]. The principle of their method is shown by the signal-flow diagram in figure 5; this approach is now often called the classical method or the *feedback* method in distinguishing it from the approach that uses the

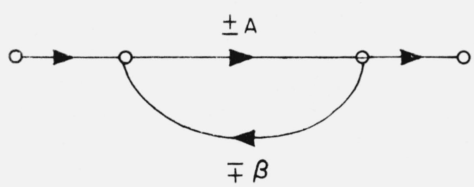

FIGURE 5. *Signal-flow diagram of the classical method of active RC network design.*

[24] A paper on these results is being prepared.

negative-impedance converter (NIC). The amplifier gain A is assumed to be a constant independent of frequency and β represents the transfer function of the RC network in the feedback path. The poles of K are the zeros of $1+A\beta$; we can thus get complex poles that cannot be achieved with the RC network alone. The assumption of idealized properties should be noted: the active element is ideal with infinite input impedance, zero output impedance and zero reverse transmission. This field of active synthesis remained dormant for a long time—for good reasons. The passive elements, namely, the positive resistance, the positive inductance, and the positive capacitance, are rugged, long-lived and can be designed to be quite stable with respect to ambient conditions; networks containing tubes, on the contrary, are less rugged, bulkier than passive networks, and have characteristics that may deteriorate for any of a number of reasons, among them being insufficient cathode emission, the variation of an active parameter, or a change in the power-supply voltage. The concept of a negative element or a negative driving-point function is, as we've mentioned above, useful in the synthesis of active circuits. It was used by Merrill to design an NIC using vacuum tubes [1951]. This, however, possessed the disadvantages of any other vacuum-tube circuit.

The advent of the junction transistor changed this situation. It made possible a small, rugged, long-lived active package that can be used in an NIC to give the characteristics of a negative driving-point function over an operating range of frequencies. J. G. Linvill was the first to exploit the transistor in the design of an NIC [Linvill, 1953, 1954]. This brought the NIC into prominence as a tool for active network synthesis.

Linvill achieved a transfer function whose poles were not restricted to the negative real axis by the use of RC networks and an NIC. The denominator polynomial with unrestricted zeros was decomposed into the difference of two polynomials whose roots are confined to the negative real axis; the subtraction is achieved by the N.C. The zeros of the transfer function were achieved by passive networks.

Kinariwala [1959] also used the NIC to achieve RLC characteristics with RC networks. He showed how to realize any driving-point impedance by means of resistances, capacitances, and only one NIC. In his configuration the NIC could achive the subtraction required in both the numerator and the denominator. Horowitz [1956] pursued a different course in applying the NIC: he extended the classical work of Brune, Darlington, and Dasher—i.e., synthesis by means of a cascade connection of canonical sections—to active RC synthesis. He didn't, however, solve the general problem, since he realizes only a positive real RC driving-point impedance in order to achieve an associated transfer impedance with unrestricted zeros; the network configuration is that of active RC ladders. The general problem of realizing a positive real RLC driving-point function, or even going further, a driving-point function that is not positive real, by means of a cascade of canonical RC sections is still unsolved. In addition, Horowitz's method does not show how to realize a large gain; achieving a large constant multiplier is often an important consideration.

Horowitz [1957, 1960 (in press)] also pursued research on the classical method. Contrary to the original attacks on this problem, where as we pointed out ideal active elements were assumed, Horowitz took into account the active-element parasitics and found the limitations due to them.

6.2. Adaptive Systems

The past three years have seen a great deal of activity in plant- or process-adaptive systems. Much of the motivation appears to be due to the large and rapidly changing parameters of modern supersonic aircraft and the resulting problems imposed on the autopilot.

Nearly all workers in this field have divided up the problem into three phases:

(1) Identification of plant or process parameters
(2) Computation of required corrective action
(3) Modification of system parameters or of signals to achieve corrective action.

The differences in the research have been in the methods used in one or more of these three phases. In phase (1), the following methods have been used:

(a) correlation of noise input with plant output to obtain the plant impulse response [Anderson et al., 1958; Goodman and Hillsley, 1958];
(b) sampling of plant input and output, and solving the difference equations relating input and output [Kalman, 1958; Mishkin and Haddad, 1959].
(c) construction of a model of the plant and using the differences between the output of the plant and its model to vary the parameters of the model so as to minimize the differences [Margolis and Leondes, 1959].

In general the resulting systems are nonlinear and thereby difficult to analyze. Hence most of the analysis has neglected the nonlinearities. Even the linear analysis comes forth with few basic conclusions on the reasons for the adaptive systems. A critique on these adaptive systems [Horowitz, 1960] implies that there has been a singular lack of continuity between this research and fundamental feedback theory. The workers give as motivation for their work the problem of large parameter variations. Some suggest that ordinary time-invariant linear feedback is unable to cope with large parameter variations. Horowitz feels this is not true. Others suggest that ordinary feedback may be inadequate because of noise or saturation limits [Staffin and Truxal, 1958]; Horowitz feels this criticism may often be valid. It appears, however, that more analytic work is needed in this area of research.

6.3. Tunnel-Diode Networks

Except for the active RC synthesis, the field of active synthesis is largely unexplored; though some problems have been solved, there does not exist a body of synthesis procedures comparable to that for passive networks. Some new approach is needed; it has often been felt by network theorists that the development of a pure negative resistance might stimulate such an approach. This is one reason why the discovery of the tunnel diode is exciting [Sommers et al., 1959; Lesk et al., 1959].

Much of the work on linear amplifiers using tunnel diodes has represented attempts to build a stable single-stage amplifier [Sommers et al., 1959]. According to most discussions in the literature it appears that the problems in the design of tunnel-diode amplifiers are how to achieve isolation in order to build two-stage amplifiers and how to make the tunnel diode unilateral. The writer questions whether these are the real problems. Perhaps the available synthesis procedures for passive networks can be adapted to the design of active networks.

What synthesis emphasizes is the realization of a prescribed gain-bandwidth by essentially a single process; or to be more specific, it attempts the exact realization of a prescribed function of frequency—its magnitude, its phase, and its constant multiplier. This requires a change in philosophy from that being used in present design; instead of worrying about isolating the active device, one should attempt to take advantage of its parameters in achieving a desired frequency and gain characteristic. Instead of bemoaning the fact that the tunnel diode is a two-terminal device, one should take advantage of this: our passive synthesis procedures employ two-terminal elements.

One method of adaptation of a synthesis procedure has been proposed by the writer [Weinberg, in press]. It applies to the synthesis of tunnel-diode networks, where the equivalent circuit is taken to to be a parallel connection of the junction transition capacitance and a negative resistance.

This technique is an adaptation of *predistortion*. The use of the predistortion technique in reverse may be useful for realizing active networks incorporating the new devices. Instead of substituting $s=p-d$ into the given system function, we substitute $s=p+d$, where d is a positive constant, $s\equiv\sigma+j\omega$ is the original complex variable, and p is a new complex variable.

It is recalled that in ordinary predistortion the pole of the given system function that is closest to the j axis limits the size of d that can be chosen. In reverse predistortion, however, stability considerations no longer limit the size of d, since poles of the original function, instead of moving closer to the j axis, move away from the axis. A number of other advantages are obtained by this shift of the critical frequencies to the left. For example, nonminimum-phase functions can be made minimum phase by choice of an appropriate value of d; thus procedures that can be used only for minimum-phase functions—like Dasher's procedure for the realization of resistance-capacitance (RC) networks, or even simple ladder networks—now become applicable.

It is not true, however, that, since the shift is to the left, there are no constraints on the value of d. As shown in [Weinberg, 1958b], for a normalized design d is the reciprocal of Q; thus the value of the Q that can be achieved with the tunnel diode may be a limiting factor for some applications. For a negative-resistance device it is desirable that the absolute value of Q be as small as possible; for example, a small capacitance and a large absolute value of negative conductance yield a high-quality tunnel diode.

A most important effect of this procedure of reverse predistortion is that the final network, which will require negative resistances for its realization, yields what could be called a *flat gain*. This gain can be computed in a manner similar to that given in the reference [Weinberg, 1958b] for computing the flat loss.

A simple example illustrates the technique. In this example the tunnel diodes, in effect, substitute for an ideal transformer. The voltage ratio

$$K\equiv\frac{E_2}{E_1}=\frac{H}{(s+1)(s+3)}$$

is realized by the network in figure 6 with $H=15$. The maximum possible H for a passive network without transformers is 3. Each of the RC parallel networks within the dashed lines can be replaced by a tunnel diode.

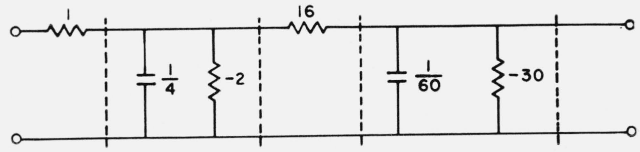

FIGURE 6. *Network realizing the given RC voltage ratio. (Values in ohms and farads.)*

Nonuniform predistortion can also be used to realize RLC networks containing tunnel diodes. In addition, it appears possible to control the number of tunnel diodes used in the design.

6.4. Future Research Activity

There are two approaches that have been explored in active RC synthesis using feedback techniques. One is the NIC approach exemplified by the work of Linvill [1953] and Kinariwala [1959]. Here complex poles that are unrealizable by RC networks and zeros that are inconvenient to realize by such networks are achieved by polynomial *subtraction*. The other approach, as carried forward by Horowitz [1957, 1960 (in press)], is basically the classical method; this is achieved by the *addition* of polynomials. This classification is interesting from the point of view of sensitivity. Because of the subtraction in the first approach the resulting sensitivity of the filter to the active- and passive-element variations is very large. The optimum NIC synthesis from the point of view of sensitivity to both active and passive elements was found by Horowitz [1959.]

The second class of procedures leads to considerably less sensitivity to the active elements but sensitivity to passive elements is of the same order of magnitude as in the first class. While the ultimate in sensitivity in the first class has been solved,[25] in the second class it has been achieved only for specific configurations. Also no study has been made to determine configurations which lead to minimum sensitivity to passive-element variations. It should be mentioned, moreover, that this has not even been done in passive network synthesis.

In the matter of extending modern network synthesis to active RC systems, a significant research problem is to apply the Brune and Darlington methods to active RC realization of any input impedance by means of a cascade of canonical sections. This appears to be a very difficult problem.

With regard to active synthesis procedures using negative elements, more study should be applied to extending and applying the work of Bello [1959]. In addition, optimum tunnel-diode synthesis procedures with respect to gain-bandwidth and other criteria will probably be worked out. An inevitable problem that will arise when active synthesis becomes practicable is the sensitivity problem. Finally, it is desirable that an understanding of the deceptively simple negative resistance become more widespread; for example, one sees again and again in the literature the incorrect statement that a negative resistance cannot be both open-circuit stable and short-circuit stable.

Much basic work remains to be done in the theory of adaptive systems. Up to now the mass effort has been on building systems. This is evident from the references previously cited and a perusal of the Proceedings of the Symposium on Self Adaptive Flight Control Systems, held at Wright Air Development Center on January 13–14, 1959. To quote Lt. Gregory of the Flight Control Laboratory, Wright Air Development Center, which government organization sponsored and organized the Symposium [Gregory, 1959]: "I think there is one general statement we can make about most of our systems and that is, they work; but why do they work? In the future we intend to try to establish the basic fundamentals of why our systems work and how we can analyze them better . . . . We intend to devote more of our program to the development of the basic fundamentals." To this statement of future plans one can only say: amen. If at least a small part of government money used to support work in adaptive systems is devoted to basic research in this area, a firm analytical base will be placed under future designs.

7. Concluding Remarks

To round out our discussion of circuit theory, we make some brief comments on books, special issues of journals, and Symposia devoted to areas of circuit theory.

Previously the student of network synthesis was forced to pick up much of his background in the field by consulting old issues of journals. This unsatisfactory situation no longer exists. The field of network synthesis has now received a wealth of documentation in book form. This will undoubtedly accelerate research in extensions of RLC synthesis techniques to active systems, to nonlinear systems, and to analogous nonelectrical systems. The driving-point problem is painstakingly treated at some length by Tuttle's book [1958], whereas Balabanian [1958] has covered both driving-point and transfer function synthesis. The book by Kuh and Pedersen [1959] attempts to introduce synthesis at the undergraduate level. These books, coupled with those of Guillemin [1957] and Storer [1957] give an adequate picture of many aspects of synthesis. At least three of the above books not only collect the significant material that could formerly be found only in technical journals, but also contain previously unpublished results or results published only as theses. Another significant event was the translation into English of Cauer's [1958] important book; this will serve as a reference and important scientific document for many years to come. It appears that there will be a continuing flow of books on the subject, now that the tap has been opened. At least two more are planned for the next year, one by Van Valkenburg [1960] and the other by Weinberg [in press]. Finally, another book that should be mentioned in this connection is the one on control systems edited by Truxal [1958]; this book contains sections on signal-flow theory, network synthesis, and sampled-data systems.

Of course, the circuit theorist will still require to read the journals in order to keep up; in fact, he will be hard put to it to keep his head above water even in his own particular area of circuit theory. The field is so fast-moving that no sooner is a new idea broached than it receives a critical comment, an extension, or a new application; an example was previously cited on Baum's introduction of the positive function and Belevitch's applying it to the realization of a Brune network. The problem of bringing circuit theorists up to date in their comprehension and application of what is now known has been a cause of some concern to the Administrative Committee of the PGCT. A number of remedies has been proposed, one of them being the sponsorship of Symposia and another being the publication of special issues of the Transactions PGCT.

Though one of the purposes of a special issue has been tutorial, most of them have in large part contained new material. The special issue on topology has already been mentioned [IRE Trans., **CT-5,** 1958b]. There have also been such issues on sequential circuits [IRE Trans., **CT-6,** 1959], active systems [IRE Trans., **CT-4,** 1957], and modern filter design techniques [IRE Trans., **CT-5,** 1958a]. Special issues are planned on the applications of electronic computers to network design and on nonlinear networks. The latter issue had Dr. B. van der Pol as Guest

[25] H. J. Orchard in a private communication to Dr. Horowitz shows an elegant and simple method of decomposing the polynomial; this method eliminates the need of the nonlinear-equation approach used by Horowitz [1959].

Editor until his untimely death; it will be published as a memorial to the late distinguished scientist.

In the past three years the Transactions PGCT has consolidated its position as the foremost network-theory journal in the country. Under Dr. W. R. Bennett, who took over the Editor's job from Dr. W. H. Huggins, the Transactions has continued to publish the outstanding papers on the circuit-theory research that is being done in the U.S. The journal has also attracted such papers from all over the world.

The International Symposium on Circuit and Information Theory held at UCLA in June 1959, has been previously noted [IRE Trans., **CT-6,** 1959b]. In addition, an important International Symposium on the Theory of Switching was held at Harvard University on April 2–5, 1957 [Vols. XXIX and XXX, Harvard University Press, 1959]. This Symposium included three Russian papers, one of which summarizes the research on relay networks in the U.S.S.R. and gives an interesting chart comparing the numbers of articles on switching theory published in various countries [Gavrilov, 1959]. There is also a paper by Belevitch that attempts to bridge the gap between the theory of contact networks and RLC network theory by taking account of equations of current flow in contact networks [Belevitch, 1959b].

Finally, a special Transactions issue on matched (or *conjugate*) filters is being planned by the Professional Group on Information Theory [IRE *Trans.*, **PGIT,** 1960]. This area appears to be one where sophisticated network design techniques are urgently needed. The TW (time-bandwidth) product for a signal or its matched filter arises in this theory; it is a most important parameter since in general a better signal requires a larger TW product. Not much has been done at the present time to realize matched filters with TW products greater than several hundred. Achieving products an order of magnitude larger by practical networks represents one of the unsolved network-theory problems. Detailed statements of the other problems in this field are given in the special issue of the Transactions PGIT.

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Part 1. Diffraction and Scattering

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The borderline of the diffraction and scattering field both with other fieldr of interest in Commission 6 and also with those of other Commissions is no longer well-defined. In order to effect a delineation, we have excluded from this report all aspects of diffraction and scattering pertaining to areas of investigation which are covered more properly under separate reports. For example, scattering from discontinuities in surface waveguides is expected to be covered in Wait's report on surface waves, while the vast area of multiple scattering and scattering by rough surfaces will be described separately by Twersky (Commission 6.3). We have also omitted any discussion of diffraction and scattering problems involving plasma media. The field of plasma physics and one of its subdivisions; namely, the transmission, scattering, and absorption of electromagnetic waves by high, low, and medium density plasmas, would appear to warrant a separate subcommission with joint membership between Commissions 3, 6, and 7. Most previous interest in plasmas has been in purely ionospheric effects and the interactions of electromagnetic waves with the ionosphere, as covered by Commission 3. However, when one considers the interaction of electromagnetic fields with plasmas caused, for example, by the motion of a high-speed vehicle through the atmosphere, problems arise which are of basic interest to the activities covered by Commission 6.3. The plasma field is growing so quickly that it would seem desirable to form at the next General Assembly a new subcommission to deal with those plasma problems of interest to several commissions.

This report will concern itself with high-frequency diffraction (involving obstacles with dimensions large compared to the wavelength), Rayleigh scattering (obstacle dimensions small compared to the wavelength), and scattering in the resonance region (obstacle dimensions comparable to the wavelength). Moreover, we mention those areas which we feel will receive attention during the next three years. The list of references appended to this report, while not presumed to be complete, is certainly representative of the current activities in the electromagnetic diffraction and scattering field in the U.S.A. through December 1959. In addition to work mentioned specifically in the body of the report we have also appended references which contain material either related to topics discussed in the text or of somewhat broader interest in diffraction theory. Additional references are to be found in recent books by Wait [1959a] and King and Wu [1959], with the latter devoted primarily to a summary of research activities in electromagnetic diffraction and scattering at Harvard University.

Primary emphasis in this report is placed on recent theoretical developments, and selected pertinent experimental results are mentioned only in conjunction with verification of certain theoretical predictions discussed in the text. The authors are well aware of the excellent experimental programs under the direction of P. Blacksmith at Air Force Cambridge Research Center, R. Kell and J. Lotsof at Cornell Aeronautical Laboratory, E. Kennaugh and L. Peters at Ohio State University, S. Silver and D. Angelakos at the University of California, Berkeley, R. King and H. Schmitt at Harvard University, and R. Hiatt at the University of Michigan. There are, of course, significant measurement programs going on at many of the major corporations. In this regard the work at Radiation Incorporated, Melbourne, Florida, should be particularly mentioned.

1. High-Frequency Diffraction

As in the preceding period, the work carried out on diffraction problems during the past 30 months can be grouped into two broad categories: (1) The solution of canonical problems, and (2) the investigation of general approximate methods of solution. By canonical problems we mean those for which exact formal mathematical solutions can be found; the asymptotic investigation of these results in the short-wavelength limit yields the rigorous asymptotic behavior of the solution. Because of the requirement of rigorous mathematical solutions, canonical problems usually involve simple configurations whose component surfaces are describable by a single coordinate in a given coordinate system.

In contrast, the aim in studying general approximate methods of solution is to provide asymptotic expressions for the scattering by objects of relatively arbitrary shape. In the limit of short wavelengths the scattering from such objects appears to arise primarily from the vicinity of certain stationary points on its surface (at least, if the object is impenetrable). The configuration in the vicinity of these points can frequently be approximated by a canonical one, as for example, a sphere, wedge, cylinder, etc., and the total scattering can then be computed by a systematic procedure combining the effects of the various canonical contributions. Thus, a study of canonical problems is indispensable for an accurate analysis of relatively arbitrary structures. Moreover, solutions of canonical problems often provide the means for checking the results of a more general approximate procedure. It is desirable in this connection to seek an interpretation of the asymptotic solution for a canonical problem in terms of simple physically meaningful contributions, such as geometrical optics, diffraction and transition effects, with the latter arising in the vicinity of geometric optical boundaries.

1.1. Canonical Problems [1]

A number of results became available during the past 30 months for the problem of diffraction by a wedge whose sides have a nonzero surface impedance. In a cylindrical coordinate representation, in terms of which this configuration is analyzed most naturally, a constant nonzero surface impedance leads to a mixed boundary condition at the wedge faces so that the usual method of separation of variables no longer applies. Following a procedure employed previously by Peters [1950] in connection with a study of water waves on a sloping beach, Senior [1959a] obtained a rigorous solution for the two-dimensional problem of diffraction by a homogeneous imperfectly conducting wedge of arbitrary angle. The method of solution is rather complicated but contains certain features which seem to indicate the feasibility of solving the wedge problem by a generalized Wiener-Hopf technique. For the special case of a right-angle wedge, Senior shows that the generally very complicated formal result reduces to a simple expression. Karp [1959a] and Karp and Karal [1958] employed an entirely different technique to solve the right-angle wedge problem for both line-source and plane-wave excitation by introducing an auxiliary problem which removes the coupling of the boundary conditions at the wedge faces. They have also employed a modification of this technique in a simple method of evaluating the diffracted far fields for a dissipative wedge with interior angle $\pi/2n$, $n=1, 2, \ldots$ [Karal and Karp, 1959a]. Apart from treating the dissipative case, Karal and Karp also considered right-angle wedge configurations, one or both faces of which have a constant surface reactance which allows the propagation of a surface wave, and they have calculated the amplitude of excitation of the surface wave [2] [Karal and Karp, 1959 a; b]. Considered in the asymptotic limit of short wavelengths, the results of Senior and Karal and Karp yield the expected decomposition of the far field into geometrical optics, diffracted (due to the presence of the edge), and, possibly surface wave contributions. The complications arising in this class of problems when an electromagnetic wave is incident obliquely have also been emphasized by the above authors [Senior, 1959b; Karal and Karp 1958].

Concerning lossless wedges, a summary of solutions for scalar steady-state and pulse excitations has been presented by Oberhettinger [1958]. Results for diffraction of pulses by a perfectly conducting wedge and by a half-plane situated on the interface between two semi-infinite dielectric media have also been obtained by Papadopoulos [1959].

As regards the numerical evaluation of the scattering from an absorbing half-plane, the formal solutions available in the literature have been suitable only for small values of surface impedance. Utilizing previous work of Fock and Gruenberg [1944], Marcinkowski [1959] has obtained a comparatively simple far-field representation from which an evaluation for arbitrary impedance values can be carried out conveniently. He presents numerical calculations for the diffracted fields of a lossy half-plane which absorbs completely a plane wave incident at a specified angle.

Although generally mixed, the boundary conditions at the wedge faces may be uncoupled in a cylindrical coordinate representation if one chooses a surface impedance (or admittance, depending on polarization) which varies linearly with distance from the edge. This problem was solved for arbitrary wedge angles and two-dimensional excitation via the separation-of-variables technique by Felsen [1959a] who showed that if the variable impedance is reactive, the surface can support a new type of surface wave which decays exponentially away from the surface along a circular arc centered at the wedge apex. Felsen [1958] also carried out a high-frequency asymptotic evaluation of the plane wave scattering by such a wedge and found that the solution is interpretable in terms of geometrical optics and edge diffracted contributions which exhibit an explicit dependence on the rate of variation of the surface impedance. For the case where the wedge degenerates into a half-plane, Shmoys [1959] has employed a separation-of-variables analysis due to Lamb [1945] utilizing both rectangular and parabolic cylinder coordinates to obtain the solution for diffraction by a half-plane with a rather specialized impedance variation differing from the linear variation mentioned above. He has carried out an asymptotic evaluation yielding geometrical optics and diffraction effects.

Concerning diffraction by a perfectly conducting semi-infinite cone, Felsen [1959b] has obtained the expected decomposition of the rigorous far-field solution due to a radiating ring source concentric with the cone axis into geometrical optics, diffraction and transition effects. Explicit formulas are given for the geometrical optics and transition contributions, while the angular distribution of the diffracted field arising from the presence of the cone tip is represented in terms of a canonical integral (which can be evaluated approximately) [Felsen, 1957a]. Felsen [1959a] has also analyzed the two-dimensional azimuthally symmetric problem of scattering by a cone with a linearly varying surface impedance and has obtained results analogous to those described above for the similar wedge configuration.

The problem of diffraction of a scalar plane wave by a large circular aperture in an infinite plane screen was investigated by Levine and Wu [1957] via an integral equation technique. By approximating the kernel of the integral equation for the aperture in a manner which highlights the straightedge-like behavior of the aperture rim in the high-frequency limit, they solved the resulting integral equation and obtained the first few terms of an asymptotic expansion for the scattering cross section of the aperture in inverse fractional powers of ka, where k is the free-space wave number and a the aperture radius. They

[1] Although this section emphasizes the short-wavelength behavior, any formal canonical solutions apply for all wavelengths.

[2] The reader is also referred to their forthcoming N.Y.U., Inst. Math. Sci. Rept., Scattering of a surface wave by a discontinuity in surface reactance on a right-angled wedge.

also present a physical interpretation of the various contributions to the scattering cross section as arising from simply and multiply diffracted geometrical rays. An analogous procedure was employed by Wu and Seshadri [3] for the electromagnetic problem involving a vector plane wave, and by Wu [1958] and Tang [1959] for the plane-wave and cylindrical-wave scattering, respectively, by an infinite slit.

The diffraction problems listed above give rise to asymptotic field solutions which contain geometrical optics, transition and either edge- or tip-diffraction effects. A considerable effort has also been expended on configurations which exhibit surface curvature. Suitable canonical structures in this category are the (two-dimensional) circular and elliptic cylinder and the (three-dimensional) sphere and spheroid. Emphasis has been placed on the extension and solidification of the approximate theory introduced by Fock [1946]. Wetzel,[4] Logan,[5] Goodrich [1958] and Wait [1959a] have reformulated the problems of diffraction by a perfectly conducting cylinder or sphere in a manner which involves directly the "canonical" functions introduced by Fock and which permits the simple asymptotic evaluation of the field on the dark side of the obstacle surface, including the transition region surrounding the light-shadow boundary. Deep in the shadow, the solution can be expressed in terms of the customary contributions from the "creeping" waves which appear to be launched at the shadow boundary, propagate along the obstacle surface into the shadow region with an exponentially decaying amplitude, and radiate energy away from the obstacle surface during their progress. In the transition region one employs the functions tabulated by Fock. Wait and Conda [1958a] have applied this technique also to formulate the scattering by imperfectly conducting cylinders and spheres and have tabulated the values of the Fock functions for this case. They treat problems with observation points situated either on the obstacle or near the light-shadow boundary off the obstacle surface. For the latter case they have exhibited a correction factor to be added to the approximate result obtained from Kirchhoff theory [Wait and Conda, 1959b]. For wave propagation between two concentric spheres, Wait [1959a] has also studied the influence of transition regions (caustics), nd has calculated and plotted the correction factors to be applied to the usual geometrical optics representations in and near the caustics. The problem of the propagation around the earth of an electromagnetic pulse produced by a vertical dipole source has been analyzed by Levy and Keller [1958]. They evaluate the distortion of the pulse shape as a function of distance and material constants.

A useful technique for obtaining directly alternative representations for the solution of separable diffraction problems is the method of characteristic Green's functions discussed by Marcuvitz [1951] and Felsen [1957b]. A refinement of this technique through the use of the Laplace transform has been carried out by Ritt [1958] and has been applied by Kazarinoff and Ritt [1959] to obtain alternative representations of the solution for the scattering of a scalar plane wave by a perfectly reflecting prolate spheroid. They have also obtained in this manner the two-dimensional Green's function for a perfectly reflecting elliptic cylinder [Ritt and Kazarinoff, 1959] and have evaluated the far field scattered in the forward direction due to an incident plane wave. The elliptic cylinder problem was also investigated via the conventional separation-of-variables technique by Levy [1958]. Upon expanding the exact solution asymptotically for small wavelengths, Levy found that the field solution on the dark side of the obstacle admits of an interpretation in terms of "creeping waves" which are launched at the shadow boundary and progress into the shadow region with an exponentially decreasing amplitude (an identical result is inferred from the solution of Kazarinoff and Ritt which corrects a different interpretation presented by Ritt [1958]). The amplitude and phase variation of the creeping waves is in agreement with that predicted by the approximate theory of Keller [1958; Keller and Levy, 1959a, c] for convex surfaces with variable curvature. Keller and Levy [1959b] have also investigated the spheroid problem and have obtained asymptotic results whose interpretatio is analogous to the above. By a different procedure involving Fourier integral techniques, Clemmow [1959a] obtained integral expressions for the scattered field, and for the total scattering cross section, of a circular cylinder. Although the results obtained are not new, several novel aspects are contained in this application of Fourier techniques [Clemmow, 1959b]. Clemmow also derived an infinite Legendre integral transform defined over an infinite domain of the angular variable and has analyzed thereby the problem of diffraction by a sphere [Clemmow, 1959c].

The above-mentioned smooth objects are defined by single-coordinate surfaces in various separable coordinate systems, with surface conditions such that the associated diffraction problems can be analyzed rigorously by separation-of-variables procedures. For nonseparable configurations no comparable methods of solution are available. However, if the surface conditions on an object depart only slightly from separable ones, one can employ perturbation methods involving a small parameter which exhibits the deviation from the separable case. Such a procedure was employed by Clemmow and Weston [1959] in the approximate analysis of the plane wave scattering by a slightly noncircular, perfectly reflecting cylinder. For a sinusoidal deviation from a circular periphery, they obtained a solution to the first order in the perturbation parameter (amplitude of the deviation) and verified, from an asymptotic eval-

[3] S. R. Seshadri and T. T. Wu, High-frequency diffraction of electromagnetic waves by a circular aperture in an infinite plane conducting screen, presented at URSI meeting at Penn. State Univ., Oct. 1958.

[4] This investigation, carried out just prior to the end of the time period covered by this report, is described in detail in King and Wu (1959).

[5] N. A. Logan, Fresnel diffraction by convex surfaces, presented at URSI meeting, Washington, D.C., May 1959.

uation of the case where the impedance varies slowly over an interval of a wavelength, that the associated creeping waves around the cylinder have a decay rate which agrees with that predicted by Keller [1958] for objects of arbitrary surface curvature. In the illuminated region the field can be constructed according to geometric optics. A perturbation method was also applied by Felsen and Marcinkowski [Felsen, 1959b] to the somewhat similar problem of diffraction by a circular cylinder with a surface impedance which varies slightly (and sinusoidally) around the periphery. A general study of diffraction by noncircular cylinders was carried out by Wu and by Wu and Seshadri.[6]

The structures considered so far have been impenetrable. Comparatively little has been done during the past 30 months on large homogeneous penetrable objects such as dielectric cylinders, spheres, etc. Work in this area has been carried out by Kodis [1959] who has studied alternative field representations for the scattering by a dielectric-coated cylinder and has obtained a formulation in terms of the perfectly conducting cylinder result plus correction terms. A somewhat greater activity has been in evidence on problems of diffraction by certain inhomogeneous structures, and by homogeneous impenetrable objects imbedded in an inhomogeneous medium. Concerning the former, Karp[7] has obtained the (two-dimensional) solution for the reflected and transmitted waves caused by a plane wave incident nose-on on a certain curved, variable dielectric medium which occupies the region between two confocal parabolic cylinders. The variation in dielectric constant is selected so as to permit a solution by a separation-of-variables technique. Levy and Keller [1959] have analyzed the scalar problem of diffraction by a sphere with a radially varying refractive index. Flammer [1958] has calculated the electromagnetic field caused by a source at infinity in a medium whose dielectric constant varies like $1+(c/r)$, c=constant, r=radial distance, and has obtained an asymptotic representation of the formal solution for large values of r. Diffraction by planar objects imbedded in a linearly stratified medium and by cylindrical objects imbedded in a cylindrically stratified medium was studied by Seckler and Keller [1959] who obtained asymptotic solutions by the WKB method. As expected these solutions were found to be interpretable in terms of geometrical optics and diffracted ray contributions. For a certain monotonic refractive index variation along a rectilinear coordinate, Felsen [1959c] has obtained exact solutions (and high-frequency asymptotic representations) for the diffraction of line source fields by various two-dimensional impenetrable objects including cylinders, wedges, half-planes, strips, etc. These formal solutions are obtained by showing the equivalence between a class of two-dimensional diffraction problems in a certain variable medium and a class of axially symmetric three-dimensional diffraction problems.

[6] This work is described in the monograph by King and Wu (1959).

[7] S. N. Karp, Reflection and transmission by a class of curved dielectric layers presented at URSI meeting, Washington, D.C., Apr. 1958.

1.2. Approximate Theories

As predicted in the last Assembly Report, the two most actively investigated approximate theories are those due to Fock [1946] and Keller [1958]. While the theory of Fock and its extensions are concerned only with diffraction by smooth convex bodies, Keller's geometrical theory of diffraction has also been applied to objects with edge and tip singularities [Keller, 1959; Siegel, 1958] and to diffraction by objects imbedded in variable media [Seckler and Keller, 1959]. In the original formulation of his theory, Fock was concerned with the behavior of the diffracted fields near the light-shadow boundary on the surface of a smooth, convex, perfectly conducting body. His solution for the field on the dark side of the body near the light-shadow boundary involves certain functions, now called "Fock functions," which contain for their distance parameter not the actual path length from the shadow boundary on the body to the observation point, but rather the projection of that path length onto the light-shadow boundary behind the object. While the difference between these distances is small for observation points near the shadow boundary, it may be appreciable for locations of observation points deep in the shadow. Keller suggested that the correct distance parameter is the actual path length on the object as measured along the geodesic and has proposed how to calculate the amplitude and phase of a wave "creeping" along the surface of the object. Goodrich [1958] has analyzed by this modified procedure the fields diffracted into the shadow region of a perfectly conducting cone. He applied his results to the reciprocal problem of radiation into the shadow region from an infinitesimal slot on a cone, and thence to the radiation from a slot array. The good agreement between the calculated results and measurements taken at the Hughes Aircraft Company [Goodrich, et al., 1959] serves as a confirmation of the validity of the procedures of Fock and Keller[8] for a configuration for which exact asymptotic solutions are not as yet available.

Because of its characterization of high-frequency diffraction effects in terms of various classes of geometric optical and diffracted rays, Keller's geometrical theory of diffraction highlights in a physically significant and systematic manner the mechanism of diffraction by a composite object. If the complete scattering properties of the various canonical constituents of the object, such as edges, corners, surface curvature, etc., are known (these can generally not be obtained from the geometrical theory) then the total scattered field at any point is obtained, according to Keller, by adding the contributions from the various geometric optical and diffracted rays passing through this point. The theory has been confirmed for a variety of simple canonical configurations, and also for some nonelementary structures, at least as far as the first-order contributions to the scattered

[8] Concerning diffraction by convex objects, it seems proper to credit Fock with the analysis of the transition range behavior and Keller with the formulation of the field behavior in the dark-shadow region.

field are concerned (singly diffracted ray contributions). Concerning higher order effects arising from the contributions of multiple diffracted rays, a discrepancy in some higher order terms was noted, for the case of scattering by a large circular aperture, between the scattering cross section computed by Keller [1958a] and that obtained by Levine and Wu [1957] from an asymptotic analysis based on the rigorous integral equation for the problem. This difficulty has now been resolved by Keller [Karp and Keller, 1959] through the use of a canonical solution [Buchal and Keller, 1959] for the high-frequency diffraction by a curved edge, in contrast to the single straight-edge result employed originally by Keller. However, this example would seem to demonstrate that a simple "local" analysis of diffraction problems,[8a] while straightforward for the determination of dominant effects, must be applied with great care for the evaluation of higher order effects associated with more general configurations. Moreover, the application of a "local" analysis to scattering by objects with variable surface properties is restricted to variations which are gradual in an interval of a wavelength. For rapidly varying surface conditions, the local analysis inherent in Keller's theory must be modified and requires the solution of a new canonical problem [Felsen, 1959b; Shmoys, 1959].

As mentioned above, Keller has made some very significant extensions and systematizations of diffracted ray theory and its application to the analysis of diffraction by objects of relatively arbitrary shape, and has thereby illuminated the basic mechanism of diffraction processes. However, approximate solutions for simple composite objects have been constructed by quasi-optical techniques for some time. The treatment of diffraction by a wide slit in terms of multiple scattering from two isolated half-planes, for example, can be considered classical. More recently, Siegel [1958] has obtained by quasi-optic considerations an approximate solution for the axial plane wave back-scattering due to a finite cone. The same problem was analyzed subsequently by Keller [1959] by a purely geometric treatment. (Due to the occurrence of algebraic errors in the course of both analyses, the solutions presented by Siegel [1958] and by Keller [9] are incorrect and differ from each other. Corrected versions of these results, in agreement, are now available [Keller, 1959; Siegel, Goodrich and Weston, 1959][9a]) Karp [1959] and Karp and Zitron [1959] have employed a self-consistent field method in the analysis of the scattering by an aperture and by isolated cylinders, respectively. In this method, which can be applied to several simple obstacles or to a simple composite object, each scattering element is excited by the incident field plus the scattered far fields from all the other elements. The amplitudes of the various scattered fields are then determined in a self-consistent manner. Results so obtained have been compared and agree with those available from other less direct analyses. It is also pertinent to mention in this context the work of Wu and Levine [1958] on the evaluation of the scattering cross section of a row of circular cylinders.

There is not need to dwell in detail at this time on the applications of the classical "Kirchhoff" or "physical optics" procedure to the calculation of high-frequency diffraction effects. A discussion of results obtained recently by this method was given by Siegel [1958] in a talk presented at the last General Assembly Meeting. A summary of this technique and its application to an analysis of the scattering by the tip of a perfectly conducting cone has been given by Goodrich et al. [1959]. Briefly, in the Kirchhoff procedure one assumes that the induced currents at a given point on a (perfectly conducting) body are the same as those excited on an infinite perfectly conducting plane tangent to the object at that point (i.e., the obstacle currents have a strength equal to twice that of the tangential component of the incident magnetic field). The scattered field is then computed as that arising from the radiation due to these known currents. In the asymptotic evaluation of the Kirchhoff integrals in the limit of short wavelengths, the major contribution to the scattered fields arises from certain stationary points on the object and admits an interpretation of the result in terms of geometrical optics and diffracted ray effects. The diffracted wave amplitudes computed in this manner will differ in general from those obtained by more rigorous techniques. A modification of the Kirchhoff procedure was employed recently with good results by Shkarofsky et al. [1958].

The Kirchhoff procedure can be refined by assuming in the vicinity of a stationary point not the physical optics currents but the rigorous current distribution appropriate to a canonical configuration which has the same local geometry. For example, to compute the scattering from the base of a finite cone, one can employ for the local current distribution near the curved edge the known currents for a perfectly conducting wedge. The resulting asymptotic evaluation should then yield the same result as would be obtained more directly by Keller's geometrical theory of diffraction. On the other hand, the Kirchhoff procedure can provide information about the field behavior in geometric optical transition regions which cannot be directly inferred from Keller's theory. In addition, a Kirchhoff analysis can yield approximate results for scatterer configurations whose canonical constituents have not been fully explored.

The Kirchhoff approach to determining the scattering properties of very complex shapes as, for example, of aircraft at small wavelengths, involves, in essence, a formulation in terms of physical optics currents. It is then important to use random phase

[8a] By a "local" analysis, the field along a diffracted ray is determined by the surface properties of the scattering object "at" the point of emergence of the ray.

[9] J. B. Keller, Diffraction by a finite cone, presented at URSI meeting Washington, D.C., Apr. 1958.

[9a] For a detailed discussion and comparison of results, see K. M. Siegel, The resonance region, to be published in Proc. URSI XIIIth Gen. Assembly.

between the different contributors [Crispin, Goodrich, and Siegel, 1959]. This is especially true for mass-produced shapes such as aircraft and automobiles which never emerge exactly the same. For a very complicated shape like an aircraft, geometrical optics would actually be all that is required, since corrections due to edge contributions would make little change in the results.

To summarize the utility of the various approximate techniques, we note that either the Keller or Kirchhoff procedure should certainly be used in preference to simple geometrical optics for scattering problems wherein the main contributions arise from edges and corners. In these situations the geometrical optics result vanishes but the actual contribution can be quite large. When the solutions for the canonical configurations which comprise the object are available and exist in simple form, then Keller's technique is more accurate and is to be preferred over the Kirchhoff procedure. A typical example is again the finite cone. On the other hand, for problems which involve complicated shapes made up of many simple shapes, or for structures whose canonical constituents have not been investigated, the Kirchhoff procedure is more appropriate. Moreover, as pointed out before, Keller's theory does not directly yield information about the field behavior in caustic, focal, and transition regions.

Summary

Results of analyses of high-frequency scattering problems during the past 30 months involving impenetrable objects with homogeneous, or certain inhomogeneous, surface conditions have served generally to confirm the previously proposed extensions of Fock theory and also the interpretation and evaluation of diffraction phenomena via Keller's geometrical theory of diffraction, thereby strengthening the understanding of the mechanism of high-frequency scattering processes for such objects.

A number of new canonical diffraction problems have been solved. Results have been obtained for wedge-shaped surfaces whose surface impedance is constant and may have a reactive component which can support a surface wave. Other problems involve such configurations as perfectly reflecting cones, elliptic cylinders, spheroids, and imperfectly conducting cylinders and spheres. In addition to these constant impedance configurations, a variety of problems involving variable surface impedances or penetrable variable media have been treated. Asymptotic evaluations in the short-wavelength range have led to representations which can be interpreted in terms of geometrical optics, diffraction, and transition effects.

2. Rayleigh Scattering

Although no new canonical problems of diffraction at long wavelengths seem to have been solved during the past 30 months, novel integral equation formulations for the scalar problem of diffraction by circular disks and apertures have received attention. Heins and MacCamy have treated the disk [Heins and MacCamy, 1959] while Bazer and Brown have considered the Babinet-equivalent problem of the aperture [Bazer and Brown, 1959]. Both studies depart from an integral equation formulation of Jones [1956]; the earlier work of Heins and MacCamy [1958] is also to be cited in this connection. Siegel and Senior [10] have shown how higher order terms in the asymptotic expansion of the scattering amplitude for perfectly conducting bodies of certain selected shapes can be constructed at long wavelengths by an algebraic technique which is relatively straightforward. This is in contrast to other methods which require the solution of differential equations for the evaluation of higher order terms. Concerning approximate theories, Siegel [1958] discussed in some detail at the last General Assembly approximate procedures for the evaluation of diffraction effects in the Rayleigh region. One of his results implies that the scattering cross section of an axially illuminated body of revolution is independent of its detailed shape. This behavior has been verified experimentally for a finite cone by Hiatt [Brysk, Hiatt, et al., 1959]. Later Hiatt found the same scattering cross section for nose-on and rear-on illumination.[11] A study of the extent of the Rayleigh region in terms of the ratio of wavelength to maximum object dimensions was also carried out [Brysk, Hiatt, et al., 1959] and detailed experimental confirmation was obtained by Keys and Primich [1959].

3. The Resonance Region

Although good progress has been made recently toward the solidification of the understanding of diffraction processes in the high and low frequency ranges, many questions remain concerning the scattering in the resonance region by certain specially shaped objects whose maximum dimension is comparable to the wavelength. This is true despite the fact that for other simple shapes (such as a sphere), the inclusion of higher order quasi-optic diffraction contributions yields good agreement with exact calculation even for values of $ka \approx 1$, where k is the free-space wave number and a is the sphere radius. It is to be hoped that increased attention will be paid in the future to this interesting electromagnetic region from both the theoretical and experimental standpoints.

Concerning contributions during the past 30 months, Weston [1959] has solved exactly the problem of a pulse which is reflected by a perfectly conducting sphere. He has obtained the solution for the resonance region and has also investigated the high and low frequency behavior of the tail of

[10] K. M. Siegel and T. B. A. Senior, The asymptotic expansion of electromagnetic scattering functions at long wavelengths, presented at the URSI-Toronto Symp., Univ. of Toronto, June 1959.

[11] R. E. Hiatt, K. M. Siegel and H. Weil, The ineffectiveness of absorbing objects illuminated by long wavelength radar, submitted to Proc. IRE.

the returned pulse as a function of sphere size and pulse length. It is found that a considerable amount of pulse lengthening can take place in the resonance region. Olte and Silver have obtained experimental results for the radar cross section of spheroids and cones in the resonance region [Olte and Silver, 1959]. The experimental work of Keys and Primich [1959] on finite cones should also be cited as well as the experimental work on cones by August and Angelakos [1959]. It might also be of interest in this connection to call attention to the recent work of Belkina [1957] on the radiation characteristics of prolate spheroids.

4. Future Activities

Concerning high-frequency scattering it appears certain that the extended theory of Fock and Keller's geometrical theory of diffraction will be applied to shapes of increased complexity. It is to be expected that the emphasis both in rigorous solutions and in the application of Keller's geometrical theory will be placed on the construction of higher order corrections (multiply diffracted ray contributions) to the asymptotic representations of the scattered field. Scattering by impenetrable objects with variable surface properties and by penetrable homogeneous or inhomogeneous objects is also likely to receive further attention.

At present, there is little evidence to expect any marked increase in activity on the study of diffraction phenomena in the Rayleigh and resonance regions, per se, although unanswered questions still remain. However, it is to be hoped that the availability of higher order diffraction contributions, as mentioned above, permitting an approach to the resonance region from the high-frequency end, will aid in the clarification of scattering phenomena in this frequency range.

One of the new problems likely to receive some attention during the next 3 years concerns the effect of model dimensions and the need for reproducing exact dimensions by precise modeling theory. Results in this area are of direct interest for electromagnetic modeling experiments for the determination of the effective mechanical tolerances required to obtain a desired scattering behavior. (The topic of surface roughness related thereto is covered separately in the report by Twersky as noted in the Introduction.) In this connection one can expect increased efforts to be devoted to the study of nonlinear modeling techniques [Belyea, Low, and Siegel, 1959] which hold promise of removing some of the basic stumbling blocks associated with laboratory experiments designed on a linear modeling basis.

There is little doubt that problems involving the interaction of electromagnetic fields with anisotropic media, such as plasmas and the radiation from, or scattering by, objects embedded in a plasma medium or surrounded by a plasma sheath will receive a great deal of attention. However, as previously suggested, work in the plasma area should really be covered under a separate report.

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(Additional references are given on page 750.)

Part 2. On Multiple Scattering of Waves

V. Twersky [1]

1. Purpose

The purpose of this report is to survey some of the recent analytical work on scattering of waves by many objects. It attempts to cover certain aspects of scattering by fixed configurations and by random distributions which have been dealt with in the U.S. literature in classical physics, applied mathematics, engineering, and chemistry. General analytical procedures and treatments based on distinct scatterers (as opposed to those dealing with perturbed continuous media) are emphasized.

Inasmuch as no analogous previous reports on multiple scattering are available, the literature survey is prefaced by (and interlarded with) background and introductory material intended to indicate the roots of current activity, to introduce special terms, and to delineate the restricted viewpoint and coverage of this report. This last consideration is an essential one since the survey is quite limited: no attempt is made to cover the large number of physical phenomena which involve multiple scattering (or even to list the larger number of labels by which they are referenced in the literature) or to discuss the variety of analytical and heuristic procedures used in their treatment.

2. General Considerations

The essential features of a scattering problem are the effects arising when a given obstacle, or collection of obstacles, is placed in the path of a specified wave. We assume that we are dealing with a source whose field "when isolated" is known, and seek the redistribution of radiation arising from the presence of obstacles. Physically speaking, in electromagnetics, the "primary wave" induces charges and currents in the obstacles, and these in turn give rise to the "secondary waves" that constitute the "scattered field."

If we restrict consideration to continuous wave excitation and fixed scatterers whose location, orientation, etc. are not affected by the applied field, then we formulate the problem analytically as seeking a solution of appropriate wave equations, subject to prescribed boundary conditions at the objects, and subject to conditions at large distances from the region containing the objects. The wave equations describe local properties of the media in question; the boundary conditions take account of the physical characteristics, shapes, and sizes of the objects; and the conditions at infinity specify the forms of both incident and scattered components of the solution. We may be interested in the field arising from a particular object, or from some configuration of objects of specified shapes, etc., or in the average field and energy flux to be expected for some statistical distribution of configurations, shapes, etc.

The "single body" wave problem as it is usually formulated corresponds to the limiting case of a practical situation involving one source of radiation and one fixed obstacle, such that the effects of the scattered radiation on the source, extraneous reflections from other objects in the environment, etc., have been minimized. In general, such simple limiting cases lead to unsolvable integral equations. In a few special cases, for which the surface of a homogeneous scatterer coincides with one or more complete coordinate surfaces in one of the systems in which the wave equations are separable, solutions are obtained as infinite series of more or less tabulated special functions. Simple closed-form solutions in terms of elementary functions are rarer still. However, through analytical approximations valid for restricted values of the parameters, and through heuristic procedures motivated by the insight obtained in more elementary problems, one can now obtain explicit results which are adequate to describe many principal phenomena of physical interest. Although this subject is far from closed,[2] it is convenient in considering multiple scattering, to assume that solutions for the component scatterers when isolated are known, and that they may be regarded as "parameters" in the more general problem.

Thus one seeks representations for scattering by many objects in which the effects of the component scatterers are "separated" from the effects of the particular configuration (or statistical distribution of configurations) in the sense that the forms of the results are to hold independently of the type of scatterers involved. Of course, such representations can usually be obtained in the range of parameters where a single scattering approximation is valid, i.e., in which the results for a distribution of identical scatterers reduce to that for an isolated object times an "array factor". We discount this range from the start, and seek in general a functional relation for the many-body solution in terms of a single-body function. Thus if one can treat a particular spatial configuration (or statistical distribution of configurations) explicitly, and independently of the component scatterers, then the results for specific isolated objects, for particular ranges of the parameters, etc., can be inserted for detailed applications.

The above, in first regarding the single-body problem as a limiting case of that of many bodies, and then regarding the distribution as composed of objects whose solutions when isolated are known, has emphasized the view to be taken in the following.

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[1] Sylvania Electronic Defense Laboratories, Mountain View, Calif.

[2] Recent activity on scattering by isolated objects is surveyed by L. B. Felsen and K. M. Siegel in a companion report to URSI.

Related treatments of distributions of distinct objects start essentially with Poisson's [1821; 1823] molecular model of magnetic induction, and its application to dielectrics by Faraday [1839], Kelvin [1845], Mossotti [1847], and Clausius [1897]; the work of Maxwell [1873a] on the "bulk resistivity" (essentially the reciprocal of the dielectric constant) of a distribution of resistive spheres in a medium of different resistivity, and on the permeability of a distribution of perfectly conducting spheres [Maxwell, 1873b]; the work of Lorentz [1880] and Lorenz [1880] on the index of refraction; and Rayleigh's [1892] investigation of scattering by rectangular arrays of parallel cylinders and spheres. These treatments were restricted to low frequencies, special scatterers, and limiting distributions; they range analytically from the intuitive development of Faraday to Rayleigh's detailed analysis of "packing effects" in terms of the ratio of scatterer size and spacing.

More generally, a formal representation for the solution of any given configuration of arbitrary scatterers (a configuration specified by a set of position vectors to reference points on the objects, and the scatterers specified by their shapes and boundary conditions) may be obtained as follows: We apply Green's theorem to the free-space Green's function and to the required unknown solution in the region external to all scatterers, and thereby represent the scattered field as an integral over some surface inclosing the region. Contracting the surface and breaking it up into individual portions inclosing a single object, leads to a representation of the total scattered field as a sum of surface integrals; it is the terms of this sum (the integrals over the surfaces of the individual scatterers) that we identify as the "elementary scattered waves". Then imposing the boundary conditions at each object leads to a determinate set of coupled integral equations for the fields on all scatterers, and could these values be obtained explicitly, the total field in space would follow on integration.

This analytical procedure, or similar ones applicable for relatively arbitrary scatterers, was used both for general considerations and specific applications by Ekstein [1951; 1953], Ignatowsky [1914a], Karp [1953], Lax [1952], Millar [1960], Row [1955], Storer and Sevick [1954], and Twersky [1956a, 1957a, 1958a, 1959a, 1959b]. An analogous procedure leading to sets of algebraic equations for the separable problems of arbitrary configurations of circular cylinders was used by Zaviska [1913], Ignatowsky [1914a], Row [1955], and Twersky [1953a, 1953b, 1954]; similarly Kasterin's [1897] formalism for the scalar problem of a periodic array of spheres holds for all wavelengths and for any of the usual boundary conditions. Thus, for example, for circular cylinders and homogeneous boundary conditions, the Green's functions procedure yields N coupled integral equations for the surface fields (or for their normal derivatives, or for linear combinations of fields and derivatives); equivalently, separation of variables yields an N fold infinite set of algebraic equations for the scattering coefficients.

There are essentially three different analytica procedures which may be used to obtain a representation taking into account the effects of multiple scattering, or the coupling of the radiation fields of the objects: One may seek to solve the boundary value problem for the "compound body"; one may use a self-consistent procedure based on the known response of the isolated elements (the single-scattered results) such that each object is considered as excited by the primary wave plus the resultant of the initially unknown total scattered fields of the other objects; or one may use an iterative procedure corresponding to the "successive scatterings" of the primary field. In the successive scattering approach, which is essentially an iterated form of the self-consistent one, each object is initially regarded as excited solely by the primary field and radiating in consequence its "first order of scattering"; next in response to the sum of the first orders of the others, each scatters its "second order," etc. The second and third methods differ essentially from the first in that they isolate the single-body solutions implicit in the problem. Thus they enable us to exploit known single-body results, or to seek them independently of the configuration (either analytically, or by direct measurement).

The class of many-body problems for which one may obtain a solution for the compound body is small; it comprises periodic arrays whose essential parameter is a simple sinusoid (e.g., the sinusoidal profile of a reflection grating treated by Rayleigh [1907a], or the sinusoidal refractive index of a medium considered by Bragg [1915] and Laue [1931]). More generally, however, one must consider "multiple scattering" by an infinite set of such sinusoids, i.e., by the components of the complete "spectral representation" (Fourier series or Fourier integral) of the appropriate parameters of the collection of scatterers. Thus, for example, Rayleigh [1907a] represented the grating of arbitrary periodic profile as a Fourier series; Laue [1931] represented the crystal of arbitrary periodic index as a "triple" Fourier series; Rice [1951] represented the randomly perturbed planar rough surface as a double Fourier integral; and Hoffman [1959] represented a medium whose index was a slowly varying random function of position by a triple integral. The Fourier representations treat the spectral components of the parameters of the distribution as the "individual scatterers"; they lead to rapidly convergent approximations for values of the parameters such that the field arises essentially from one sinusoid (e.g., Bragg reflections in a crystal), or when the parameters are only slightly perturbed from those of a uniform region (e.g., slightly rough plane).

A large variety of heuristic, self-consistent procedures (starting with those of Mossotti [1847], Clausius [1897], and Maxwell [1873]) have been applied to determine the macroscopic parameters of the coherent field for distributions of scatterers. Analytical self-consistent procedures for periodic structures are illustrated by Rayleigh's [1892] work on lattices of cylinders and spheres, Ignatowsky's

[1914a] treatment of the grating of arbitrary cylinders, and Ewald's [1917] analysis of the lattice of dipoles. Anologous procedures to treat the coherent field in sparse random distributions were used by Born [1933] for dipoles, and by Foldy [1947] for monopoles; dense distributions of dipoles were treated by Brown [1950] (static case) and by Mazur and Mandel [1956].

The successive orders of scattering approach (discussed by Heaviside [1893]) was used by Reiche [1916] (who also gave a self-consistent treatment) to derive the coherent field for a slab region of randomly distributed dipoles. Twersky [1950a] obtained a criterion for the range of validity of Schaefer and Reiche's [1911] single-scattering treatment of the grating of circular cylinders, and constructed a series solution for an arbitrary configuration, and series and closed form approximations for two cylinders and gratings [Twersky, 1952a, 1952b, 1952c]. Similarly Yvon [1937, 1935], Kirkwood [1936], and Jansen and Mazur [1955] averaged the scattering series for dipoles to treat the dielectric constants of dense gases.

The papers mentioned above serve to illustrate approaches for treating many-body scattering problems. Additional work will be cited in the sections on particular configurations. Thus we reserve discussion of essentially particle scattering procedures based on transport equations until the topic of "incoherent scattering" arises in its appropriate context.

The above also serves to indicate the main lines we follow. Thus we do not consider "multiple scattering treatments of single-body problems" in the following sections. However, since we mentioned single-body treatments of many scatterers, it may be appropriate to sketch the "inverse" situation. Thus a finite scatterer with sharp edges may be treated by exploiting Sommerfeld's and Macdonald's solution for the field on the semi-infinite wedge. For example, an infinite cylinder having a triangular cross section with sides large compared to wavelength may be regarded as a collection of three "infinite wedge edges" plus specularly reflecting planes. As a first approximation, each of the three edges may be treated as excited solely by the plane wave; then the "coupling effects" of the "single scattered edge waves" on each other may be developed in terms of higher order scattering processes (by regarding each edge as excited by the asymptotic forms of the waves leaving the other two in response to the primary excitation, etc.). More directly, the infinite wedge result may be used in a self-consistent procedure which treats each edge of the finite wedge as excited by the incident wave and by two cylindrical edge waves of initially unknown amplitude. The solution for the degenerate case of the wedge of zero angle (i.e., the half-plane) was first used by Schwartzchild [1902] to construct the series solution and a single scattering approximation for a wide aperture in an infinite plane screen, and higher order scattering of the edge waves was recently treated by Clemmow [1956], Karp and Russek [1956], and Keller [1958, 1957]. Similarly, Braunbek, Clemmow, Keller, and Levine treated the wide circular aperture by assuming that the edge field was approximately that on a half-plane locally coincident with the edge of the aperture, and Keller, and Siegel used the infinite wedge result to approximate the local field on the curved edge of the base of a finite cone; these treatments range from "single scattering" approximations, to Keller's detailed consideration of the "multiple scattered" edge rays. (References yo these papers, and to analogous treatments of scattering by isolated objects are given in Felsen's and Siegel's URSI report, Diffraction and Scattering.)

Breakdown of the Many-Body Problems

In the preceding paragraphs, we more or less jumped into the literature in order to associate this general topic with such familiar names as Poisson, Faraday, Maxwell, Rayleigh, Lorentz, etc. Then, papers were cited to illustrate different procedures for taking into account the effects of multiple scattering. Since a representative selection of methods was insured at the expense of a systematic presentation of problems having physical interest, we now mention classes of problems; citations to the literature are reserved for the following sections.

We distinguish two categories of multiple scattering problems: In one, we deal with a fixed configuration; and in the other, with a statistical ensemble, or distribution of configurations. This breakdown is primarily for convenience; it serves to single out the well-defined boundary value problems of several scatterers, as well as the periodic structures for which a large variety of special analytical techniques are available. However, subsequently, we regard the fixed configuration as a limiting case of a general distribution.

Fixed configurations: Several two-body situations, general collections of N-bodies, and periodic arrays, have been treated in some detail. Explicit solutions for N-bodies have been derived for planar scatterers (e.g., infinite slabs, discontinuities on transmission lines, etc.). Explicit approximations (series and closed forms) have been obtained for completely bounded scatterers in ranges where, (1) the wavelength is large compared to the scatterer's size, and the spacing is arbitrary, and (2) for arbitrary bodies and spacing large compared to wavelength; here the literature ranges from meson-deuteron scattering in quantum mechanics to coupling effects between transmitting antennas and scatterers arising in "single body" microwave measurements.

The literature of scattering by periodic arrays covers diffraction gratings, planar lattices, dielectric constants, indices of refraction, crystal analysis, "artificial dielectrics", obstacles in rectangular waveguides, as well as the analogous antenna arrays. Single periodic layers, gratings, etc., are of interest in connection with their use as spectrum analyzers, polarizers, open-mesh reflectors, etc.; and the analysis of scattering by three-dimensional arrays facilitates studies ranging from the exploration of crystal structure by X-rays to the design of practical microwave components.

In general, the field of an infinite periodic structure consists of an infinite number of discrete plane waves; some are propagating (e.g., the usual spectral orders of a grating), and the rest are exponentially damped normal to the face planes. Analytically, one seeks to relate the amplitudes of the propagating modes of the transmitted and reflected fields to the spacings of the array, and to the single scattering characteristics of its elements. (The Fraunhofer form for a finite array is more or less a "blurred" version of the set of propagating modes.)

To a large extent, the general problems of three dimensional periodic arrays hinge on the solution for a single planar lattice; once the results for the isolated component planes are known, one can use difference equations, matrix algebra, group theoretic procedures, and other equivalent "multimode transmission line" approaches to treat the crystal. Because of this (as well as because of its intrinsic interest) the planar lattice has merited special consideration. Special attention has also been given to the essentially one-propagating-mode situations which arise when the spacings parallel to the face planes are small compared to wavelength (artificial dielectrics, obstacles in waveguides), or when the Bragg conditions are fulfilled.

Statistical distributions: The other large class of many-body problems deals with statistical distributions of scatterers. Such problems are basic in the use of scattering and propagation measurements as a diagnostic tool in discovering the fundamental properties of matter, and in various practical problems related to the transmission of information via radiation. The special distribution corresponding (more or less) to that of an "ideal gas" of elastic objects has received most extensive consideration, and some progress has been made in treating the "packing effects" in "dense gasses" and "liquid state" distributions.

The previously mentioned formal representation for the field scattered by an arbitrary fixed configuration may be applied to treat scattering by statistical distributions. One introduces an ensemble of configurations defined by an appropriate distribution function (giving the probability of occurrence of the component configurations) and seeks the expectation value of the field by averaging over all variables (positions of scatterers, scatterer sizes, etc.). One may attempt to first solve or approximate the original integral equations (e.g., by an iterative procedure), and then introduce the "statistics"; or one may average the formal solution for a single configuration over the specified ensemble, and then attempt to solve the resultant set of equations. Similarly one averages the corresponding representations for the power density (and energy flux) over the ensemble, and obtains equations for the "coherent component" (essentially the absolute square of the average wave function), and for the "incoherent" or fluctuation scattering. Although not even the simplest of such problems has been treated rigorously by these procedures, useful approximations have been obtained for various ranges of parameters. (See Foldy's [1947] basic paper for a detailed introduction, and for a discussion of the relevance of such averages to quantities obtained by measurement.)

We may regard the periodic configuration and the ideal gas, as special cases of a general "liquid state" distribution. Thus in terms of appropriate distribution functions we may start with the limiting case of an ideal gas, and introduce "local order" in the distribution of a scatterer's neighbors to model some of the characteristics of dense gases and liquids of elastic particles. Proceeding to the limit of an appropriate parameter (essentially "compressing" the distribution) yields results corresponding to a periodic array.

The intensity pattern for the liquid lies "between" those of a gas and crystal. If we visualize a narrow beam incident on a slab region of a distribution of identical objects, then for the gas case we obtain coherent transmitted and reflected beams and a "background" of incoherent scattering (more or less resembling the single-scattered intensity pattern of a component object). As the ratio of average to minimum separation of scatterers is decreased (a minimum in general greater than the scatterer's size, and, say, of the order of several wavelengths) and the liquid state approached, the incoherent scattering becomes peaked at angles in the vicinity of those corresponding to the propagating modes of the periodic limit. With increasing local order, these additional "beams" becomes better defined, and finally go over to the propagating modes of the appropriate crystal.

Alternatively, instead of dealing with ensemble averages of configurations of distinct scatterers, one may seek to model statistically inhomogeneous regions by means of an appropriately perturbed continuum. Approximations for the coherent field may be specified in terms of the index of refraction, and corresponding approximations for the intensity depend on the autocorrelation of the values of the index at two different points. Representative papers are cited in subsequent sections.

3. Survey

3.1. Fixed Configurations of *N* Scatterers

The static limits for two parallel circular cylinders, a grating of N parallel cylinders, two parallel coplaner strips, and two spheres, are conveniently found in Wendt's [1958] recent review article. He has many references to the recent literature, and a bibliography of texts going back to Maxwell's. Other static problems of interest include Maxwell's [1873c] treatments of stratified conductors (N slabs with characteristics alternating as ABAB . . .), and composite dielectrics (N slabs ABC . . .).

Silver [1949] discusses coupling between transmitter and receiver antennas from a multiple scattering point of view. King [1956] treats a variety of problems involving coupling between two linear antennas (parallel, colinear, perpendicular), between N parallel linear antennas, between three parallel antennas at the vertices of an equilateral triangle, between four at the corners of a square, etc.

Redheffer gives the solution for scattering by two parallel slabs of arbitrary physical parameters [Redheffer in Montgomery, 1947], and an elegant analytical discussion of N arbitrary parallel slabs [Redheffer, 1950, 1954]; for N slabs (arbitrary spacings, and arbitrary reflection and transmission coefficients of the isolated slabs) he uses group theory, abstract multiplication, as well as conventional matrices, and difference equations. A detailed systematic successive scattering treatment for N arbitrary parallel slabs is given by Marcus [1946]. Their closed-form solutions for N identical equally spaced slabs were obtained originally by Darwin [1914] in his basic paper on the scattering of X-rays by crystals. Redheffer [1956] also gives an exact treatment of "limit-periodic dielectric media" (the limit for $N\rightarrow\infty$ of N identical inhomogeneous slabs of thickness $1/N$), and Bazer [1959] considers the conditions for which an arbitrary configuration of N identical slabs may be analytically approximated by an appropriate continuum.

The above papers on collections of planar scatterers are but a few among the many to be found in the literature; see Hartree [1929], Luneberg [1947a, b], Lurye [1951], Russek [1951], Keller [1953], Keller and Keller [1951], Schelkunoff [1951], Bremmer [1951], Landauer [1951], and Kay [1958]. The limiting case of the arbitrary stratified region is that of an inhomogeneous medium: recent work includes that of Kay [1955], Kay and Moses [1956, 1955a, 1955b, 1955c, 1957], Saxon [1957; 1959], Schiff [1956], Saxon and Schiff [1957], Seckler and Keller [1959], and Hall [1958]. Additional references to the recent literature and discussions of procedures for treating such problems are given in Bremmer's [1958] Handbuch review of radio wave propagation; and in the same volume (Electric Fields and Waves), King's [1958] review of electric circuits, and the review of Borgnis and Papas [1958] on waveguides, include germane transmission line procedures for treating collections of planar scatterers.

Turning to arbitrary collections of arbitrary, parallel circular cylinders, the separations of variables procedure of Zaviska [1913], Ignatowsky [1914a], Row [1955], and Twersky [1953a] gives infinite sets of linear algebraic equations which relate the multiple scattered coefficients of a cylinder to known single scattered values and to Hankel functions of the spacings. A Neumann iteration of these "self-consistent" equations leads to the "orders of scattering" series whose successive terms involve higher products of single scattered coefficients; Twersky [1952a] also obtained this series by successive application of the boundary conditions.

For radii small compared to wavelength, Zaviska [1913] gives closed form approximations for two and three (equally spaced, coplanar) cylinders, such that E is parallel to their axes (henceforth E parallel), and the direction of propagation is perpendicular to the plane of their axes (henceforth k perpendicular); for this polarization, he also considers two cylinders and k parallel (i.e., E, k, and the axes all coplanar). In these approximations, the isolated scatterers are treated essentially as monopoles (isotropic scatterers), and all orders of scattering are taken into account; e.g., Zaviska's multiple scattered coefficients for one of the two identical cylinders for E parallel and k perpendicular may be written as $A=a/(1-aH)$, where a is the single-scattered value, and $H=H_0(kb)$ (the "configuration factor") equals the zeroth order Hankel function of spacing b and wavenumber k. [This elementary multiple scattering solution (in fact, the most elementary) serves to illustrate the terminology used previously. Thus the "self-consistent" equation leading to the closed form may be written $A=a+aH_0(kb)A$: the first term on the right is the response of the cylinder to the incident plane wave (the single scattered value a), and the second is its response to the field of its neighbor (i.e., to a cylindrical wave of strength A originating from a source at distance b). Iterating the self-consistent equation, or expanding the closed form, gives the "orders of scattering" series $A=a+a^2H+a^3H^2$ . . . , which one would obtain directly from considerations of successive scattering processes. Also note that the single scattered coefficient a is essentially a "parameter" of the multiple scattered value A: the closed form holds for one element of all symmetrically excited pairs of identical monopoles. (Differences between results for various pairs arise from different single scattered coefficients; e.g., for perfect conductors in two dimensions, the circle involves the logarithm of the radius, and the ellipse, of the arithmetical mean of the major and minor axes.)] Zaviska [1913] also considers two identical cylinders for E perpendicular and k perpendicular and gives the first two orders of scattering for the monopole and dipole terms. He also shows that if two arbitrary sized cylinders are in each other's far fields, then the problem reduces essentially to that of one cylinder excited by two plane waves (the incident wave, and a wave traveling in the plane of the axes); however, he fails to notice that this can be exploited to obtain closed forms.

Twersky [1952b, c] obtains closed forms for several cases by retaining only the largest terms involving the separation b in each order of scattering; e.g., a generalization of the above A for two different isotropic scatterers (not necessarily cylindrical) for arbitrary angle of incidence; closed forms for two scatterers with radii and spacing small compared to wave length; closed forms for two arbitrary cylinders each in the far field of the other (call this "far-multiple-scattering"), and an analogous result that holds for N equispaced coplanar cylinders (a finite grating) when end effects are neglected. He also applies an image technique to these results to consider the analogous multiple scattering problems for semicylindrical protuberances on a ground plane; he shows that for E parallel, there are no far-multiple-scattering contributions for an arbitrary configuration of arbitrary semicylindrical protuberances, and derives the first nonvanishing terms for special cases.

Row [1955] applies his general results to two identical perfect conductors for E parallel and k perpendicular. He obtains numerical values by several methods (including truncating the system of algebraic equations, and a diagonal approximation of their matrix). Comparisons with experiments are made for several wavelengths less than or equal to the diameter, for fixed spacing and a varying field point, and vice versa. In particular, he finds detailed agreement for a truncation procedure which keeps as many multiple scattered coefficients as single scattered coefficients called for by the analogous isolated cylinder problem.

Storer and Sevick [1954] apply the variational procedure of Levine and Schwinger to their integral representation for N scatterers, and obtain a stationary form for the far-field scattering amplitude. They specialize their results to treat two finite identical, parallel circular cylinders (radius small compared to spacing and to length) for E parallel. Using a "shifted cosine" trial function, they find good agreement between theory and experiment for backscattering and k perpendicular to half-wave and full-wavelength scatterers. Minkowski and Cassedy [1956] use the analogous procedure to treat the case of colinear cylinders.

Karp [1959] gives a general discussion of the integral representation for N arbitrary scatterers, and considers the conditions for convergence of the orders of scattering series. Subject to far-multiple-scattering, the integral representation for an arbitrary configuration of N arbitrary cylinders yields N simultaneous equations for the multiple scattered amplitudes in terms of their single scattered values; in particular, Karp [1959] gives the closed form for two arbitrary cylinders (the generalization of a result in Twersky's [1952b] result for two circular cylinders).

As discussed by Twersky [1952b], the closed forms that hold for far-multiple-scattering retain only the largest terms in $kb>>1$ of each order or scattering. Thus the series in powers of $1/\sqrt{kb}$ obtained on expanding the closed form for the multiple scattering amplitude is not the rigorous expansion of the function. Zitron and Karp [1959] show that for two cylinders, the "far-multiple-scattering, orders of scattering" series is correct in its three leading terms (i.e., to $1/kb$), and obtain the next term of the series for two arbitrary cylinders: in distinction to the leading terms, which involve only the far-field scattering amplitudes of the isolated cylinders (say f), the new term also involves derivatives of f. They specialize their result to the case of arbitrary circular cylinders, and show it agrees to appropriate order with the result obtained on approximating the complete series derived by separation of variables in Twersky [1952a]. They also obtain the corresponding number of terms of the analogous series arising for the scalar problems of two arbitrary scatterers in three dimensions.

Wu and Levine [1958] consider a row of large circular cylinders for k parallel, and obtain multiple-scattering corrections to the geometrical optics value of the total scattering cross section.

Millar [1960] considers the N simultaneous integral equations obtained for the two cases of E parallel and E perpendicular to a row of perfectly conducting cylinders of arbitrary shape. For elliptic cylinders with major axis small compared to wavelength, and arbitrary separation, he reduces the integral equations to linear algebraic equations (using a procedure similar to Bouwkamp's for the single strip). For two cylinders and E parallel, his closed form approximations for the multiple-scattered coefficients are identical with the forms for two arbitrary isotropic scatterers given in Twersky [1952b]. His plots of the real and imaginary parts of one coefficient as a function of kb, for three directions of incidence (k perpendicular, and k parallel, say, from the right, and from the left) show the effects of multiple scattering and "shielding."

The scattering of waves by two objects has also been recently considered in the literature of quantum mechanics. Thus Brueckner's [1953] closed form for the impulse approximation for S-state scattering from a two-body system, is a result for two monopoles in three dimensions (identical with the form for two isotropic scatterers given in Twersky [1952b]); and his result for P-state meson scattering, is that for the scalar problem of two dipoles in three dimensions. Representative related papers are those of Watson [1953], Takeda and Watson [1955], Brueckner [1955], and Drell and Verlet [1955]. More complex systems are discussed by Gerjuoy [1958], and in the proceedings of the recent Grenoble lecture series on many-body problems edited by Dewitt [1959].

3.2. Infinite Planar Lattices

Here we begin with the more general treatments, and then consider special procedures.

As is well known, the field of an infinite grating of arbitrary identical cylinders excited by a monochromatic plane wave consists of an infinite discrete set of plane waves; some of these waves are "propagating modes" (the usual spectral orders) and these carry energy in specific directions determined by the wavelength, the spacing, and the angle of incidence; the remaining are "surface waves" (or "evanescent modes") which are exponentially damped normal to the plane of the grating. The existence of these waves follows directly from the periodicity of the structure; i.e., the field must be representable as a Fourier series, and this has been the starting point of most rigorous approaches to the problem. But this is merely the starting point: it is the amplitudes of these waves which must be determined. Thus Rayleigh [1907a] represents the mode amplitudes in terms of an algebraic set of equations involving the Fourier components of the grating's profile, and Ignatowsky [1914a] expresses them in terms of an integral equation for the current distribution on one element; and most expansion procedures in the literature are based on one of these two classic representations. However, alternative representations prove more tractable when strong coupling occurs. Thus Twersky [1956a] starts with the set of multiple scattering surface integrals for the elements of the array, and proceeding initially in analogy with Ewald's [1917] treatment of a lattice of dipoles, derives a "mixed representation." The mode amplitudes are expressed in terms of the multiple scattered amplitude of a cylinder in the grating, and specified through a new functional equation involving its single scattered value (as the inhomogeneous term, and in the kernel of the operator). Differing from both Ignatowsky's integral equation and Rayleigh's "sum equation" (i.e., the set of algebraic equations), the operator in the new equation equals an integral minus the analogous sum (a relatively rapidly convergent representation). The new formulaism is applied [Twersky, 1956a, 1957b] to treat the grating resonances investigated experimentally by Wood and Strong; and the enhancement of one spectral order, or the diminution of another, as well as other "anomalies" with respect to single scattering theory, are interpreted in terms of surface wave coupling between the propagating spectral orders. Multiple scattering effects are significant for such resonances (which for normal incidence occur when the grating spacing is nearly an integral number of wavelengths), for near grazing incidence, and for relatively closely packed scatterers; for other situations, the multiple-scattered amplitude reduces to its single scattered value (the inhomogeneous term of the equation). Twersky [1958b] applies the general theory to circular cylinders (and obtains, for example, simple explicit results for the "packing effects" at low frequencies up to multipoles of order 2^5), and Burke and Twersky [1960] apply it to elliptic cylinders. Analysis of such grating problems is facilitated by Ignatowsky's [1914b] elementary function representations for the Schlömilch series that arise for normal incidence, and by the analogous forms for the more general series arising for arbitrary angle of incidence derived by Twersky [1958c].

The procedure discussed above is one of the few multiple scattering treatments of a grating of general elements which expresses the field in terms of the behavior of the elements when isolated. The first of this kind was Ignatowsky's [1914a]. In addition, the variational procedure discussed by Marcuvitz [1951], and applied to circular and elliptical elements with spacing and cross-sectional dimensions small compared to wavelength, is also quite general. Another initially general procedure is Karp's [1955], which was applied by Karp and Radlow [1956] to grating resonances subject to "far-multiple-scattering" (i.e., each scatterer in the far field of all others).

The principal anomalies with the single scattering approximation for the grating (an approximation obtained originally by Schwerd [1835]) indicated by the experiments of Wood [1902], Ingersoll [1921], Strong [1936], Palmer [1952], and others, are the "resonances" mentioned previously. With reference to these, Rayleigh [1907a] treats a perfectly conducting grating, and finds that his representation of the perpendicular polarized amplitudes diverges if there is a grazing mode. Fano [1938] considers the same range for a grating of finite conductivity. Artmann [1942] begins with Rayleigh's model (Fourier series expansion of the profile), but derives an alternative, convergent series representation for near grazing modes. Artmann's expressions for the maxima correspond to the maxima of the usual Wood anomalies; and although he does not consider the associated minima (lying between the Rayleigh wavelength and the maxima), these may also be treated using this model. Fano [1938] also presents a general expression (suggested by a quantum mechanical analogy) which can be adjusted to describe the anomalies, and he is the first to stress the role of the surface waves. Karp and Radlow [1956], and Lippmann and Oppenheim [1942] consider the anomalies, and a relatively detailed discussion is given by Twersky [1956a, 1957b, 1958b].

A perhaps more intuitive approach may also be applied to consider the grating anomalies. Thus Twersky [1952c] gives an "orders of scattering" treatment of the anomalies for a finite grating (perfectly conducting semicylinders on a plane, end-effects neglected): the extrema are interpreted as occurring at wavelengths which optimally fulfill the conditions that each order of scattering is a maximum, and that successive orders are either in or out of phase. Here a suggestion made by Wood [1902], and originally elaborated by Artmann [1942], is developed into a "vibration curve" method based on a discrete analog of the Fresnel integral [Russek and Twersky, 1953].

The method of images provides a convenient means for obtaining solutions for reflection gratings from results for analogous transmission problems. The first treatments of the reflection grating based on this approach originally took into account single scattering [Twersky, 1950a, b, c], and then analogous multiple scattering results [Twersky, 1956a, 1957b, 1958b] were obtained. The image technique itself (for wave problems) was first used by Rayleigh [1907b], who applies the results for a perfectly conducting cylinder with radius small compared to wavelength to obtain the analogous functions for a semicylinder on a perfectly conducting plane.

In addition to papers mentioned above, there are a large number of treatments of transmission gratings of specific scatterers: For example, strips are treated by Vainshtein [1955], Heins [1954], and Miles [1949]; circles are treated by Shmoys [1951], Shmoys and Sollfrey [1952], and Reiche [1953]; and fine wires are treated by Wessel [1939], Honerjager [1948] and Franz [1949]. Fine wires, closely spaced, are also treated by Lamb [1945], and Gans [1920] (both giving incorrect results for polarization perpendicular to the axis—see Twersky [1958b] for details), and by Lewin [1951], and Marcuvitz [1951]. The most detailed treatment of circular cylinders appears to be that of Twersky [1958b], which also gives comparisons with previous work.

Additional treatments of gratings inclued those of

Voigt [1911] (who extended Rayleigh's [1907a] procedure to a lossy interface), Tai [1948], Lippmann [1953], Meecham [1957, 1956a, b], Snow [1956], Primich [1957], Heaps [1957], Proud [1957], Parker [1957], Theissing and Caplan [1956], Hatcher and Rohrbaugh [1958, 1956], Palmer [1956], Rohrbaugh and others [1958] Wait [1958, 1955, 1959], Senior [1959], and Felsen [1959].

As for other two-dimensional planar lattices, Marcuvitz [1956] has given a general formulation for the planar lattice of arbitrary scatterers in terms of the periodicity factors of the array, and in terms of the amplitude of one element. Low frequency results for planar lattices of spheres, disks, etc., have been derived in connection with artificial dielectrics, and in connection with the related problems of obstacles in rectangular waveguides: see the recent review by Cohn [1960] for the literature of the first, and the texts by Marcuvitz [1951], and Lewin [1951] and others for the second. General scattering theorems for such structures are given by Schwinger, Dicke [1948], Redheffer [1950], Friedrichs [1949], and Twersky [1956b].

Additional papers dealing with planar periodic arrays are cited in recent reviews by Harvey [1959] and by Lysanov [1958].

3.3. Planar Random Distributions

As in the previous section, we begin with the most general treatment of the problem; this minimizes repetition.

a. Sparse Distribution (Two-Dimensional "Rare Gas")

The scattering of a plane wave by a planar random distribution of arbitrary objects may be treated by averaging the set of multiple-scattering surface integrals for one configuration over an appropriate distribution. In particular, for identical scatterers whose average separation is large compared to their minimum separation, we may assume that the one-particle and two-particle distribution functions are constant (as for a rare gas).

For such sparse planar distributions of arbitrary scatterers, Twersky [1957a, 1955], using a procedure analogous to Foldy's [1947], shows that the coherent scattered field consists of two plane waves—one in the direction of incidence, and one in the direction of specular reflection (with respect to the plane of the distribution). The amplitudes of these waves are proportional to corresponding values of the average multiple scattering amplitude of a scatterer fixed in the distribution; i.e., to the response of one fixed object to the incident field plus the fields of all other objects averaged over the configurations these other objects may assume. The average with one fixed scatterer is given [Twersky, 1957a] by an integral relation whose kernel involves the same function averaged with two scatterers held fixed; approximating one by the other (as first done explicitly for a volume of monopoles by Foldy [1947], and as done "instinctively" in earlier less analytical treatments of dielectric constants, etc.) leads to a simple expression for the unknown amplitudes in terms of their single-scattered values. (The validity of this approximation requires that the number of scatterers be large; see Foldy [1947] for discussion, and Bazer [1959] for an analytical treatment of the one-dimensional case.)

To this approximation, the total excitation of a scatter within the distribution is proportional to the average of the coherent transmitted and reflected plane waves; and since the response of an isolated scatter to a plane wave is known, one obtains two algebraic equations which can be solved directly. This gives simple expressions for the multiple-scattered amplitudes in terms of their presumably known single-scattered values. The final transmission and reflection coefficients take into account the major effects of coherent multiple scattering; in particular, whereas their single-scattered values would become infinite as grazing incidence is approached, the total scattered field approaches the negative of the incident wave (which merely means that only surface wave, or inhomogeneous plane wave, solutions exist in the limit), i.e., the coherent field of the distribution becomes that of a perfect reflector.

The value for the multiple scattering amplitude obtained by taking into account the coherent effects is also used [Twersky, 1957a, 1955] in the corresponding incoherent scattering (i.e., excitations arising from multiple incoherent scattering are neglected); this leads to an approximation for the total scattered field which explicitly fulfills the energy theorem. Thus, the final results (expressed solely in terms of the known single scattered amplitude, the number of scatterers per unit area, the angles of incidence and observation, and the wavelength) state simply that the average power reflected, transmitted, absorbed, and scattered by the area of distribution illuminated by unit area of incident wave is equal to the incident power density.

The multiple-scattering amplitude of the "random screen" (for the case of scatterers symmetrical to the plane of the distribution) is also imaged [Twersky, 1957a, 1955] to obtain the corresponding function for the analogous distribution of arbitrary protuberances on a ground plane; this amplitude gives directly the reflection coefficients and differential scattering cross sections per unit area for a relatively general model of "rough surfaces". It is shown that for such surfaces the coherent field fulfills an "impedance boundary condition" on the plane of the distribution (i.e., the scalar field is proportional to its normal derivative, or, equivalently, the tangential component of E is proportional to the tangential component of H), and the impedance is expressed simply in terms of the scattering amplitude of an isolated protuberance. For both "vertical" and "horizontal" polarizations, the ratios of reflected to incident fields approach minus one as grazing incidence is approached; the corresponding coherent intensity reflection coefficients approach unity, and the incoherent backscattering cross sections approach zero. More explicitly, for arbitrary protuberances on a ground plane, if the "horizon angle" (or grazing angle) approaches zero, then the reflection coefficients approach unity linearly and the backscattering for polarization perpendicular/or parallel to the plane of incidence vanishes like the fourth/or second power of the angle respectively.

This model for reflection and scattering from rough surfaces appears to be the only one which treats both coherent and incoherent scattering phenomena in parallel, and which relates them explicitly to each other through the energy principle. Theorems are derived to show that the sum of average powers coherently reflected, incoherently scattered, and absorbed by the area of distribution "illuminated" by unit area of incident wave equals the incident power density; and (using a general theorem for an isolated protuberance [Twersky, 1954b]), it is shown that the forms for reflection coefficient and scattering cross section (in terms of single scatterer results) explicitly fulfill the required theorems. Illustrative examples are obtained by specializing the general results to arbitrary hemispheres, and semicylinders, and explicit approximations in terms of elementary functions are given for scatterers very small or very large compared to wavelength [Twersky, 1957a, 1955].

The above model is a generalization of the one introduced by Rayleigh [1907b] to consider incoherent scattering from a striated surface; his paper gives a single-scattering treatment based on the field of a fine semicylindrical protuberance. Rayleigh's work was initially extended to obtain single-scattered coherent and incoherent intensities for distributions of small semicylinders and hemispheres [Twersky, 1950a, b, c, 1953c], and of large semicylinders [Twersky, 1952d); then multiple-scattering effects for separations large compared to wavelength were taken into account for semicylinders [Twersky, 1953a, d]. These special scatterers are considered as illustrations in the more general treatment mentioned previously [Twersky, 1957a, 1955].

The phase of the coherent reflected wave for the special case of small hemispheres on a plane is also considered by Biot [1958]. Neglecting incoherent scattering, Biot [1957] considers a monopole source, and Wait [1959] a dipole source exciting small hemispheres on a ground plane. In particular, Wait [1959] considers surface wave effects for lossy bosses and shows that the first approximation for the coherent effects may be described by a plane having an inductive surface reactance; he also considers the analogous problems for a curved ground plane, and for parallel (plane, and curved) guides.

A variety of other models for random screens and rough surfaces exist in the literature. However, it is not the purpose of the report to consider these topics, except as they relate to multiple-scattering problems of distinct objects. The reader is referred to the works of Rice [1951], Booker, Ratcliffe, and Shinn [1950], Beckman [1957], Miles [1954], Magnus [1952], Schouten and De Hoop [1957], Ament [1956], Hoffman [1955], Lysanov [1958], Senior and Siegel [1959], Beard, Katz, and Spetner [1956], Beard and Katz [1957], Spetner [1958], LaCasce and Tamarkin [1956], LaCasce [1958], Berning [1957], Meecham [1956], Heaps [1956], Parker [1956], Jones and Barton [1958], Katzin [1957], Pollak [1958]. Additional papers are cited by Lysanov [1958] and by Twersky [1957a].

b. General Statistical Distribution

The grating and the random screen of arbitrary cylinders are essentially the "crystalline" and "rare gas" limits of a one-dimensional "liquid" of perfectly elastic scatterers. To treat this general statistical distribution, Zernike and Prins [1927] take the one-particle distribution to be constant and use probability considerations to derive a pair distribution function; the pair-function is expressed essentially in terms of an "elbow room parameter" (L) equal to the ratio of average to minimum separation of scatterer centers, a minimum generally greater than a scatterer's width. They obtain a single scattering approximation for a large number of scatterers on a line, and show numerically that for $L \to 1$, both pair-function and intensity become sharply peaked; and that for $L \gg 1$, both become relatively smooth. (Their basic paper introduces the now standard "inversion procedure" used in X-ray scattering by liquids. Inverting a corresponding approximation for the three-dimensional case, enables one to construct approximations for the "radial distribution function" in terms of scattered intensity measurements; see Gingrich's [1943] review.)

Twersky [1959b] considers the scattering of waves by a one-dimensional liquid of coplanar, parallel, arbitrary cylinders; he obtains a continuous transitional formalism from the rare gas limit [Twersky, 1957a] to the periodic one [Twersky, 1956a]. The analysis is based on a Poisson one-particle distribution function, and on a more convergent transform of the pair-function introduced by Zernike and Prins [1927]. Representing the field of one configuration as a sum of surface integrals, and averaging over the distribution, gives an integral relation between the average fields with one and two particles held fixed; equating these to each other yields an integral equation involving the known distribution functions and the presumably known scattering amplitude of an isolated cylinder. The absolute square of the average field specifies the "coherent intensity." A corresponding approximation is constructed for the "incoherent" differential scattering cross section by taking the average field with one scatterer held fixed as the excitation of a scatterer within the distribution.

The total average intensity for this distribution depends critically on L, the relative "elbow room" per scatterer. As $L \to 1$, the "local order" increases; in the limit, it is shown [Twersky, 1959b] that the one and two particle distributions go over to δ functions, and that the scattered field reduces to the solution for the grating of equispaced arbitrary elements [Twersky, 1956a]. For this "crystalline" case, the field is all coherent and consists of the transmitted and reflected propagating spectral orders plus the infinite set of evanescent surface waves. At the other limit $L \to \infty$, the local order disappears; the distribution functions become constants, and the results reduce to those of the analogous "rare gas" [Twersky, 1957a]. For this case, the coherent field consists of the directly transmitted and specularly reflected plane waves, and the differential cross section is relatively smoothly varying (as determined by choice of scatterers). The coherent field for the general case of the "liquid" (of infinite "length") has the same form as for the gas, but the incoherent intensity is more or less a smudged version of the intensity pattern for the periodic case: it is peaked in the vicinity of the parameters corresponding to the noncentral spectral orders of the grating, and these maxima broaden and decrease away from the directly transmitted and specular directions.

The results for scatterers symmetrical to the plane of the distribution are also imaged [Twersky, 1959b] to obtain corresponding functions for the general striated surface of arbitrary protuberances on a ground plane. Applications are given to illustrate multiple-scattering effects in certain resonance phenomena ("near-grating" anomalies), in the behavior near grazing incidence, and in the effects of packing for small scatterers.

3.4. Periodic Volume Distributions

The main lines for treating scattering of X-rays by crystals follow the works of Ewald [1917], Darwin [1914], Bragg [1915], and Laue [1931]. Ewald [1917] obtains a multiple-scattering solution for the lattice of dipoles. Laue [1931, 1935] works with a Fourier series representation for a general periodic index of refraction. Darwin [1914] uses a single-scattering approximation for the fields of the planar lattices parallel to the interface of a semi-infinite lattice, and takes into account multiple scattering between planes. He introduces the "transmission line" procedure for treating such problems; and, for the situation corresponding to Bragg [1913] and Laue [1913] resonances, Darwin's 1914 paper gives the associated pair of coupled difference equations, since rediscovered many times in connection with one-mode propagation in periodically loaded lines, guides, and artificial dielectrics. Prins [1930] applies Darwin's procedure to take into account absorption.

Laue's procedure is applied by Kohler [1933] to treat the one-mode case for the bounded periodically perturbed medium, and Mayer [1928] and Lamla [1939] consider the case of three strong modes. Essentially Darwin's procedure (but taking into account some multiple-scattering effects in the component planes) is used by Twersky [1954a] to treat the lattice of circular cylinders; and the general case is treated by this means by Marcuvitz [1956], whose results are applied to special problems by Barone and Schneider [1956].

A broad, relatively elementary survey of analytical techniques for treating scattering by periodic structures (methods introduced for scattering of X-rays by crystals) is given by James [1950], and some additional results are included in Partington's [1951] comprehensive treatise on physical chemistry (particularly vol. 3). Fourier methods for treating scattering (of anything) by periodic structures are reviewed by Slater [1958].

The lattice of spheres is treated by separating variables in the static limit by Rayleigh [1892], and for arbitrary wavelengths, by Kasterin [1897], and by Morse [1956]. Various approximations also exist in the literature of artificial dielectrics, a subject whose modern aspects start essentially with the work of Kock [1948]. The literature to 1952 is surveyed in Brown's [1953] monograph on microwave lenses, to 1957 in Cohn's [1960] review chapter of the Antenna Engineering Handbook, and to 1958 in Harvey's [1959] survey article on optical techniques at microwave frequencies. Recent papers in the U.S. literature include those of Lippmann and Oppenheim [1954], Storer [1952], Jones, Morita, and Cohn [1956], Morita and Cohn [1956], Collin [1959], Ward, Puro, and Bowie [1956], Kaprielian [1956a, 1956b, 1956c], Cohn [1956], and Hickman, Risty, and Stewart [1957].

3.5. Random Volume Distributions

The earliest analytical treatment of the scattering of waves by random distributions of objects (or potentials) is essentially Rayleigh's theory of the color of the sky [Rayleigh, 1899]. The subject has since received much attention in the literature, but much of the work has been heuristic.

Foldy's [1947] treatment of scattering by monopoles serves as a model for those seeking to treat more arbitrary scatterers. Thus Lax [1952], and Twersky [1958a, 1959a] give different generalizations for the coherent field. Foldy's self-consistent treatment of monopoles is extended essentially three different ways to obtain the propagation coefficient (say K) of the coherent field for a random distribution of relatively arbitrary scatterers excited by a wave having propagation coefficient k. Each procedure expresses K in terms of an isolated object's scattering amplitude, say f; but Twersky [1958a] uses $f(k\rightarrow k)$, the amplitude of the object in free space; Lax [1952] uses $f(K\rightarrow K)$, the amplitude in the new medium associated with the coherent field; and later Twersky [1959a] uses $f(K\rightarrow k)$, the amplitude of an object excited in K-space but radiating into k-space. Twersky [1958d] obtains $f(K\rightarrow k)$ by introducing a new class of single-body scattering problems, in which the source and radiated terms of the solution satisfy different wave equations. [This type of scatterer may be more palatable if its limiting form for a monopole is recognized in the usual volume integral representation for the field scattered by a constant potential k^2-K^2, i.e., in the integral whose kernel comprises a monopole (the free k-space Green's function) weighted by the local field: since the local field travels in K-space, these monopoles radiating into k-space are elementary forms of the "schizoid scatterer" characterized by $f(K\rightarrow k)$.]

The new formalism is applied [Twersky, 1959a] to a slab region of large tenous scatterers, and simple explicit forms are obtained for the coherent and incoherent intensities, and for the average phase. Theoretical results are compared with a series of detailed experiments by Beard and Twersky [1958, 1960a, 1960b] on a large scale dynamical model of a "compressible gas" of spheres. Measurements were made from a relatively rare gas (average separation of centers 10 times scatterer diameter) to practically a "liquid state" case (average separation about one-eighth larger than diameter). [Here the "molecules" were 1½-in. styrofoam spheres and the measurements were made with ½-cm radiation; a system of blowers and turbulence-creating wedges produced the distribution, and the required statistical functions were measured separately by optical methods.] The recent computed and measured intensities are in accord over the full range investigated.

The three extensions [Lax, 1952; Twersky, 1958a, 1959a] of Foldy's [1947] procedure are among the more recent extensions of Rayleigh's original model. For a volume distribution of small scatterers, Rayleigh [1899] gives a leading term approximation involving $f(k \to k)$. Other results for wave scattering in terms of forms of $f(k \to k)$ appropriate for special objects, are given in Reiche [1916] (slab region of dipoles), Urick and Ament [1949] (slab of small spheres), and Twersky [1953b] (slab region of cylinders). Similarly, expansions in terms of $f(k \to k)$ are used in the work on dense distributions of dipoles by Yvon [1937], Kirkwood [1936], Brown [1950], Mazur and Mandel [1956], Green [1952], Jansen and Mazur [1955], Jansen [1955], Fixman [1955], Born and Green [1946], and Green [1957]; these papers are particularly noteworthy for their care with the probabilistic aspects of the problem. Alternative approaches, still based on well-defined elementary scatterers, are discussed by Onsager [1936], Böttcher [1952], De Loor [1956], and others.

Brief, relatively comprehensive, introductions to various aspects of the subjects involved in the above, and additional references, are given in several articles of the Handbook of Physics edited by Condon and Odishaw [1958]: see "Dielectrics" by von Hippel; "Molecular Optics" by Condon; "Principles of Statistical Mechanics and Kinetic Theory of Gases," and "Vibrations of Crystal Lattices and Thermodynamic Properties of a Solid" by Montroll; and "The Equations of State and Transport Properties of Gases and Liquids" by Bird, Hirschfelder, and Curtiss. A detailed review of the literature of dielectrics is given by Partington [1951–1955], vol. 4 and 5. See also Frenkel [1956], Debye [1945], Hartshorn and Saxton [1958], Van Vleck [1932], Von Hippel [1954], and, in particular, the excellent recent review by Brown [1956]. Fournet [1957] gives the latest review on the structure of liquids; and the recent article by Montroll and Ward [1958] on the statistical mechanincs of interacting particles, and the references it gives to the classical statistics literature, indicate the more fundamental models of matter now under study.

Much work on multiple scattering of incoherent radiation has been done from essentially a particle scattering viewpoint. Instead of the wave equation, one works with the Boltzman integro-differential equation for transport processes. See Hopf [1934], Chandrasekhar [1950], Case [1957], Woolley and Stibbs [1953], Fano, Spencer, and Berger [1959], and Goldstein [1959] for fundamentals, applications, and reviews of computational procedures.

An alternative approach to problems of scattering and propagation in random media is to work with a perturbed continuum—see papers of Einstein [1910], Smoluchowski [1908], Pekeris [1947], Debye [1954], Booker and Gordon [1950], Villars and Weisskopf [1954], Silverman [1957, 1958], Staras [1955], Wheelon [1957], the scatter-propagation issue of the Proceedings of the IRE [1959], Bremmer's Handbuch article [1958], and the recent review of tropospheric propagation by Staras and Wheelon [1959]. References to the literature of physical chemistry involving this approach are cited by Fishman [1957] and Stacey [1956].

Recent papers on the topic of this section include Booker [1956], Kraichman [1956], Chu and Churchill [1956, 1955], Gordon [1958], Zink and Delsasso [1958], Skydrzyk [1957], Silverman [1956, 1957, 1958], Stein [1958], Phillips [1959], Smith [1956], Zweig [1956], Buckingham [1956], Buckingham and Stephen [1957], Yvon [1958], Prins and Prins [1957], Longuet-Higgins and Pople [1956], Goldstein and Michalik [1955], Fixman [1955], Jefferies [1955], Megaw [1957], Peterlin [1957], Nakagaki and Heller [1956], Stevenson [1957], Richards [1955], Sekera [1957], Richards [1956], Ament [1952], Meeron [1960], Digest of Literature of Dielectrics, Conference on Electrical Insulation, National Academy of Sciences—National Research Council, Vols. 20, 21, 22 [1956, 1957, 1958].

Additional categories of phenomena involving "multiple scattering", and additional references (particularly to the literature of quantum mechanics), are given by Lax [1951, 1952].

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Part 3. Antennas 1957–59

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Developments in antenna theory during 1957 to 1959 are summarized, with emphasis on the definitive papers. Work of U.S. authors published in United States and English language foreign journals is included. The survey is divided into four sections: Broadband antennas, dynamic antennas, large aperture antennas, and small aperture antennas. Surface wave antennas are not included in this paper.

Major progress has been made in the broadband antenna field with the appearance of log-periodic structures and unidirectional spirals. Pattern bandwidths of 10:1 have been achieved with the former.

Newest in the field are the developments arising from application of communication theory to antennas, treating the antenna as a spatial filter. Developments include: Exchange of bandwidth for aperture size or density, time modulation of certain antenna parameters to obtain multiple simultaneous modes of operation or to obtain enhanced performance, and time processing of multiple antenna outputs to obtain increased resolution or decreased array density. Thus the antenna is in general a multiterminal time varying (dynamic) device which must be considered as an integral part of the system.

Important advances in large antennas include: Application of array and electronic scanning techniques to conical geometries; electronically scanned two-dimensional arrays using frequency shift or ferrite phase shifters; use of unequal spacing between elements in an array to obtain depressed secondary responses and to utilize lower array density; annual slot arrays consisting of annuli of half-wave slots, with the advantage of a simple mechanical structure; a UHF dipole array coupled electromagnetically to a two-wire transmission line; focusing and control of radiation in the Fresnel region; determination of the constituents of antenna noise temperature.

Another important accomplishment has been the evaluation of HF aircraft antennas considering pattern, efficiency, and bandwidth.

1. Introduction

The three-year period between the 12th and 13th General Assemblies has seen substantial progress in many of the fields recommended for study in the "Resolutions and Recommendations" of the URSI Proceedings. In addition to surveying these topics, this report endeavors to cover those aspects of antenna research and development which are of primary interest to URSI and on which significant progress has been made.

The progress in the U.S.A. during 1957, 1958, and 1959 on antennas is broken down into the following topics: Broadband antennas; dynamic antennas, including data processing arrays and modulated antennas; large aperture antennas, including radio astronomy, array, and scanning antennas; and small aperture antennas, including those for space vehicles. Surface and leaky wave antennas and scattering and diffraction are covered in separate reports. The survey is based mainly on the definitive papers and reports in the field with a bibliography of supporting developments, and is not a catalog of all antenna papers. All important U.S. journals have been covered, but only papers with U.S. authors are included herein. In addition, papers by U.S. authors in certain English language foreign journals have been included, along with unclassified technical reports from major antenna establishments in the United States.

2. Broadband Antennas

Introduction. The term "broadband antenna," by definition, denotes an antenna having essentially constant pattern characteristics, as well as input standing wave ratio, over at least an octave of frequencies and usually several octaves. Intrinsic in this definition is the assumption that the efficiency of the antenna remains above some specific value, for efficiency is as pertinent a characteristic of broadband antennas as impedance and directivity. Often the term "broadband antenna" also carries a connotation of omnidirectionality, since it is an order of magnitude more difficult to design a broadband array of broadband elements than the broadband element alone. Thus, as is evident from Proceedings of the 12th General Assembly, early work tended to concentrate on the easier of the two problems; namely, single element broadband antennas of limited directivity. It is encouraging to note that some headway is being made on the problem of more directive broadband antennas.

Spiral antennas. The infinite equiangular spiral antenna is a device which is specified entirely by angles and is obviously frequency independent in its ideal state. Rumsey [1957] has reviewed frequency independent antennas of which the plane spiral and conical spiral represent particularly useful specializations. These antennas have been investigated on a continuing basis during the past three years primarily

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by Rumsey [1] and Dyson [1959a]. The necessity of having an antenna of finite size, of course, requires the specification of at least one length. Thus there is a rather definite "cut-off" frequency below which the antenna functions either with very low efficiency or not at all. The effective size of the antenna at higher frequencies is apparently bounded quite effectively by radiation damping, since excellent pattern constancy is obtained (as well as low standing wave ratio) over bandwidths of 20 or 30 to 1.

Dyson [1959b] has also shown that by extending the planar equiangular spiral into a conical spiral, the usual disadvantage of bidirectional radiation can be effectively overcome. Using this technique, bandwidths of the order of 12 to 1 have been obtained with 20 or 30 to 1 probable in the future. If flush mounting is required, some sacrifice in bandwidth or efficiency is necessary, at least with the current state of the art.

Log periodic antennas. The previous section dealt with circularly polarized radiating elements which are, theoretically, made independent of frequency through the application of the "angle concept" and the "self-complementary principle." The practical antenna, however, is frequency sensitive by virtue of the inability to construct an ideal model. If one starts with a mathematical model which is not quite frequency independent, the practical approximation can sometimes produce results superior to those of the practical approximation to the theoretically perfect antenna. Such a device is the log periodic antenna which is defined as a radiator having characteristics which vary periodically as the logarithm of the frequency [DuHamel, Isbell, 1957]. The basic log periodic antenna can be obtained by a simple modification of the angular antenna and the result is a predominantly linearly polarized antenna (although circularly polarized versions are available) having reduced end effects caused by the necessary finite size.

While there are an unlimited variety of log periodic antenna configurations, the class which has received the greatest amount of attention is the self-complementary "bow-tie" structure having tooth-like discontinuities along its radial edges. DuHamel and Isbell [1957] and DuHamel and Ore [1958] have obtained bandwidths of over 10 to 1 with such antennas, and in addition have found that the beamwidth could be controlled over a considerable range by varying the periodicity of the teeth. In general, however, the latter effect is also accompanied by a change in the low frequency "cut-off" wavelength. DuHamel and Berry [1958, 1959] have started investigation of several other designs including three dimensional versions and trapezoidal toothed structures which have promise for antennas of higher gain.

High gain broadband antennas. Two approaches to highly directive broadband antennas have recently been used. These are expansion of the effective aperture size of conventional antennas and the forming of a broadband array of broadband elements.

[1] Cheo, Rumsey, and Welch, A solution to the equiangular spiral antenna problem, paper presented at the 1959 Fall IRE–URSI Meeting, San Diego, Calif.

An example of the use of a resonance mechanism to control the effective aperture is the "Pin Wall Horn" of Parker and Anderson [1957] wherein two walls of a horn radiator are serrated with rectangular holes of ever increasing size as one proceeds from the throat to the mouth. Constant patterns in both principal planes have been obtained over a 4 to 1 bandwidth.

DuHamel and Ore [1959] and Isbell [1959] have shown that the effective aperture of a log periodic antenna can be increased by optical magnification. Log periodic feeds were constructed for paraboloidal reflectors, giving bandwidths between 10 and 20 to 1 and a VSWR of 2. Gains up to 30 db were obtained.

DuHamel and Berry [1958] have also investigated arrays of log periodic antennas of trapezoidal type which have gains of 15 to 20 db. These arrays are ingeniously designed so that the element spacing is given in terms of angles rather than distances. As a result, excellent patterns and VSWR of the order of 2 are obtained over a bandwidth approaching 10 to 1. The typical very narrow bandwidth of an array of many elements has been greatly exceeded by McCoy et al. [1958]; they have obtained a 35-percent bandwidth with a linear array of 80 waveguide horns with corporate feed structure. Sidelobe ratios near 25 db and VSWR of the order of 1.3 are maintained throughout the band (S-band). Hybrid junctions are not used for power division, hence the efficiency is high.

It is encouraging to see that someone has finally realized the advantages of an array of unequally spaced elements. D. D. King [1959] has shown analytically that a linear array, capable of being steered $\pm 90°$ with respect to broadside over a 2 to 1 frequency band, can be designed to maintain its collimate characteristics with no sidelobe higher than -7 db. Also, fewer elements are needed than with an array of equally spaced elements.

Summary. In reviewing the progress represented by the above mentioned reports, several conclusions seem evident. In spite of some recent attempts, a satisfactory theory describing the operation of the newer types of broadband antennas is still lacking. At the moment it is impossible to choose between the infinite number of theoretical broadband configurations except from a constructional viewpoint.

The most important area for further work, however, would appear to be in really broadband, electrically scannable, two dimensional arrays for such applications as radio astronomy and space communications.

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3. Dynamic Antennas

Introduction. The past three years have seen a modest but concerted effort throughout the country to adapt many of the successful techniques of circuit theory to antenna design. Of particular interest have been attempts to incorporate the concepts and formulations of communications theory to the analysis of antenna performance. Under this philosophy an antenna is considered to be a spatial filter whose characteristics are completely defined in terms of a transfer function which is identical in form to that used in conventional circuit theory. The objective of the communication theory approach is to optimize the transfer function in terms of criteria which are determined on the basis of the operating antenna environment. For example, terms are introduced such as the "fidelity defect," which is a root mean square measure of the ability of a mapping or scanning antenna to provide an output signal which is an accurate reproduction of the target distribution. Unfortunately, many of the error criteria which are based on circuit theory concepts are not pertinent in applications to antenna design. The "fidelity defect" as an illustration is much too restrictive in some cases since it doesn't weigh the parameters which are important in system operation. As a result, the fundamental system concepts are often lost sight of because of the mathematical formulation. There is an urgent need, then, to reformulate the communication theory concepts in a way that takes into account the inherently different characteristics of antennas and their associated systems. In addition, care should be taken to keep from utilizing communication theory techniques in areas where conventional methods are clearer and simpler.

In addition to the use of communication theory as a *tool* in the rating of antenna performance, the basic concepts have been utilized to achieve *new operating techniques* which are capable of considerably improving the information gathering efficiency of an antenna system. This achievement has come about through the recognition that the antenna, viewed as a spatial filter, may be made nonstationary by the process of time modulation of the transfer function, in such a way that a direct correlation is obtained between spatially dependent and time dependent signals. More accurately, a series of orthogonal time dependent signals is generated, each one of which is modulated in accordance with a different spatial pattern. Time domain processing of these signals then produces a multitude of spatial patterns which can be used in the conventional way. These techniques result in a considerably greater quantity of spatial information than would be obtained with conventional antenna operation, and it is felt that future antenna systems will place a greater reliance on these techniques.

Unfortunately, in reporting on this exciting new field of antenna theory, the authors must be content with giving merely a blanket acknowledgement to a substantial amount of work done in connection with various military projects which, obviously, is unavailable as reference material.

Communication theory applied to antennas. One of the earliest attempts to utilize the concepts of communication theory in the analysis of antenna performance was by White [1957]. The objective of this work was the determination of the fundamental limits on the information available from antenna systems. White demonstrated that an antenna (a linear array in particular) viewed as a spatial filter has a bandwidth which is determined by the aperture extent and that, therefore, it will reproduce only a finite number of the space harmonics representing a desired spatial pattern. By the same reasoning, in a two-way radar situation, the received voltage $G(\theta)$ will not exactly reproduce the target distribution $F(\theta)$ because of the finite spatial bandwidth of the antenna system. Thus from a basic standpoint, the antenna resolution is limited by the highest space harmonic within the bandwidth of the spatial filter; this bandwidth is in turn determined by the aperture size. It should be pointed out, as White has neglected to mention, that antenna resolution can be increased theoretically without limit, by the use of supergain techniques. This is equivalent to artificially increasing the space bandwidth of the antenna by producing additional spatial harmonics which contribute large amounts of reactive power. Although this technique is of little practical value for well-known reasons, it is of interest with respect to the concept of spatial filtering. Additionally, this well-known limitation on antenna resolution may be overcome by the use of correlation type processing of the antenna signals. For radar operation, White points out that improved resolution may be accomplished by correlating the signal returns with any a priori information that is available about the target distribution. In other applications, such as radio astronomy, the interferometer antenna structures are utilized in conjunction with correlation processing to achieve high resolution with low gain. Unfortunately, in addition to their low-gain characteristics, interferometric-correlation schemes are unreliable in radar operation because of the three-dimensional type of target distribution which permits the possibility of false correlations.

Raabe [1958] has also considered the antenna as a spatial filter, and he has utilized this concept in a discussion of antenna pattern synthesis. His ideas are based on the recognition that the finite spatial bandwidth of an antenna limits the highest harmonic variation which can be contained in the radiation pattern. It is thus proposed that the bandwidth limited pattern be utilized as the desired waveform rather than the pattern of infinite harmonic content. The sampling theorem is then used to determine the optimum sample characteristics; the samples are taken as properly spaced "sinc" beams which are weighted according to the desired waveform. The spectrum represented by these samples is then said to be matched to the filter (antenna) characteristics, and the desired waveform is reproduced. This technique is very similar to Woodward's method of synthesis and Raabe presents an analysis of the similiarity. Unfortunately, detailed numerical examples of actual synthesized patterns are not presented for comparison. Nevertheless, Raabe's ideas are fundamentally correct and represent an interesting application of communication theory to one of the more familiar aspects of antenna theory.

Two attempts to utilize the communication theory concepts of processing and filtering in the actual design of antennas for special application have been reported by Anderson [1958] and Dausin et al. [1959]. Essentially, Anderson considers a displaced-phase-center antenna with correlation type processing to reduce platform and scanning noise in airborne moving target radar. The work of Dausin, et al., deserves a more detailed discussion since it presents concepts which have not been mentioned previously. In this work, the spatial frequency bandwidth (and hence the angular resolution) of an antenna system is shown to depend not only on the aperture extent but also on the time frequency bandwidth of the received signal. This is not too strange, however, since the aperture extent is only uniquely defined in terms of wavelengths and, hence, the signal bandwidth should play a part in determining the spatial filter characteristics. The major conclusion from this work, then, is that signal bandwidth can be utilized in place of aperture size or density of sources to produce equivalent radiation patterns. Operationally, this is achieved by summing the autocorrelated spectral outputs from each element, weighted according to a given aperture illumination. Formulas are derived which permit the determination of the required aperture and source distribution for a given signal bandwidth characteristic. An illustrative example of this technique is presented in which a 10-percent signal bandwidth is sufficient to produce a normal 200-element pattern from a 66-element array—a 65-percent reduction in the number of elements! Obviously, the same percentage reduction will not hold for arrays with a few number of elements.

Time domain antennas. Shanks and Bickmore [1959] have presented an excellent tutorial discussion of the use of time modulation techniques in advanced antenna design. The basic concept introduced in this presentation is that of periodic time modulation of selected antenna parameters to improve the operating characteristics of an antenna system. This modulation technique produces a correlation between spatial information and time dependent signals which permits time-domain processing to provide increased spatial information. Not only does this concept provide improved pattern control, as in sidelobe suppression, but it also indicates new operating techniques which are shown to have application to electronic scanning and multipattern operations. In addition, Shanks and Bickmore consider possible physical configurations which are capable of producing this type of operation and present some of the system problems of detection and processing which must be studied. Finally, an elementary experimental demonstration of the basic concepts is reported.

In an elaboration of the above work, Shanks [2] has discussed in detail the application of time modulation techniques to electronic scanning. It is shown that with the proper aperture modulation applied to a linear array, a series of directive-beam patterns is generated, and that the information from each may be separated by time domain processing of the received signals. The required aperture excitation is equivalent to a coherent pulse, of length much shorter than the antenna length, sweeping across the aperture. In practice this is achieved by on-off devices which are switched progressively. This technique promises to overcome many of the disadvantages which are normally associated with the conventional control devices used in electronic scanning operations.

The concept of modulated antennas has been utilized in a more restricted sense by other researchers in the field. Drane [1959] has applied periodic modulation to the relative phases of a multiple antenna system to achieve improved resolution. In particular, he considers a system comprising a number of 2-element interferometers alined colinearly with a single linear array; the overall length of this system is many times the length of the single linear array. When the output signals from each interferometer are phase modulated and added (in a nonlinear detector) to the array output, simple time domain processing produces a pattern having no angle ambiguities and a resolution which is equivalent to that obtained from a continuous linear array with a length of the entire system. Drane also demonstrates that correlation type devices can be used in place of modulation and time domain processing to achieve the same type of results. In this case, a direct multiplication of the various element outputs is performed and this allows the addition of as many interferometer elements as desired. The major advantages in Drane's type of system is the large saving in the number of elements which are required to achieve a given angular resolution. However, two disadvantages are apparent which limit the range of application of this technique. First, the system is inherently low-gain, thereby restricting its use to applications such as radio astronomy, where long integration times can be used. Secondly, the system is unilateral because of the processing methods which are used.

[2] H. E. Shanks, A new technique for electronic scanning, paper presented at 1959 URSI Fall Meeting, San Diego, Calif.

Bracewell [3] has presented an interesting review of switched interferometer techniques which have long been used in the field of radio astronomy. This presentation was largely an attempt to provide a better physical understanding among antenna people of the concepts which are common in the astronomy field. Of particular interest is his graphical method of constructing a spectral sensitivity diagram for interferometer-type structures. This technique provides a simple method of visualizing the effectiveness of these systems.

A study of correlation techniques in antenna pattern control has been reported by Band and Walsh [4] in what is actually a companion paper to Drane's work. No new operating principles are introduced, but rather a description of several practical correlation and multiplication devices is presented. The work of Band and Walsh represents the first known effort to develop correlation devices which are specifically suited for the special signal output characteristics found in antenna applications; other work is by Smythe.[5]

Summary. Based on the above survey of the past three years of work in the field of dynamic antennas, it may be concluded that antennas are acquiring a "new look." The conventional concepts of antennas are gradually being pushed aside to make way for the new philosophy of integrated antenna systems. Whereas the processes of correlation, filtering, and integration have in the past been associated with systems design, these same ideas are now rightfully within the province of the antenna art. This new philosophy has important implications to antenna people, in that they can no longer rely on supporting personnel for systems inputs, but must adopt an integrated antenna-systems approach to their problems.

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4. Large Aperture Antennas

Radio astronomy and large antennas. Emberson and Ashton [1958] have reviewed in detail the design of a 140-ft paraboloidal antenna for the telescope program of the National Radio Astronomy Observatory; the paper presages the design of a 600-ft dish. Another large radio telescope antenna is the fixed standing parabolic reflector and tiltable flat sheet reflector which has been designed by Kraus [1958]. This antenna allows elevation scanning through tilting of the sheet reflector on an E.-W. axis and azimuth scanning through primary feed rotation. Scale model tests have been completed and are summarized [Kraus, 1958].

Bracewell [1958], in an important paper on radio interferometry, derives the relation between spacing of ground observation points and resolution of discrete sources as a function of frequency. His conclusions are that in the case of the sun, independent data are available only at points on the ground separated by at least 100λ. Bracewell [1957] has also considered the design problems of cross interferometers using dishes as elements. Swarup and Yang [1959] have also studied phasing problems for cross interferometers at microwave frequencies. A general theory in which switching antennas, such as cross-arm types, may be included is covered under Dynamic Antennas. An interesting scanning technique reported by Miller et al. [1958] uses a linear array of dishes, each of which is fed by a helix. Rotation of the helices produces phase shift and consequent scanning of the beam. Scans of the order of ± 4 beamwidths have been obtained. Another array for radio astronomy purposes consists of two parallel line sources with Yagi elements [Gallagher, 1958].

Sletten et al. [1958a] have developed shunt slot arrays for use as corrective line source feeds for paraboloids. These feeds allow an elevation fan beam yet maintain well-focused narrow beam azimuth patterns over the entire elevation interval. Another use of a line source feed is for the 1000 ft diam spherical reflector to be erected in Puerto Rico for radar astronomy studies. This fixed bowl with movable feed will allow wide angle scanning and is being designed by AFCRC and Cornell. The bowl should be useful at 21 cm, and if the tolerances can be achieved, this bowl will have at 21 cm the highest gain of any antenna, over 68 db, allowing 2 db for gain loss due to spherical aperture. Also to be mentioned is the very low sidelobe parabolic horn developed at Bell Laboratories [Friis, May, 1958], with first sidelobes below -40 db and back loves below -70 db.

Array antennas. The previous triennium (1954 to 1956) saw completion of a very extensive program on the properties and design of array antennas, particularly waveguide slot arrays. This work is reported in the 1957 Commission 6.3 report [Cottony et al., 1959]. Efforts in the triennium 1957 to 1959 have been concentrated on extending both the theory and practice to shapes other than planar, and on new configurations, e.g., annular slot arrays and arrays of unequally spaced elements. Goodrich, Siegel, Chernin, et al. [1959] have summarized the work by University of Michigan and Hughes Aircraft Company authors on producing a pencil beam from an array on a conical surface. Although this is also

[3] R. N. Bracewell, Switched interferometers, paper presented at 1959 URSI Fall Meeting, San Diego, Calif.
[4] H. E. Band, and J. E. Walsh, Correlation techniques applied to antenna pattern control, paper presented at 1959 URSI Fall Meeting, San Diego, Calif.
[5] J. B. Smythe and S. Weisbrod, Utilization of space frequency filters in anenna design, paper pr esented at 1959 URSI fall meeting, San Diego, Calif.

an electronic scanning problem, the conical surface is the unique feature of the problem. The theoretical analysis of radiation from current distributions on a cone used Geometrical Optics and Fock theory, the latter being used in the shadow region. Physical Optics was used to account for tip diffraction. A more complete discussion of the analysis problem is a paper by Goodrich et al. [1957]. A multiplicity of array configurations were studied: axial, circumferential, spiral, etc. The final design [Goodrich et al., 1959] consisted of a stack of parallel plate transmission lines, all coaxial with the cone and terminating on the conical surface. Excitation of the stacked parallel plates was provided by a central slot array, arranged to be sufficiently dispersive as to allow elevation frequency scan. Azimuth scan was accomplished by a rotation of the central waveguide feed structure. Other work on conical surface elements near the tip is by Held et al. [1958]. A paper which concerns synthesis over a conical surface is by Unz [1958]. Also investigated is the equivalence of a slot array and a continuous current distribution with particular application to a conical surface [Mayes, James, 1958]. Other papers on synthesis include a technique which uses multiple sets of elements in an interference or supergain fashion [Sletten et al., 1957a] and strip sources [Mittra, 1959]. Wait and Householder have extended the Tschebyscheff array design of Dolph to an array of axial slots disposed circumferentially about a circular cylinder [Wait, Householder, 1959].

Secondary main beams of two-dimensional slot arrays due to alternating inclination or displacement of elements have been studied by Kurtz and Yee [1957]. They treated the array as having virtual elements consisting of a pair of adjacent elements; all virtual elements were then alike. Also covered in this paper is the successful use of baffles to reduce the secondary beams.

As mentioned earlier, arrays of unequally spaced elements constitute a promising configuration. D. D. King and others [6] have demonstrated that such effects as amplitude taper (in an equally spaced array) can be simulated with proper spacing. Irregular spacing suppresses undesirable effects such as secondary beams (usually caused by regular spacing) and may reduce the number of elements needed. This work is a continuation of earlier work of Unz.

A novel departure from the conventional array of half-wave elements wherein the element and array factors can be separated is given by Ronold King [1959]. Here, an array of full-wave dipoles is considered and the pattern derived from an integral equation of dipole current distributions. King shows that the assumption of equal, sinusoidal current distributions may produce appreciable errors in the region of minor lobes. However, the extreme difficulty of synthesizing patterns will probably severely restrict use of the analysis.

Constructional advantages over the rectangular array are offered by the annular slot arrays developed by Kelly [1957]. Several rings of discrete slots are fed by a single radial line, offering an extremely simple yet flexible design. Excitation of the $n=1$ circumferential mode produces a beam in the normal direction; the modes for $n=0$ and higher than $n=1$ produce nulls on the axis. Schell and Bouche [1958] have developed a concentric loop array in which the loops are large in wavelengths and in which two feeds are used, allowing rotation of the pattern.

Some interesting developments have appeared in arbitrarily polarized slot arrays. Hougardy and Shanks [1958] have developed a linear array consisting of crossed slots in a square waveguide fed with two dominant orthogonal modes. Appropriate microwave plumbing allows the relative phase and amplitude and hence radiated polarizations to be adjusted. The annular arrays of Kelly [1959] above can also be excited with two modes for variable polariaztion. Hines and Upson [1958] have developed an interesting concept wherein a parallel plate pillbox is fed with a line source guide containing 45° inclined slots. The spacing between guide and mouth controls the polarization since the two modes have different phase velocities.

Although the emphasis has been on slot arrays, one important development has arisen in the field of dipole arrays. Sletten et al. [1957b] developed a dipole array wherein the dipoles are coupled electromagnetically to a two-wire transmission line. The shorted folded dipoles are in a plane parallel to the line and are spaced as in an array. Coupling is controlled by the angle and spacing between the dipole and the line; an analysis of the coupling is given by Seshadri and Iizuka [1959]. This array offers simplicity of construction in the UHF region comparable to that of microwave slot arrays. Cottony and others [1959] at the National Bureau of Standards have fed a large corner reflector by a collinear dipole line source, obtaining 40-db first side lobes, with a narrow azimuth beam and a broad elevation beam. The collinear array allows close realization of the design values; the 40 db is better than that obtained with waveguide slot arrays.

Electronic scanning. Electronic scanning of two-dimensional slot arrays has been achieved in practice by several means including frequency shift scanning and dielectric and ferrite phase shifters. An array which is scanned by frequency in one plane and by dielectric stub phase shifters in the other plane is described by Spradley [1958]. A serpentine (snake) main feed guide couples energy to the branch guides, allowing a small frequency swing to produce the large phase progression needed for large scan angles. Goodwin and Senf [1959] have developed a prototype 10,000–Mc/s ferrite phase shifter scanning array where a set of main line phase shifters produce elevation scanning and a set of phase shifters, one at each element, is used for azimuth scanning. All phase shifters are relay programed, with a TV-type raster scan. The term "volumetric scanning" has been used to describe these arrays, and means that

[6] Steerable antenna focusing techniques, Electron. Comm., Inc., reports during 1959 to Rome Air Development Center.

the beam can be scanned in two planes so as to sweep out a volume. Gabriel et al [1957, 1958] have developed an "organ pipe scanner" in which a large lens is fed by a two-dimensional array of horns which in turn is excited by a feed moving across the matrix of waveguide ends. An improvement on the use of ferrite phase shifters with attendant nonlinearities is to use quadrature coils and progressive frequency harmonics to produce linear progressive phase shifts [Clavin, 1959].

Another phase shifting device is the helical trombone phase shifter of Stark [1957] where movable double coupling loops are used. This device has been very successful in the UHF region. A general investigation of beam scanning for large antennas has been conducted at Stanford Research Institute [1958]. A technique which allows scanning to be accomplished through amplitude variations rather than through the usual phase front adjustments has been proposed [Sletten et al., 1958b]. However, amplitude scanning is similar to a supergain phenomenon in that adjacent elements operate with fixed phase in an interferometric fashion to produce a net small radiating current in the proper direction, except exactly at broadside and end fire.

One of the most promising developments in the electronic scanning field is the ferrite excited slot developed by Shanks [1959]. This is a radiating element suitable for inclusion into two-dimensional arrays, in which the phase and amplitude of the element can be controlled. Slot coupling changes result from a shifting and rotation of the field inside the waveguide by means of two ferrite post irises. This is an extension of the slot developed by Tang which uses movable mechanical irises. Individual control of each element will allow maximum flexibility for both scanning and data processing type antennas. Although nonlinearity is a severe programing problem, this development offers great promise. The end-fed array appears to incur serious mutual impedance changes for large scan angles. Blasi and Elliott [1959] show that the changes of mutual impedance for uniform amplitude linear phase arrays make end feeding unsuitable due to the change in coupling as the wave proceeds down the feed line. However, corporate feeding does not suffer from this disability. Another investigation has shown that the popular cos θ approximation for effective aperture must be modified for large scan angles. Bickmore [1958] has derived the correct result which yields the end fire value in the limit as it must. The cos θ result is very good to a point (typically 60°), beyond which the value drops rapidly before entering the end fire region. Tolerances continue to be an important subject in antenna array design. Elliott [1958] has summarized the quantitative effects of mechanical and electrical tolerances. Of these, translation errors in element position are most important. The effect of random errors on beam pointing has also been investigated [Rondinelli, 1959]. A quality factor has been derived for evaluating the system performance of search scanning antennas, taking into account such things as scan rate and hits per scan [Gardiner, 1957].

Considerable progress has been made in the reflector field. One of the most notable examples is the parabolic torus antenna [Mavroides, Provencher, 1958]. A particularly interesting version of the torus, developed by Barab et al. [1958] of Melpar, and Flaherty, and Kadak [1958] of Westinghouse, is made of wires inclined at 45° so that an internal rotating horn affords 360° scan. Another version uses a ring array of dipoles outside the torus [Fullilove et al., 1959]. These devices should be applicable to astronomy.

Li [1959] has shown that a spherical reflector can be used for wide angle scanning with good side lobes at a cost of gain. A typical gain loss for wide angle scanning is 9 db. This can be reduced below 2 db using a corrective line source feed as mentioned earlier. Another development in optical scanning devices is the double-layer pillbox of Rotman [1958]. The feed is placed in one layer with the mouth in the other, with a consequent reduction of shadowing and reflection, which in a single-layer pillbox is due to energy reflected back into the feed. Also, aberrations can be corrected either in object or image space. Zoned mirrors can be corrected to be coma-free according to Geometrical Optics. This conclusion has been refined on the basis of diffraction theory [Dasgupta, Lo, 1959]. Additional work on mirror lenses for scanning has been done at the Naval Research Laboratory [Marston, Brown, 1958].

An important generalization of symmetric lens design has been made by Morgan [1959] of BTL. He has given a general solution for the spherically symmetric lens with variable index of refraction which includes Luneberg and Eaton lenses. Proctor has given a design technique for constrained scanning lenses, i.e., lenses in which the wave propagation direction is constrained to be parallel to the beam axis [Proctor, Rees, 1957].

Near zone studies. A focusing concept developed by Bickmore has led to some interesting applications. This allows optimum transfer of energy between two unequal size apertures [Bickmore, 1957a]. Furthermore, it allows measurements with a narrow beam antenna inside the Fresnel region with the same resolution obtained in the far field region [Bickmore, 1957b]. This is accomplished in a linear array simply by introducing a slight spherical curvature of the appropriate amount into the array. In two-dimensional arrays, the requisite phase change can be introduced into the phase shifter or feed devices. With this technique, it has been possible to measure far field patterns as near as 2 percent of the normal distance $2D^2/\lambda$. Another technique for measurement of far field patterns in the Fresnel region is that of Cheng [1957]; this uses a defocused primary source. Goodrich and Hiatt [1959] have considered the transfer of energy from a point source to a point sink using an ellipsoidal reflector. A complete ellipsoid would yield according to scalar theory 100 percent transfer. The focusing properties of the ellipsoid are concomitantly studied. Harrington, Villeneuve and Hu [1959] and Harrington [1958] investigated near field gain and derived a near field synthesis technique. The near field gain study obtained the widely used physical limitations on antenna gain and Q, originally derived by Chu, in a different fashion. This interesting method expands the antenna pattern in spherical harmonics and relates the maximum gain to the number of harmonics used. This is then heuristically related to physical aperture size. For the synthesis problem, the transform of the Fourier series for the aperture field is matched point by point to the pattern transform, over a surface in the near field; this is similar to Woodward's method. Both the Michigan work (Goodrich) and the Syracuse University work (Harrington et al) were supported by subcontract from General Electric Company.

Although the subject of Fresnel diffraction dates to the nineteenth century, several interesting analytical techniques have appeared. Barrar and Wilcox [1958] used the Sommerfeld $1/r^n$ expansion with success in the Fresnel region. Hu [1957] has applied Fresnel approximations to the problem of coupling between circular aperture antennas with tapered illuminations. Hansen and Bailin [1959] have computed circular aperture near field data by a computer evaluation of a series derived from the exact field formulation. This formulation produces angular integrals independent of illumination and radial integrals independent of source attitude. Results are compared with various Fresnel formulas and side lobe behavior in the Fresnel region. The side lobe ratio may actually increase over the far field value in some regions, as the side lobes decay more rapidly than the main beam amplitude. How to calculate safe radiation regions (safe against irreversible tissue damage) in the Fresnel region of high power antennas has been shown by Bickmore and Hansen [1959]. On-axis power density and defocusing factors are given.

Antenna noise and breakdown. The advent of low noise preamplifiers such as parametric amplifiers and Masers has made the evaluation of antenna or radiation noise temperature essential. Hansen [1959] has made a survey of methods for finding the effective noise temperature of a microwave antenna. Hogg [1959] gives the effect of oxygen and water vapor absorption upon antenna temperature. De Grasse et al. [1959] report very careful measurements of antenna effective noise temperature at 5,600 Mc/s using a parabolic horn antenna with very low side and back lobes (see Friis, May, 1958). A zenith temperature as low as 18 °K has been achieved. At the other end of the frequency spectrum, the precipitation particle noise mechanism for dielectric covered antennas has been shown by Tanner [1957] to be an acquisition of charge upon impact by individual precipitation particles. This has resulted in the development of successful static reduction devices. The space age has necessitated a more careful study of antenna breakdown due to high power and high altitudes. Chown et al. [1959] report on the effects of breakdown upon VSWR, pulse shape, power, and pattern. Linder and Steele [1959] provide data for calculating breakdown for various antenna configurations as a function of frequency. An additional paper surveys the earlier state of the art [Ashwell et al., 1957].

Summary. Antennas for radio astronomy are advancing along two fronts: Mechanical design improvements allowing construction of larger single aperture dish-type antennas; and switching or data processing antenna systems wherein multiple antennas are used to obtain some performance parameters of a larger single aperture. The advance obtained by the Mill's Cross should be furthered by more sophisticated systems. These are discussed further in *Dynamic Antennas*. The quasi-optical fixed reflector scanning devices such as the parabolic torus should find use in the astronomy field.

In the array field, new techniques are needed for feeding and constructing millimeter wavelength arrays. Effort on nonconventional slot arrays such as the radial line annular slot arrays should be extended to other configurations.

The largest problems remaining in electronic scanning arrays are how to obtain the requisite phase shift, for arrays with all elements coupled together, and how to simplify the components in data processing arrays. The individually controlled element, of which the ferrite slot is a prototype, appears to offer most promise and should be broadly investigated. Application of solid state circuitry to data processing arrays, where various mixing or amplifying functions could take place at each element without severe space and weight penalties, represents another promising area.

Focusing of antennas is most easily done in data processing antennas, especially those of the time-processing type.

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5. Small Aperture Antennas

Low gain antennas for air and space vehicles. Antennas discussed in this section are primarily simple element types such as loops, dipoles, and slots. In general, boundary value problems, such as dipoles over reactive surfaces, are covered in the companion paper on Surface Waves. Ronold King and others at Cruft Lab. have investigated loop antennas carefully and have shown that the small loop contains small electric multipoles as well as the magnetic dipole and that the loop equations reduce to those for the folded dipole as the length approaches zero [King, 1959; Prasad, 1959]. Small antennas of both loop and dipole type have been treated by Wheeler [1959, 1958a]. For small coils or small loaded dipoles, the performance available is essentially dependent upon length and volume and independent of configuration within the two types. Oliner [1957] has obtained an improved formulation for series and shunt waveguide slots using a variational technique. Wall thickness is taken into account by a microwave network. Shape of the slot end is also considered. Radiation from the end of a waveguide loaded with ferrite has been attacked [Tyras, Held, 1958] in a manner similar to an earlier paper of Angelakos.

Radiation from many types of cylindrical structures including wedges, cylinders, half planes, and sheets has been covered in a book by Wait [1959a]. This work is an excellent compendium of the state of the art and describes in detail the mathematical techniques and solutions. Other papers include slots on spheres [Mushiake, Webster, 1957], spheroidal dipoles [Weeks, 1958; Flammer, 1957], and radial dipoles on a circular cylinder [Levis, 1959]. Wait and collaborators have studied slotted circular [Wait, 1957], elliptic [Wait, Mientka, 1959], and dielectric coated [Wait, Conda, 1959] cylinders. The circularly polarized element consisting of crossed slots on a waveguide broad wall has been investigated by Simmons [1957]. The way in which a curved and/or lossy surface effects an antenna pattern has been studied by Wait and Conda [1958], using a combination of residue series, Fock functions, and geometrical optics. An electric monopole exciting a finite cone has been studied by Adachi and Kouyoumjian [1959]. Cruzan [1959] and Weeks [1957] have treated the receiving loop antenna with a ferrite core. The important parameters for the receiving loop are the area, number of turns, and effective permeability [Wheeler, 1958b]. Polk [1959] has studied ferrite loaded biconical dipole antennas and shows, as predicted by Schelkunoff and Friis [1952], that in general, effective length is decreased by the addition of ferrite or dielectric loading except for high loss supergain conditions which are undesirable. Grimes [1958] reaches similar conclusions. A closely related subject is that of an antenna immersed in a lossy medium. Wait [1959b, 1958] shows that the field of a buried loop is essentially that of a loop on the surface plus an exponential attenuation with depth.

Air frame antennas using shunt or notch feeding have been put on a sound engineering basis by Tanner [1958]. This is a definitive paper and summarizes these types of antennas. An investigation similar to that by Infeld [1947] of a few years back on the input admittance singularity of a dipole antenna due to the delta function generator has appeared [Wu, King, 1959]. In this paper, as in Infeld, the admittance is separated into a gap capacity term and a bounded term.

In the larger realm of antenna systems, a paper by Turner [1959] summarizes several types of submarine communication antenna systems. Antenna multicoupler systems, so important in LF and HF ranges where antenna efficiencies are typically very low, have been extensively studied in a series of reports from Stanford Research Institute [Cline]. This series is an excellent summary of the state of the art in exciting HF airframe or satellite antennas.

Medium gain antennas. Klopfenstein [1957] and Woodward [1957] have carefully reinvestigated corner reflectors with various dipole and apex angles both from the sophisticated dyadic Green's function and from the image point of view. Cottony and Wilson [1958] present excellent design curves; other limited data also are available [Neff, Tillman, 1959]. Rhombic antennas of large size have also been investigated in a paper by Decker [1959]. which gives design for maximum gain. Design data for helical antennas for lengths up to 10λ has been augmented by Maclean and Kouyoumjian [1959]. They applied Sensiper's infinite helix solution and obtained results valid up to a length of 10λ. A new and very interesting antenna configuration is the trough waveguide invented by Rotman and Oliner [1958]. A continuous trough waveguide is suitable for end fire radiation and a periodic asymmetrical design covers a number of radiation directions including broadside [Rotman, Oliner, 1959]. Although the transverse resonance method is best applied to nonleaky structures, it has yielded good values for propagation characteristics in this case.

An excellent survey of the printed technique is given by McDonough et al. [1957]. They cover such different types as ladders, rhombics, cigars, and capacity-coupled collinear arrays. Another type of printed antenna is the sandwich wire antenna of Rotman and Karas [1957, 1959]. This is an array of undulating wire strips wherein each wire acts as a quasi-discrete leaky radiator.

Evaluation. A most important paper on the effect of satellite spin on radiation performance has been contributed by Bolljahn [1958]. He shows that when the satellite spin axis and the antenna axis are not alined, the ground-received signal (with a CW signal radiated) splits into three spectral components. The variation of these with the geometry of the configuration is derived.

The evaluation of aircraft and satellite antennas has always been difficult, especially the comparison of different types of antennas, since impedance, pattern, and gain performance vary widely among types and even among variations within each type. This evaluation problem has now been satisfactorily solved by a series of papers. Lucke [1958] uses a channel capacity formula to weight patterns and impedances over frequency and space. Moore [1958] compares the various rating schemes, the three most important of which are the average channel capacity method of Lucke, the radiation pattern distribution function of Ellis, and the radition pattern efficiency method of Granger. He shows that if carefully applied, all methods give essentially similar results, and he therefore recommends use of the simplest method, that is, the radiation pattern efficiency technique. This defines the quality in terms of the fraction of power radiated in useful directions. Impedance compensation or broadbanding of HF antennas has long been an art without suitable boundaries. However, the broadbanding potential of such antennas has now been bounded by two important papers. Vassiliadis and Tanner [1957] have approximated the impedance by a rational algebraic function from which the broadbanding capability is readily determined. Levis [1957] has used a different approach, that of relating the impedance bandwidth to the far field polarization characteristics and to the stored energy. These papers allow determination of the best broadbanding available so that a bound can be placed on attempts to realize this performance physically. A final paper gives numerical integration computer techniques for antenna pattern calculations [Allen, 1959].

Summary. As the effect of the counterpoise shape and size upon antenna radiation pattern and impedance becomes better understood, it should be possible to devise quasi-empirical synthesis techniques which would allow optimum advantage to be taken of this effect. The search for new and advantageous radiator configurations, e.g., the sandwich-wire, should continue.

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A Bibliography on Coherence Theory

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The inadequacy of the concepts of complete coherence and complete incoherence for the description of physically interesting phenomena was recognized by Verdet in 1869 when he showed that sunlight could produce fringes in a Young's interference experiment. After Verdet the development of coherence theory before 1940 was associated with the names of Von Laue [1907], Van Cittert [1935], and F. Zernike [1937]. Each of these investigators introduced his own, apparently different, formulation of the theory—each formulation being well suited to the problems considered by the particular investigator. In [1951], H. H. Hopkins again reformulated the theory in a manner which was particularly suited to the treatment of imaging problems. While each of these theories took account of intermediate states (partial coherence), they each suffered from one or more of the following restrictions: (1) They were applicable only to fields created by incoherent sources; (2) they were applicable only to *nearly* monochromatic fields; (3) they were formulated in terms of undefined complex functions.

These shortcomings were all removed in the new formulations of the theory of partial coherence introduced independently by Wolf [1955], and by Blanc-Lapierre and Dumontet [1955]. While these formulations are equivalent, it is much more convenient to work with the definitions introduced by Wolf. Working with the Wolf theory of partial coherence, Parrent [1958–1960] has extended the theory by finding several of the implications of the formulation and existence theorems for the basic functions of the theory and by showing how the approximate propagation laws of earlier theories are related to the solution of the wave equations that describe the propagation of partially coherent radiation. Using these theorems it was possible to formulate the imaging or mapping problem in a general and rigorous way for partially coherent illumination of arbitrary spectral width.

Thus, finally, the formulation and structure of a rigorous theory of partial coherence for scalar fields is complete enough to be considered as an available tool for the solution of problems involving statistical radiation. Part A of this bibliography provides a reasonably complete survey of the principal works on the subject.

The problem of discussing the behavior of, and formulating a calculus for, the description of vector fields is considerably more complex than the corresponding scalar problem. Consequently in this area very little has been accomplished by comparison. The general problem of discussing statistical vector fields consists in two essential concepts: Partial coherence (the correlation between the disturbance at two different points), and partial polarization (the correlation of the various components at the same point). Limiting our attention to a plane wave eliminates coherence problems and isolates partial polarization effects. Wolf has treated this class of problems at some length in the last few years, and recently Parrent and Roman have used the results of Wolf's work as a basis for constructing a matrix calculus for the study of partial polarization effects. Nonplane waves have not been extensively discussed as yet; however, Roman has succeeded in generalizing the Stokes parameters to a set applicable to nonplane waves. This is, of course, an important first step in the understanding of this field. Part B of this bibliography is an attempt to list the most important papers related to the description of statistical vector fields.

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A Bibliography of Automatic Antenna Data Processing

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Many source detection applications have required ever higher antenna resolving power to distinguish distant sources from adjacent sources. In the optical region, the limitation to the increase of resolution has been the fluctuations existing in the earth's atmosphere. In the radio region, on the other hand, the most immediately apparent limitation to high resolution has been largely an economic one, the high cost of materials, as well as the difficult problem of construction tolerance. Additionally, recent work by [Skinner, 1960] has brought to light certain limitations on the gain and resolving power of antennas used for the reception of randomly varying signals due to statistical fluctuations of the source distribution and/or of the intervening medium characteristics.

In both the optical and radio regions interferometric techniques of one sort or another have yielded greater resolution than obtainable with equal size dishes or mattress arrays. Conventional interferometry suffers from pattern ambiguity, but diverse methods of data processing can be used to overcome this ambiguity, and to optimize different aspects of the antenna systems performance.

In view of the fact that the resolving power of an interferometer is generally proportional to l/λ, where l is the separation of interferometer elements while λ is the wavelength of radiation, it would seem that one should either increase the separation (baseline) or the frequency or both. To increase the frequency without limit would be impractical, first because of the decrease in source intensity with wavelength, then because the construction of large antennas and sensitive receivers is more difficult as the wavelength decreases, and finally because one sometimes wishes to study the diameter of a source as a function of frequency. All of these reasons favor increasing the interferometer baseline.

Mills [1952] describes a radio transmission link as a means of increasing the baseline of the interferometer—and thus the resolution of this instrument—with phase preserved in the following manner. The received signal frequency at one element of the interferometer is converted to a radiofrequency which is transmitted along with the local oscillator frequency over the same path to a receiver located near the other element. This signal is then reconverted to the original frequency and combined with the signal from the other antenna, the latter signal having been delayed by an amount equivalent to the propagation time across the radio link. There is, however, a limitation on the length of this radio link, which is introduced by the effect of turbulence of the intervening medium on the phase stability of the transmitted signal. When converting the radio frequency signals at each antenna element of the interferometer to a low frequency, transmitting by a radio link one of these low-frequency signals as an amplitude modulation of a radiofrequency carrier, and cross-correlating the two low-frequency signals, it has been shown [Brown & Twiss, 1954; Brown et al., 1952] that several advantages arise. The relative phase of the two low-frequency signals is more easily preserved than that of the radio frequency signals, and it is equal to the latter in this particular arrangement. In view of this, the baseline of this interferometer can be made much larger, possibly indefinitely so by recording the interferometer element signals separately on magnetic tape and cross-correlating later. The system also happens to be less sensitive to ionospheric disturbances. One disadvantage is that the antenna yields information only about the amplitude distribution across the source. It is also relatively insensitive to weak sources inasmuch as the signal-to-noise ratio is proportional $(P_s/(P_r+P_c))^2$, whereas for the usual interferometer it is proportional to just $P_s/(P_r+P_c)$, where P_s=power in source signal, P_r=receiver noise power, P_c=cosmic noise power.

To improve the detection of weak "point" sources in the presence of much more intense extended sources or continuous background radiation, Ryle [1952] suggested the periodic introduction of a half wavelength of cable into one of the antenna lines of an interferometer. The interference pattern has an alternating component in addition to a steady component as a result of the alternately in-phase and out-of-phase relationship between the two antennas. Upon separating the alternating term from the steady one by means of a phase-sensitive detector, one can separate the background radiation from the "point" sources. Additionally, this system provides a means of more accurate determination of the position of radio sources in such a way as to be reasonably independent of rapid variations in the intensity of the radiation. The improvement of the ability to detect and localize weak signals by using correlators has been investigated by Faran and Hills [1952]. They have pointed out that in some instances signal-to-noise ratios can be improved in some interferometers, but by no more than 3 db, while in others a decrease in this ratio is seen, compared to a conventional antenna system. Any disadvantage here may be offset in part at least by the opportunity to use much higher gain recording instruments after the correlator in view of the fact that the amount of background noise does not contribute largely to the average output of the correlator. They also suggest the possibility of trading signal-processing time for physical antenna size.

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These systems previously described possess multiple principal lobes and are, hence, ambiguous when several sources are present. By considering interferometer elements whose patterns differ essentially from one another, unidirectional interferometer patterns can be obtained. Ryle suggested that a decrease in the solid angle of the principal lobe of the reception pattern could be obtained without necessarily increasing the total antenna area thus permitting an increase in the number of detectable sources. Mills and Little [1953] have emphasized that the number of discrete sources with intensities above the detectable threshold will normally greatly exceed the number which may be separately resolved, so that one may attempt to design antennas of increased resolving power but relatively low gain that may lack very little of the usefulness of conventional antennas, and cost a great deal less. They have introduced a system consisting of two linear arrays mutually perpendicular in the form of a cross, such that phase centers are coincident. When the technique of phase switching of the signal in one antenna channel is coupled with synchronous detection of the product of the fan-shaped patterns of the two antennas, a pencil-shaped single-lobed pattern is produced. Covington and Broten [1957] have investigated an interferometer similarly composed of two dissimilar linear antenna elements; however, these are arranged along the same axis end-to-end. The two elements, one a nonresonant slotted waveguide array, the other a two-element interferometer, are coupled by a rotary phase shifter, to produce upon synchronous detection of the alternating component in the radiation pattern a single-lobed fan-shaped beam with a twofold increase in resolving power in one plane over that of a uniform array of equal dimension. To produce a nonambiguous radiation pattern, also with an economy of the number of antenna elements, Band and Walsh [1959] have used two linear additive arrays of uniformly spaced nondirectional elements—the common spacing being different in each and greater than a wavelength—as inputs to a correlator. Nonambiguity was also achieved by them when they replaced one of the linear arrays by a closely spaced or continuous aperture antenna. These techniques have resulted in the use of fewer elements, as well as an improvement in signal-to-noise ratio over an equivalent additive array. For reasons of stability of the multiplication process, amplitude modulation is imposed on the radio frequency signal. The desired correlation signal is the output of an audio-filter tuned to the modulation frequency and following the multiplier.

Berman and Clay [1957] have considered non-uniformly-spaced omnidirectional detectors whose outputs are selectively multiplied together and time averaged according to a prescribed plan, such that a directional pattern results that is equivalent to that of a linear additive array of a larger number of elements. Here, too, the length of the multiplicative array often turns out to be half that of the equivalent one.

A comparison between the arrays of Faran and Hill and those of Berman and Clay has been made by Fakley [1959] for three applications: (a) The detection of a "point" source in a noisy background; (b) the resolution of two closely spaced point sources; and (c) the measurement of intensity distribution across an extended source. It was shown that the type of arrays described by Berman and Clay for four receiving elements, under idealized conditions, has no particular advantage over the other for the applications mentioned. It was also suggested that this conclusion could be extended to arrays consisting of more than four elements.

Drane [1959] has studied the coupling of a directional array with nonuniformly spaced omnidirectional elements after the fashion of Berman and Clay, but modified by the addition of continuously rotating phase shifters selectively used in conjunction with synchronous detection to yield nonambiguous radiation patterns. The suggested application has been to the tracking of moving targets which can be considered essentially "point" sources. Walsh and Band [1960] have also investigated such systems.

Time can be used as a degree of freedom supplementary to the three dimesions of space to achieve greater flexibility in the design of antennas. For example, it has been shown by Shanks and Bickmore [1959] that, in general, by periodic modulation of one or more of the antenna parameters (phase distribution, physical size, frequency, etc.) one obtains a temporally fluctuating radiation pattern. This pattern can be analyzed as an infinite sum of harmonics, and associated with each such frequency channel is a characteristic spatial distribution. They have applied such techniques to multipattern operation, simultaneous scanning [Shanks, 1959a], and sidelobe suppression [Shanks, 1959b].

Barber [1958] points out that on interpreting the "compound interferometer" of Covington and Broten as an array of essentially omnidirectional elementary detectors, two widely spaced ones forming the simple interferometer with several closely and uniformly spaced elements comprising the slotted waveguide, the receiving pattern can be considered the sum of all possible mean products of one element of the interferometer and one element of the long array. He suggests that one can also obtain the same members of the sum with several other configurations, all consisting essentially of two arrays in a line each having uniformly spaced elements of common spacing different from the other array of the configuration. The system with the fewest number of elements (for constant overall length) is that in which the number of detectors of one array differs from that of the other by at most unity. Covington and Broten [1958] have extended their system in just this fashion by adding two elements separated from each other and the extreme element of the simple interferometer by a distance equal to the length of the long array, i.e., by the separation of the interferometer elements. To ensure that all necessary signal products are obtained appropriate switches are used. It is to be noted that since the length of the long array remained the same, the resolving power was doubled, but the analytical properties of the radiation pattern remained unchanged. In the system discussed by Drane the overall length is extended not by producing two different but uniformly spaced arrays, but by using a uniformly and closely spaced array (to simulate the continuous array), another array whose interelement spacing increases in accordance with a geometric progression, as well as appropriately placed frequency shifters. Dausin, Niebuhr, and Nilsson [1959] while examining the problem of the reception of wideband signals have arrived at just such an "optimum" spacing on considering elemental arrays of variable spacing with the elements coupled by matched filters.

The work of Kock and Stone [1958] on the equivalence between dimensional properties of antennas and frequency content of signal in the production of a given response is in essential agreement with the results of Dausin, Niebuhr, and Nilsson for multielement arrays, wide-band sources, and with those of Covington and Broten, Drane, Walsh, and Band for complex interferometric arrays and monochromatic sources (artificially made multifrequency). They have shown that in a detection system antenna size and space complexity can be reduced for the detection of wide-band (continuous or discrete spectrum) signals by using a two-receiver cross-correlation antenna system. With such a system directional patterns equivalent to those characteristic of multielement, additive, narrow-band arrays are obtainable. Here, there exists the limitation imposed by the requirement that the antennas used in the interferometer complex be fairly broadband.

White, Ball, and Deckett [1959] have made a comprehensive study of nonlinear antennas of the various types considered above, comparing each one with linear antennas. They have found that the performance of any nonlinear antenna in the presence of continuous interference is inferior to a linear one in the same environment. Power gain and directivity of the nonlinear antenna are likewise generally inferior to those of the linear antenna. By artificially broadbanding the transmitted signal or confining the application to low-duty-cycle transmission these disadvantages may be made less significant. However, a nonlinear antenna of the space-coincidence type is described to provide a spatial selectivity not obtainable with a linear antenna.

We have been talking about situations in which it may be said that the data processing is done essentially automatically by the antenna system. Much work has been done and thoroughly discussed in the literature [Astia AD117067, 1957; Bracewell & Robert, 1954; Arsac, 1957] on the subject of the extraction of information about and reconstruction of the source intensity distribution from a knowledge of the information actually received by a conventional antenna, as well as of the properties (shape and aperture field distribution) peculiar to the antenna itself. Here, the term space-frequency—periodic spatial intensity distribution—is introduced. In both the optical and radio regions the receptor acts effectively as a low pass space frequency filter whose cutoff frequency is proportional to the physical extent of the receptor's aperture. This provides a distinct limitation to the extent to which the source characteristics may be reconstructed from the signal distribution actually observed.

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Progress During the Past Three Years In Surface and Leaky Wave Antennas

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This summary report begins where the previous URSI report on traveling wave antennas [Cottony and others, 1959] left off. The bibliography partially overlaps that in the previous reference: papers that were previously listed as reports, but have since appeared in the open literature, are listed here again with their journal references [Pease, 1958; Plummer, 1958; Hansen, 1957; Hougardy and Hansen, 1958; Plummer and Hansen, 1957; Friedman and Williams, 1958; Weeks, 1957; Goldstone and Oliner, 1959; Elliott, 1957; Kelley and Elliot, 1957; Hines and Upson, 1958].

Though both surface and leaky wave antennas belong to the general class of traveling wave radiators, they differ essentially in radiation mechanism, design principles, and performance characteristics. Surface waves are guided by the real or artificial dielectric structure along which they travel, and radiate only at discontinuities; usually there is just one of these, the termination. The total antenna pattern, which is endfire, is formed by superposition of terminal radiation and direct radiation from the feed. Beam shaping possibilities are limited. This type of radiator is nevertheless important whenever antenna height (as of a dish) must be traded for length. Leaky waves, by contrast, radiate continuously as they travel along the aperture, and very precise pattern control can be achieved. The beam is non-endfire and can be scanned over wide angles with negligible pattern deterioration.

1. Surface Wave Antennas

The *excitation* of surface waves, a problem that received considerable attention in the period before 1957 [Cottony and others, 1959; Friedman and Williams, 1958], was further examined. Wait [1957; 1958] gave a unified treatment of surface wave excitation by a dipole over diversely modified interfaces. While it had been known previously that efficiency of excitation depends on endfire directivity of the feed [Kay and Zucker, 1959], Brown now shows [1959] in a paper with interesting design implications that in the absence of supergaining, the efficiency of a source is limited to a maximum value that is a function of its physical size. In continuation of earlier work, Reynolds and Sigelman [1959] report that very clean sin ξ/ξ patterns are obtainable by using feeds that are distributed over the first third of the antenna length. Turning our attention to more specific structures, we find a precise analysis, using Wiener-Hopf techniques, of the launching of TM surface waves by a parallel plate waveguide (Angulo and Chang [1959a]). Duncan [1958] gives a Sommerfeld-type treatment of the excitation of dielectric rods; the mode he considers (lowest TM) is not that used in antenna applications, but this is the first time that the excitation problem on a rod has been tackled at all. In a paper of considerable practical interest, DuHamel and Duncan [1958] measure the efficiency with which diverse slot and wire feeds excite the HE_{11} mode on a rod.

The terminal discontinuity of a dielectric slab was examined in detail by Angulo [1957], who used variational techniques to find the terminal impedance and stationary phase methods for the radiation pattern. Angulo and Chang [1959b] calculated the terminal impedance of the lowest TM mode on a dielectric rod, Arbel [1959] analyzed the terminated dielectric disk, and Kay [1959] gave a detailed description using Wiener-Hopf methods, of discontinuities on reactive surfaces, including the terminal discontinuity. He calculated and plotted radiation fields and the surface wave reflection and transmission coefficients.

The radiation mechanism of surface wave antennas can be viewed in two ways: As the superposition of radiation from two quasi-point sources—the feed and the terminal discontinuity—or as the Fourier integral over the current distribution along the antenna structure. As one finds them discussed in the literature, these two approaches lead to pattern calculations and design recommendations that partly contradict each other. Zucker [1958] showed, for a simple case, what approximations are involved in deriving each from the rigorous Green's function formulation and indicated how the two approaches are reconciled by taking these approximations into account. Schlesinger and Vigants [1959] improved the conventional aperture integration approach and were able to predict the pattern of dielectric rods with higher accuracy than before. Kay [1960] examined the near field of Yagis experimentally and gave physical details that connect the two approaches.

Optimum design principles of surface wave antennas are still largely based on cut-and-try methods. Ehrenspeck and Poehler [1959] showed how the Hansen-Woodyard condition for optimum gain must be modified for surface wave antennas, and Ehrenspeck and Kearns [1957] used parasitic side rows to suppress the sidelobe level of a Yagi to 30 db. Bandwidths of 2:1 were achieved with polyrods by Parker and Anderson [1957]. Optimum design principles based on this and earlier work were collected for systematic presentation in the Handbook of Antenna Engineering [Zucker, 1960].

One approach to *pattern control* is to place radiating discontinuities at discrete intervals along a surface wave antenna, for example by spiking a polyrod with

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short pieces of wire, coupling being controlled by the depth and angle of insertion [Duncan and DuHamel, 1957]. A Goubau wire was similarly spiked [Scheibe, 1958]. A two-dimensional slot array excited by the surface wave on a dielectric image line, was shown by Cooper et al. [1958] to be capable of producing a broadside, endfire, or sidefire pattern, depending on the arrangement of the slots.

A second approach to pattern control consists in the use of variable impedance surfaces. Felsen [1957; 1959] gave the first rigorous solution to a problem of this type: he showed that on a surface with linearly increasing admittance (impedance) a TM (TE) surface wave propagates at the velocity of light with cylindrically spreading phase front, and without loss in total energy. This result is a key to the understanding of long tapered sections on surface wave antennas. Oliner and Hessel [1957] performed a detailed modal analysis of sinusoidally varying impedance sheets, showing that for periods shorter than about half a wavelength the surface supports a wave that is wholly trapped, while longer periods produce a leaky wave. An exact procedure for the design of an interface that supports a prescribed spectrum of waves (a "modulated" surface wave) has been given by Bolljahn [1959]. This important group of papers is the bridge between earlier work on surface wave modulation [Cottony and others, 1959], which did not concern itself with physical realizability, and the ultimate goal, which is pattern control—including the generation of non-endfire beams—with parasitically excited antenna structures.

A third approach to pattern control employs a distributed feed that is coupled to the antenna along its entire length. In continuation of earlier work Cottony and others [1959], Weeks [1957], and Giarola [1959] analyzed this problem in terms of coupled waveguide theory and obtained experimental results on a 40 λ long Yagi coupled to a two-wire line. It is, however, doubtful whether practically useful means for independently controlling phase and amplitude along the structure can ever be found in this way.

Turning now to more specific structures, we find that the *dielectric* rod continues receiving attention. Kornhauser [1959] gives general results on the modal characteristics of rods of very general cross sections, and Mickey and Chadwick [1958] worked with rods of dielectric constants up to 165, which are very much thinner than polyrods (though just as long, for equal pattern performance). Reggia, Spencer, et al. [1957] excited arrays of ferrimagnetic rods inserted in a cavity or the narrow wall of a waveguide, and show diverse arrangements for rapid switching, turning the plane of polarization, lobing, etc. Work on broadband polyrods [Parker and Anderson, 1957] has already been mentioned.

The relation between the phase velocity of a surface wave on a Yagi and the height, diameter, and spacing of the elements was found experimentally in [Ehrenspeck and Poehler, 1959], supplementary data being furnished by Frost [1957] and Spector [1958]. Sengupta [1959], using a loaded transmission line model, and Serracchioli and Levis [1959], using an approximate coupled element approach, calculated these relations theoretically; their results agree quite well with the experimental data. Very long Yagis are treated in Kay, [1960], and twisted Yagis (for circular polarization) in Reynolds and Sigelman, [1959].

A number of *new structures* were examined. Hyneman and Hougardy [1958] invented an array of contiguous below-cutoff waveguides with closely-spaced, nonresonant, transverse slots. Sengupta [1958] discussed a zigzag antenna, and Querido [1958] gave an approximate treatment of the fakir's bed antenna (array of pins).

Area sources permit scanning in azimuth. Goldstone and Oliner [1959b] pointed out a general relation for surface waves that travel obliquely across a corrugated surface, supplementing earlier work [Hougardy and Hansen, 1958] on the scanning of such an antenna. Walter [1957] obtains 360° scan from a dielectric sheet Luneberg lens whose elevation pattern is shaped by the surface wave.

Volume arrays of endfire line sources have diverse applications. Ehrenspeck and Kearns [1959] used a Yagi-Adcock arrangement for satellite tracking. Kamen and Bogner [1959] are interested in the advantages, under certain circumstances, of arrays of cigar or Yagi antennas over dishes and have built several satellite tracking and communication arrays of this type. An interesting new structure, called the *backfire* antenna [Ehrenspeck, 1960], looks like a Yagi with a large flat reflector at the end opposite the feed, and produces gains up to 6 db above that of an equal-length surface wave line source.

The influence of a *finite ground plane* on the pattern of an endfire surface-wave antenna has been considered by Wait and Conda. The model they used was a conducting half-plane which itself could be located in the interface of an imperfectly conducting half-space [Wait and Conda, 1958]. The main effect of the truncation is to tilt the beam upward and to degrade the side lobe level.

Another related problem, treated by Cullen [1960], is the excitation of a corrugated cylinder by an axial slot. He showed for certain combinations of cylinder dimensions and surface impedance that a very pure cos $m\phi$ pattern may be produced. This work has been extended by Wait and Conda [1960] who also treated elliptic cylinders with a nonuniform distribution of surface impedance.

2. Leaky Wave Antennas

Earlier work by Marcuvitz [IRE Trans., 1959] and Barone [IRE Trans., 1959] has clarified the manner in which leaky waves, in spite of their nonspectral nature, enter in the description of the total field of a source above an interface. Barone and Hessel [1958] continue this work for the case of an electric line source over a dielectric slab.

To calculate the parameters of leaky waves, Goldstone and Oliner [1959a; 1958] introduce a perturbation procedure that is very much simpler than solving directly the complicated transcendental equations that arise in these problems.

Attention has focused principally on four groups of leaky wave structures. The *asymmetric trough waveguide* was analyzed by Rotman and Oliner [1959], and applications were made by Rotman and Naumann [1958] that include positioning the beam in the broadside region by periodically reversing the deep and shallow side of the trough. Unlike conventional slot antennas, the periodically asymmetric trough guide can be scanned through broadside.

The *transverse wire grid* antenna developed by Honey [1959] has excellent frequency scanning properties (no beam deterioration from 30° to 70° off endfire), and allows precise pattern control. It has been used as an X-band area source [Honey, 1959], as a millimeter waveline source [Honey, 1960], as an area source curved on a cylindrical surface [Shimizu and Honey, 1960], and as a flat center-fed disk [Hill and Held, 1958].

Jones and Shimizu [1959] designed an area array of thick *transverse slots* which, in contrast to the wire grid, is vertically polarized. Hyneman [1959] gave a careful treatment of closely-spaced transverse slots in thin-walled rectangular waveguide. Earlier work on the "serrated" waveguide [Elliot, 1957; Kelly and Elliot, 1957] had treated the thick-walled case.

The longslot in waveguide, which had received much attention in Cottony and others [1959], was examined by Nishida [1959a] for the case when it is covered by a thin dielectric sheet. Nishida also analyzed the effect on leaky wave phase velocity and attenuation of coupling two parallel long slots in a plane [Nishida, 1959b] or on a cylinder [Nishida, 1959c].

As in the case of surface wave antennas, feeds can be designed for leaky waves that couple along the entire length of structure. The advantage in this instance is that the initial section of the leaky wave antenna would not have to carry as much power as it must when fed from one end. Barkson [1957], with this goal in mind though confining himself to a shielded case, analyzed the coupling of rectangular waveguides through a common broad wall with non-resonant transverse slots. MacPhie [1959] examined a radiating coupled structure, and by varying the coupling region achieved mechanical beam scanning.

3. Assessment and Predictions

Although the launching of surface waves, and their radiation from the terminal discontinuity, have each been separately analyzed in considerable detail, the combined and much more difficult problem of a source exciting an impedance structure of finite length has not yet been tackled. This ought to be done.

Attempts will probably be made to place the optimum design of surface wave antennas on a firm theoretical basis. Now that rigorous results are available on tapered impedance surfaces, for example, there is hope that an explanation can be found for the cut-and-dry rule that, for maximum gain, the taper should be short, for minimum sidelobes longer, and for wide bandwidth as long as the antenna itself.

Antenna structures that combine broadside aperture and endfire line source features (such as the backfire antenna) will receive attention. An effort should be made to synthesize artificial or natural dielectrics with more broadband dispersion characteristics than those of present structures.

Variable impedance surfaces will be used in attempts to diversify the pattern potentialities of surface wave antennas. Structures that permit independent control of amplitude and phase along the aperture are especially needed if modulation techniques are to become a practical reality.

The principal item of unfinished business in the theory of leaky wave antennas is the solution of a source problem over a complex impedance interface, on which the leaky wave—unlike in the case of the dielectric slab previously considered—is the dominant part of the total field.

An interesting problem that could be examined is the synthesis of complex impedance interfaces whose dispersion is such as to result in some prescribed variation of scan angle with frequency. Alternatively, the dispersion could perhaps be controlled (ferroelectrically or mechanically) to allow programed scanning. Scanning through broadside with periodically asymmetric structures will no doubt be fully exploited.

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