Use of Logarithmic Frequency Spacing in Ionogram Analysis

G. A. M. King

(April 7, 1960)

The use of logarithmic frequency spacing brings several advantages to the reduction of ionograms to electron density profiles. Among them is the fact that, when computing factors for the analysis, one need not determine the group refractive index. Formulas involving only the phase refractive index are presented; for the ordinary component one exact and one approximate formula are given, while for the extraordinary component there is an approximate formula valid over a wide range of geomagnetic latitudes. There is a brief discussion of quasi-longitudinal approximations to the extraordinary phase refractive index.

1. Introduction

In his important paper on obtaining electron density profiles, Budden [1] used for illustration a linear sampling of the frequency scale of the ionogram. He also mentioned the possibility of other samplings, including logarithmic. At about the same time, King [2] presented an illustration based on logarithmic sampling, and later [3] showed that this spacing simplifies inclusion of the earth's magnetic field in the analysis, by the use of an accurate approximation.

The methods of real height analysis using summation of spaced ordinates due to Kelso [4], Shinn (described by Thomas [5]), and Schmerling [6], are much easier to apply if the ionograms have a logarithmic frequency scale, for then overlays can be used. This derives from the fact that the properties of the propagation equation depend mainly on \( f_N/f \), the ratio of the plasma frequency to the exploring frequency, and to a much less extent on the absolute frequencies (determined by \( f_H \), where \( f_H \) is the gyro-frequency. As is usual in treating this subject, the effects of electron collisions are neglected.)

The purpose of this note is to inquire more closely into the advantages of logarithmic spacing and to present an approximation using it for analysis of the extraordinary ionogram trace.

2. General Equations

The basic equation relating the virtual height, \( h' \), and the real height, \( h \), at the reflection point (assuming geometrical optics) is, [2]

\[
h' = \int_0^{h'} \mu' dh,
\]

where \( \mu' \) is the group refractive index. This can be rewritten

\[
h' = \int_{\phi_0}^{\Phi} \mu' \frac{dh}{d\Phi} d\Phi,
\]

where \( \Phi \) is any single valued function of the electron density. The integration can now be divided into a series of steps within which \( dh/d\Phi \) is varying sufficiently slowly to be taken out of the integral sign.

\[
h_m = \sum_m \left( \frac{\Delta h}{\Delta \phi} \right)_m \int_{\Delta \phi_m}^\phi \mu' d\phi
\]

The integral \( \int \mu' d\Phi \), which we shall denote by \( F_m \), defines constant factors used in the analysis of the ionograms to obtain \( h \). Once the table of factors is prepared, the reduction to real heights is simply a matter of solving a set of simultaneous equations in \( (\Delta h/\Delta \phi)_m \).

If now \( \Phi \) is identified with the logarithm of \( f_N \), a simplification results. (The natural logarithm, \( \ln \), will be discussed here to simplify the presentation, although common logarithms may be used in practice.)

For, at fixed \( f_N \),

\[
\mu' = \left[ \frac{\partial (\ln f)}{\partial f} \right]_{f_N}
\]

\[
= \mu + \left[ \frac{\partial \mu}{\partial \ln f} \right]_{f_N}
\]

where \( \mu \) is the phase refractive index; it is a function of both \( f \) and \( f_N \).

Using the theorem

\[
\left[ \frac{\partial \mu}{\partial \ln f} \right]_{f_N} = \left[ \frac{\partial \mu}{\partial \ln f} \right]_{f_N} + \left[ \frac{\partial \mu}{\partial \ln f} \right]_{f_N}
\]

where \( \mu' \) is the group refractive index. This can be rewritten

\[
h' = \int_{\phi_0}^{\Phi} \mu' \frac{dh}{d\Phi} d\Phi,
\]

where \( \Phi \) is any single valued function of the electron density. The integration can now be divided into a series of steps within which \( dh/d\Phi \) is varying sufficiently slowly to be taken out of the integral sign.

\[
h_m = \sum_m \left( \frac{\Delta h}{\Delta \phi} \right)_m \int_{\Delta \phi_m}^\phi \mu' d\phi
\]

The integral \( \int \mu' d\Phi \), which we shall denote by \( F_m \), defines constant factors used in the analysis of the ionograms to obtain \( h \). Once the table of factors is prepared, the reduction to real heights is simply a matter of solving a set of simultaneous equations in \( (\Delta h/\Delta \phi)_m \).

If now \( \Phi \) is identified with the logarithm of \( f_N \), a simplification results. (The natural logarithm, \( \ln \), will be discussed here to simplify the presentation, although common logarithms may be used in practice.)

For, at fixed \( f_N \),

\[
\mu' = \left[ \frac{\partial (\ln f)}{\partial f} \right]_{f_N}
\]

\[
= \mu + \left[ \frac{\partial \mu}{\partial \ln f} \right]_{f_N}
\]

where \( \mu \) is the phase refractive index; it is a function of both \( f \) and \( f_N \).

Using the theorem

\[
\left[ \frac{\partial \mu}{\partial \ln f} \right]_{f_N} = \left[ \frac{\partial \mu}{\partial \ln f} \right]_{f_N} + \left[ \frac{\partial \mu}{\partial \ln f} \right]_{f_N}
\]

where \( \mu' \) is the group refractive index. This can be rewritten

\[
h' = \int_{\phi_0}^{\Phi} \mu' \frac{dh}{d\Phi} d\Phi,
\]

where \( \Phi \) is any single valued function of the electron density. The integration can now be divided into a series of steps within which \( dh/d\Phi \) is varying sufficiently slowly to be taken out of the integral sign.

\[
h_m = \sum_m \left( \frac{\Delta h}{\Delta \phi} \right)_m \int_{\Delta \phi_m}^\phi \mu' d\phi
\]

The integral \( \int \mu' d\Phi \), which we shall denote by \( F_m \), defines constant factors used in the analysis of the ionograms to obtain \( h \). Once the table of factors is prepared, the reduction to real heights is simply a matter of solving a set of simultaneous equations in \( (\Delta h/\Delta \phi)_m \).

If now \( \Phi \) is identified with the logarithm of \( f_N \), a simplification results. (The natural logarithm, \( \ln \), will be discussed here to simplify the presentation, although common logarithms may be used in practice.)

For, at fixed \( f_N \),

\[
\mu' = \left[ \frac{\partial (\ln f)}{\partial f} \right]_{f_N}
\]

\[
= \mu + \left[ \frac{\partial \mu}{\partial \ln f} \right]_{f_N}
\]

where \( \mu \) is the phase refractive index; it is a function of both \( f \) and \( f_N \).

Using the theorem

\[
\left[ \frac{\partial \mu}{\partial \ln f} \right]_{f_N} = \left[ \frac{\partial \mu}{\partial \ln f} \right]_{f_N} + \left[ \frac{\partial \mu}{\partial \ln f} \right]_{f_N}
\]
relating the partial differential at fixed $f_N/f$ to partial differentials at fixed $f_N$ and at fixed $f$, we get
\[ \mu' = \mu - \left[ \frac{\partial \mu}{\partial \ln f_N} \right]_f + \left[ \frac{\partial \mu}{\partial \ln f} \right]_{f_N/f}. \]  
(6)

The factors used in the ionogram analysis then become
\[ \int_{\Delta \Phi} \mu' d\Phi = \int_{\Delta \ln f_N} \mu' d\ln f_N = \bar{\mu} \Delta \ln f_N - [\Delta \mu]_f + \left[ \frac{\partial \mu}{\partial \ln f} \right]_{f_N/f} \Delta \ln f_N, \]  
(7)

where the "bars" denote mean values of the quantities over the interval.

All terms of eq (7) can be evaluated from a table of $11$, so that one can avoid the tedious calculation of the group refractive index. In a recent paper, Titheridge [7] has given an expression with the same advantage.

3. Ordinary Component

While one would normally use the complete eq (7), it is worth noting that, for routine analysis of the ordinary component, the last term can be dropped [3]. Experience shows that the heights deduced from the ionograms are not seriously affected.

The assumption here is
\[ \left[ \frac{\partial \mu}{\partial \ln f} \right]_{f_N/f} \approx 0. \]  
(8)

4. Extraordinary Component

While eq (7) applies formally to either the ordinary or the extraordinary component its use with the latter is made difficult by the rapid change of the last term near the gyrofrequency. By making the transformation
\[ \xi^2 = f^2(1 - y), \]  
(9)

where $y = f_N/f$, one can adopt an assumption similar to eq (8),
\[ \left[ \frac{\partial \mu}{\partial \ln \xi} \right]_{f_N/\xi} \approx 0, \]  
(10)

and so simplify the computations.

The approximation of eq (10) will be considered in the next section where it will be shown to hold extremely well down to geomagnetic latitudes as low as $30^\circ$ (where the propagation angle, $\theta$, between the earth's magnetic field and the vertical is as large as $50^\circ$).

When eq (7) is put in terms of $\xi$ and eq (10) is applied, one obtains
\[ F_{xm} = \int_{\Delta \ln f_N} \mu' d\ln f_N = \bar{\mu} \Delta \ln f_N - \frac{d \ln \xi}{d \ln f} [\Delta \mu]_f, \]  
(11)

where the subscript $x$ in $F_{xm}$ denotes the extraordinary component.

As constant $\xi$ implies constant $f$, and
\[ \frac{d \ln \xi}{d \ln f} = \frac{1 - y/2}{1 - y}, \]  
(12)

the factors for analysis of the extraordinary component become
\[ F_{xm} = \bar{\mu} \Delta \ln f_N \frac{1 - y/2}{1 - y} [\Delta \mu]_f. \]  
(13)

5. Phase Refractive Index for the Extraordinary Mode

In order to justify use of the approximation (10), a few remarks on quasi-longitudinal approximations to the phase refractive index of the extraordinary mode are appropriate.

Common approximations are [8],
\[ \frac{1}{1 - \mu^2} \approx \frac{f^2}{f_N^2} (1 - y \cos \theta) \]  
(14)

and [9],
\[ \frac{1}{1 - \mu^2} \approx \frac{f^2}{f_N^2} (1 - y) = \xi^2/f_N^2. \]  
(15)

The exact expression is
\[ \frac{1}{1 - \mu^2} = \frac{f^2}{f_N^2} \left(1 - y \cos \theta \cot \frac{\psi}{2}\right), \]  
(16)

where
\[ \tan \psi = \frac{2(f^2 - f_N^2) \cos \theta}{f f_N \sin^2 \theta}. \]  
(17)

Equations (14), (15), and (16) can be compared as follows: For the extraordinary trace, the least value of $\tan \psi$ occurs at the reflection point, where it is $2 \cos \theta/\sin^2 \theta$, so that the greatest value of $\cot \psi/2$ is $1/\cos \theta$.

That is to say,
\[ 1 < \cot \frac{\psi}{2} < \frac{1}{\cos \theta}. \]  
(18)

Therefore the exact value for $1/1 - \mu^2$ (16) always lies between those given by the two approximate expressions (14) and (15), except at the reflection point, where (15) is exact.

Now if we let $A$ represent the ratio of (16) to (14), and $B$ the ratio of (16) to (15), i.e.,
\[ A = \frac{1 - y \cos \theta \cot \frac{\psi}{2}}{1 - y/2}, \]  
(19)

and
\[ B = \frac{1 - y \cos \theta \cot \frac{\psi}{2}}{1 - y}. \]  
(20)
the following table compares the ratios for $y = \frac{1}{2}$ and various values of $(f_N/f)^2$, at different geomagnetic latitudes.

<table>
<thead>
<tr>
<th>Latitude</th>
<th>$\frac{(f_N/f)^2}{\mu}$</th>
<th>0.5</th>
<th>0.4</th>
<th>0.3</th>
<th>0.2</th>
<th>0.1</th>
<th>0.0</th>
</tr>
</thead>
<tbody>
<tr>
<td>14°</td>
<td>A</td>
<td>0.644</td>
<td>0.714</td>
<td>0.762</td>
<td>0.797</td>
<td>0.823*</td>
<td>0.844*</td>
</tr>
<tr>
<td></td>
<td>B</td>
<td>1.000</td>
<td>1.109</td>
<td>1.184</td>
<td>1.238</td>
<td>1.278</td>
<td>1.310</td>
</tr>
<tr>
<td>23°</td>
<td>A</td>
<td>0.743</td>
<td>0.792</td>
<td>0.825</td>
<td>0.850</td>
<td>0.856*</td>
<td>0.883*</td>
</tr>
<tr>
<td></td>
<td>B</td>
<td>1.060</td>
<td>1.065</td>
<td>1.110</td>
<td>1.143</td>
<td>1.168</td>
<td>1.187</td>
</tr>
<tr>
<td>30°</td>
<td>A</td>
<td>0.804</td>
<td>0.840</td>
<td>0.864</td>
<td>0.883</td>
<td>0.897*</td>
<td>0.908*</td>
</tr>
<tr>
<td></td>
<td>B</td>
<td>1.060</td>
<td>1.045</td>
<td>1.065</td>
<td>1.088</td>
<td>1.116</td>
<td>1.129</td>
</tr>
<tr>
<td>45°</td>
<td>A</td>
<td>0.904</td>
<td>0.921</td>
<td>0.933</td>
<td>0.941</td>
<td>0.948</td>
<td>0.954</td>
</tr>
<tr>
<td></td>
<td>B</td>
<td>1.160</td>
<td>1.078</td>
<td>1.081</td>
<td>1.091</td>
<td>1.098</td>
<td>1.095</td>
</tr>
<tr>
<td>60°</td>
<td>A</td>
<td>0.992</td>
<td>0.999</td>
<td>0.973</td>
<td>0.957</td>
<td>0.979</td>
<td>0.982*</td>
</tr>
<tr>
<td></td>
<td>B</td>
<td>1.000</td>
<td>1.067</td>
<td>1.011</td>
<td>1.015</td>
<td>1.018</td>
<td>1.020</td>
</tr>
</tbody>
</table>

Only in those positions of the table marked by an asterisk is $A$ closer to unity than $B$. Clearly, then, the approximation (15) is the better, especially near the reflection point $(f_N/f)^2 = 1 - y$ where the need for accuracy is greatest. Accepting eq (15), eq (10) follows.

Writing the exact expression, using eqs (16), (20), and (9), as

$$\frac{1}{1 - \mu} = \frac{\xi^2}{f_N^2} B, \quad (21)$$

it can be shown that, because of the slow rate of change of $B$ with $(f_N/f)^2$ (and hence with $\xi^2(f_N)^2$), eq (10) is applicable to ionogram analysis over a wider range of latitudes than eq (15). Rydbeck [9] used a relation equivalent to eq (15) in a method of ionogram analysis for $\theta < 20^\circ$ (latitudes greater than $54^\circ$), where it certainly holds within the accuracies to which the ionograms can be read. Use of eq (10) extends the range of application down to geomagnetic latitude $30^\circ$.

6. A Numerical Example

The following tables illustrate the computation of the factors $F_{zm}$ for analyzing the extraordinary ionogram trace.

The first table gives $\mu$ as a function of $\log_{10} f_N$ and $\log_{10} \xi$. This was calculated using eq (16).

<table>
<thead>
<tr>
<th>$f_M$/sec</th>
<th>$f_N = 1,4739$ Me/s</th>
<th>$\theta = 21^\circ 53'$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\log \xi$</td>
<td>$\log f_N$</td>
<td></td>
</tr>
<tr>
<td>2.95</td>
<td>0.32</td>
<td>0.0000</td>
</tr>
<tr>
<td>2.75</td>
<td>0.25</td>
<td>0.0000</td>
</tr>
<tr>
<td>2.63</td>
<td>0.21</td>
<td>0.0000</td>
</tr>
<tr>
<td>2.49</td>
<td>0.20</td>
<td>0.0000</td>
</tr>
<tr>
<td>2.36</td>
<td>0.16</td>
<td>0.0000</td>
</tr>
<tr>
<td>2.25</td>
<td>0.12</td>
<td>0.0000</td>
</tr>
<tr>
<td>2.15</td>
<td>0.08</td>
<td>0.0000</td>
</tr>
<tr>
<td>2.06</td>
<td>0.04</td>
<td>0.0000</td>
</tr>
<tr>
<td>1.98</td>
<td>0.00</td>
<td>0.0000</td>
</tr>
</tbody>
</table>

Lines parallel to the diagonal in the table define values of $\mu$ for constant values of the ratio $f_N/\xi$. Differences between the values along such a line, therefore, give $\partial \mu/\partial \log f_N/\xi$; and as this is very small compared with $\partial \mu/\partial \log \xi$, the assumption made in section 4 (eq (10)) is quite satisfactory.

From this table one can obtain the components $\mu \Delta \log f_N$ and $M [(1 - y/2)/(1 - y)] [\Delta \mu]$, of the factors $F_{zm}$ (eq (13)). Here, $M = 0.4343$ is the conversion factor to change from natural logarithms to common logarithms. Thus for $\log \xi = 0.24$, we obtain the factor for the interval $\Delta \log f_N$ $(0.16 - 0.12)$ by adding $\mu \Delta \log f_N (0.0246)$ and $- M \frac{1 - y/2}{1 - y} (0.0680)$ giving a value of 0.0926.

The table of $F_{zm} \times 10^4$ is then,

<table>
<thead>
<tr>
<th>$\log \xi$</th>
<th>Interval of $f_N$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.28</td>
<td>0.24</td>
</tr>
<tr>
<td>0.20</td>
<td>0.16</td>
</tr>
<tr>
<td>0.08</td>
<td>0.04</td>
</tr>
<tr>
<td>0.12</td>
<td>0.08</td>
</tr>
<tr>
<td>0.04</td>
<td>0.00</td>
</tr>
<tr>
<td>0.08</td>
<td>0.00</td>
</tr>
<tr>
<td>0.12</td>
<td>0.00</td>
</tr>
<tr>
<td>0.16</td>
<td>0.00</td>
</tr>
<tr>
<td>0.20</td>
<td>0.00</td>
</tr>
<tr>
<td>0.24</td>
<td>0.00</td>
</tr>
<tr>
<td>0.32</td>
<td>0.00</td>
</tr>
</tbody>
</table>

This example is taken from part of a table prepared for testing methods of ionogram analysis which use both the ordinary and extraordinary traces to give information on the unobserved parts of the ionosphere [10], [11], [12].

7. Conclusions

The use of logarithmic frequency spacing allows easy computation of factors for the real height analysis of ionograms, using tables of phase refractive index; the group refractive index need not be calculated.

For the ordinary wave component, there is an exact formula for the factors eq (7), and an approximate formula good enough for most work. The formula for the extraordinary wave component eq (13) contains an approximation, but it can be used with negligible error for analysis of ionograms from any but the lowest latitudes.

In the justification of the extraordinary wave approximation it was shown that the simplest quasi-longitudinal approximation to the phase refractive index eq (15) is better than one in common use eq (14).

The work described in this paper was partly supported by the International Geophysical Year program of the National Academy of Sciences.
8. References