Impedance Characteristics of a Uniform Current Loop
Having a Spherical Core

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The radiation impedance is derived by the electromotive force method in a convenient form as the sum of the self-radiation impedance of a loop in the free space and an additional term due to the reaction between the loop and the sphere which is proportional to the well-known expansion coefficient of a magnetic-type scattered wave from a sphere in an incident plane wave. The first antiresonance frequency has been given in the form of a universal curve for a very small uniform current loop with core of an arbitrary composition of \( \mu_r \) and \( \varepsilon_r \), subject to the condition that the refraction coefficient \( N_\varepsilon=\sqrt{\mu_r\varepsilon_r} \) is extremely large. Some numerical calculations show that high-\( \mu_r \) core is desirable for a comparatively lower frequency region, and high-\( \varepsilon_r \) core is rather desirable in an antiresonance region.

1. Introduction

By use of a sufficiently high refractive index, large scattering or absorption cross sections may be obtained from spheres small in terms of wavelength. For example, it has been shown that the echoing area can be increased to approximately three-quarters of a square wavelength for resonance of the electric or magnetic dipole type, and the required index of refraction can be given as a function of sphere size.

In a small loop antenna the ohmic resistance normally exceeds the radiation resistance. An increase in radiation resistance is highly desirable to improve the radiation efficiency. It can be expected that a high induced voltage, and therefore a high-radiation resistance, can be obtained by encircling a small resonant sphere by such a loop. Of course, it is desired that the radiation reactance be as low as possible. The ratio of the radiation resistance to the radiation reactance is also an important criterion.

Very recently, the most general theoretical analysis for a thin loop with a spherical core has been given by Herman [1] and Cruzan [2]. In these analyses, however, it is not easy to derive general relationships between the input impedance and the medium of the spherical core because of the difficulties of the numerical computations.

The purpose of the present paper is to correlate the medium constants of the core with the impedance characteristics under the assumption of a uniform current distribution. In order to derive the radiation impedance, a conventional emf method is applied to the electromagnetic field solution of a uniform loop current in the presence of a sphere which has been treated independently by Tai [3] and Wait [4]. The radiation impedance is given as the sum of the self-radiation impedance of the air loop and an additional term due to the reaction between the loop and the sphere which is proportional to the well-known expansion coefficient of the magnetic-type wave scattered from a sphere in an incident plane wave.

2. Impedance Characteristics of a Uniform Current Loop Having a Spherical Core

The general solution for the electromagnetic field due to a uniform current loop in the presence of a sphere has been given by C. T. Tai. Geometrical configurations are illustrated in figure 1. The radii of the sphere and the loop are the same and are indicated by \( a \); therefore, the wire of the loop is partially immersed into the core. Wavenumbers of the free space and the medium of the sphere are indicated by \( k_0 \) and by \( k=\sqrt{\mu_0\varepsilon_0N} \) and \( k=\sqrt{\mu_\varepsilon\varepsilon_\varepsilon(N)} \), respectively. As the time dependence, \( e^{j\omega t} \) is used. The primary fields due to the uniform loop current, \( I \), are expressed by the following equations:

\[ P(r, \theta, \phi) \]

\[ k_0 = \omega \sqrt{\mu_0 \varepsilon_0} \]

\[ k = \sqrt{\mu_\varepsilon \varepsilon_\varepsilon(N)} \]

\[ \text{Spherical Core} \]

\[ \text{Uniform Current Loop} \]

\[ \text{Figure 1. Uniform current loop having a spherical core.} \]
\[
E' = \sum_{n=1}^{\infty} C_n \bar{M}_e^{(1)}(k_0) \quad (r \leq a), \quad (1)
\]
\[
H' = -\frac{k_0}{j\omega\mu_0} \sum_{n=1}^{\infty} C_n \bar{N}_e^{(1)}(k_0) \quad (r \geq a),
\]
and
\[
E'' = \sum_{n=1}^{\infty} C_n \frac{j_n(\alpha)}{h_n^{(2)}(\alpha)} \bar{M}_e^{(4)}(k_0) \quad (r \geq a), \quad (2)
\]
\[
H'' = -\frac{k_0}{j\omega\mu_0} \sum_{n=1}^{\infty} C_n \frac{j_n(\alpha)}{h_n^{(2)}(\alpha)} \bar{N}_e^{(4)}(k_0) \quad (r \leq a).
\]

The vector wave functions \( \bar{M}_e \) and \( \bar{N}_e \) are the same with \( \bar{m}_e \) and \( \bar{n}_e \) defined in the text by Stratton [5]. The fields reflected to the external region and the field transmitted into the sphere are, respectively, given by

\[
E'' = \sum_{n=1}^{\infty} C_n R_n \bar{M}_e^{(4)}(k_0) \quad (r \leq a)
\]
\[
H'' = -\frac{k_0}{j\omega\mu_0} \sum_{n=1}^{\infty} C_n R_n \bar{N}_e^{(4)}(k_0) \quad (r \geq a).
\]

The vector wave functions \( \bar{M}_e \) and \( \bar{N}_e \) are defined in the text by Stratton [5]. The fields reflected to the external region and the field transmitted into the sphere are, respectively, given by

\[
E'' = \sum_{n=1}^{\infty} C_n T_n \bar{M}_e^{(4)}(Nk_0) \quad (r \leq a)
\]
\[
H'' = \frac{k_0}{j\omega\mu_0} \sum_{n=1}^{\infty} C_n T_n \bar{N}_e^{(4)}(Nk_0) \quad (r \geq a).
\]

where

\[
R_n = \frac{\nu_0 j_n(\alpha) [N\alpha j_n(N\alpha)]'}{\nu_0 j_n(N\alpha) [N\alpha j_n(N\alpha)]'} - \frac{\nu_0 j_n(\alpha) [\alpha j_n(\alpha)]'}{\nu_0 j_n(\alpha) [\alpha j_n(\alpha)]'}
\]
\[
T_n = \frac{j_n(\alpha) [h_n^{(2)}(\alpha)]'}{\nu_0 j_n(\alpha) [h_n^{(2)}(\alpha)]'} - \frac{\nu_0 j_n(\alpha) [\alpha j_n(\alpha)]'}{\nu_0 j_n(\alpha) [\alpha j_n(\alpha)]'}
\]

It is very interesting to note that the coefficient \( R_n \) is exactly the same as the coefficient associated with the magnetic-type wave scattered by a sphere in an incident plane wave. The coefficients \( R_n \) and \( T_n \) express the effect of the spherical core on the performance of the loop antenna. Neither the electric field nor the magnetic field (the surface current) are uniform around the wire cross section. If, however, the wire is sufficiently thin, it is reasonable to assume the electric field at the point \( Q \) (on the surface of the sphere, and at a distance \( b \), the radius of the wire, from the current filament which is located at the equator of the sphere) as the average value of the electric field around the wire cross section; namely,

\[
P_{\text{av}} = -2\pi a E_0^e + E_0^m \int \frac{(H_1 + H_2) * ds}{r^2}
\]

where \( \theta_0 = \pi/2 - \delta \approx \pi/2 - \frac{b}{a} \).

Thus, the radiation impedance obtained from the emf method is expressed by

\[
Z = \frac{P}{|I|^2} = Z_0 + Z_s
\]

where

\[
Z_0 = \pi \alpha a^2 \sum_{n=1}^{\infty} \frac{2n+1}{n(n+1)} P_n^1(0) P_n^1(\cos \theta_0) j_n(\alpha) h_n^{(2)}(\alpha).
\]
\[
Z_s = \pi \alpha a^2 \sum_{n=1}^{\infty} \frac{2n+1}{n(n+1)} P_n^1(0) P_n^1(\cos \theta_0) R_n h_n^{(2)}(\alpha)^2.
\]

In the above equations, \( Z_0 \) represents the radiation impedance of a loop without a spherical core, \( Z_s \) represents the variation of the radiation impedance due to the reaction between the spherical core and the loop, and \( \eta \) is the intrinsic impedance of free space. The imaginary part of series (10) does not converge rapidly enough when \( \theta_0 \) approaches \( \pi/2 \), or the thickness of the wire decreases to zero. By using the asymptotic expression of Legendre and Bessel functions, the radiation reactance \( X_0 \) in (10) can be transformed into the following alternative equations:

\[
X_0 = \eta a \left\{ \ln \frac{a}{2b} + \frac{\pi}{4} \right. \}
\]
\[
\left. -\pi \alpha a^2 \sum_{n=1}^{\infty} \frac{1}{n+1} \left[ \frac{2n+1}{n} P_n^1(0) P_n^1(\cos \theta_0) j_n(\alpha) n_n(\alpha) + \frac{1}{\pi a} \left\{ 2 \cos \frac{b}{a} (n+1) + \frac{\sin \theta_0}{a} (n+1) \right\} \right] \right\}
\]

The detailed derivation of the above equation is given in reference [6]. For a small loop, i.e., \( \alpha \ll 1 \), the radiation impedance is approximated by retaining the term \( n = 1 \) only; namely,
As for the constant $R_n$, many discussions have been made relating to the problem of the scattering of a plane wave by a sphere. These results can be directly applied to the present problem. For the lossy spherical core, i.e., the complex values of $N$, it is convenient for the computation to transform eq (11) by using the logarithmic derivative functions [7] with respect to spherical Bessel functions into the following form:

$$Z_n = -\pi \alpha \sum_{n=1}^{\infty} \frac{2n+1}{n(n+1)} \sum_{n=1}^{\infty} \frac{N}{\mu} \frac{\mu}{\mu} \sigma(N\alpha) \rho_n - N \mu \sigma(N\alpha)$$

where $\rho_n(\alpha)$ and $\sigma_n(\alpha)$ are the logarithmic derivative functions with respect to spherical Hankel and Bessel functions, respectively, i.e.,

$$\rho_n(x) = \frac{d}{dx} \ln [xj_{n}(x)]$$

and

$$\sigma_n(x) = \frac{d}{dx} \ln [xj_{n}(x)].$$

The function $\sigma(N\alpha)$ of any order with complex arguments $N\alpha$ can be computed by the recurrence formula of the logarithmic derivative functions.

Consider the maximum value of $R_n$ and its condition. A discussion has been given for special cases of pure dielectric ($\mu_s = 1$) or magnetic ($\epsilon_s = 1$) materials by E. M. Kennough. General cases for an arbitrary composition of $\epsilon_s$ and $\mu_s$ can be discussed in a similar way. When a sphere is sufficiently small in terms of wavelength, $\alpha \ll 1$, only the term $R_1$ is significant. It is concluded that the maximum absolute value of $R_1$ is $-1$ at the condition of

$$\frac{\mu}{\mu_0} j_1(N\alpha)[\alpha n_1(\alpha)]' - n_1(\alpha)[N\alpha j_1(N\alpha)]' = 0.$$  

For $\alpha \ll 1$, the above equation is approximately rewritten in the following transcendental equation:

$$\frac{1}{N\alpha} + \frac{N\alpha}{\mu_0} - 1 = \cot N\alpha.$$  

This equation is graphically solved with respect to $N\alpha$. Figure 2 shows $N\alpha/\pi$ as a function of $\mu_s$. Since $\alpha$ is assumed to be very small compared to unity, $N$ is necessarily very large, i.e., $\sqrt{\mu_0 \epsilon_s} \gg 1$. This value gives the first antiresonance frequency of a small uniform-current loop with a lossless spherical core having arbitrary values $\epsilon_s$ and $\mu_s$. It should be noted here that the above discussion can be applied to the problem of the plane wave scattering by a small sphere; namely, the curve of figure 2 gives the magnetic-type resonance frequency of the sphere, and also gives the electric-type resonance frequency by replacing $\mu_s$ by $\epsilon_s$.

The maximum radiation resistance at the antiresonance point is given by

$$R = R_0 + R_s = \frac{3}{2} \pi \eta \alpha^2 n_1(\alpha)^2 \approx \frac{3}{2} \pi \eta \alpha^2 \frac{1}{\alpha^2} (\alpha \ll 1).$$

The maximum resistance increases in the order of $\alpha^{-2}$ with the decrease of $\alpha$. It should be noted that in an actual one-point-fed loop, unless infinitesimally small, the frequency of its antiresonance deviates from that predicted by figure 2 due to the non-uniform current distribution. This situation is illustrated in table 1.

The diameter of the antenna in terms of wavelength in the medium at an antiresonance is found to vary between 0.159 and 1.43 by the constants of a medium. The results in figure 2 give the limiting case of an infinitesimally small loop, i.e., $\mu_s \epsilon_s \approx \infty$. Similarly, the expression for $Z$ in eq (9) is correct only for an infinitesimally small loop, if it is fed at one point. The radiation impedance $Z$ has been calculated for the two cases $\epsilon_s = 100$, $\mu_s = 1$, and $\mu_s = 100$, $\epsilon_s = 1$. For simplicity, the core is assumed to be lossless. Figure 3 and 4 show the radiation resistances and the radiation reactances. It must be noted here also, as mentioned before, that these figures do not give quantitative results for a one-point-fed loop.

<table>
<thead>
<tr>
<th>Medium of core</th>
<th>$\alpha$</th>
<th>$2\pi$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Air</td>
<td>0.5</td>
<td>0.159a</td>
</tr>
<tr>
<td>$\mu_s = 1.08 \approx 0.072$</td>
<td>$\epsilon_s = 9.5 \approx 0.072$</td>
<td>0.1535</td>
</tr>
<tr>
<td>$\mu_s \approx 1$</td>
<td>$\epsilon_s \approx 1$</td>
<td>0.409a</td>
</tr>
</tbody>
</table>

*From ref [1] by Julius Herman, $\lambda$ is the wavelength in the medium of the core.
cause of its nonuniform current. As for the case of 
\(\varepsilon = 100\), the antiresonance occurs at \(\alpha = \pi/10\), and 
the curves approach asymptotically to those of an 
air loop with the decrease of \(\alpha\). The \(\mu\)-core is effective 
in order to increase the radiation resistance in the 
lower frequency region. The radiation impedance 
for an arbitrary composition of \(\mu\) and \(\varepsilon\) ought to show 
a behavior intermediate between these two extreme 
cases. In figure 5 the ratios of the radiation resis-
tances to the radiation reactances are plotted. 
From these figures it can be concluded that high-\(\mu\) core is 
desirable for a comparatively low-frequency region, 
and high-\(\varepsilon\) core is rather desirable in order to use it 
in a antiresonant frequency region. This can be said also from the viewpoint of medium losses, that 
is, losses of usual magnetic materials increase more 
rapidly with the frequency than do the losses of 
dielectric materials.

3. Conclusion

The impedance characteristics of a uniform current 
loop (see footnote 3) with a spherical core of an 
arbitrary composition of \(\mu\) and \(\varepsilon\) have been theo-
retically investigated. The radiation impedance is 
derived by the emf method in a convenient form as the 
sum of the self-radiation impedance of an air loop 
and an additional term due to the reaction between 
the loop and the sphere which is proportional to the 
well-known expansion coefficient of a magnetic-type 
scattered wave from a sphere in an incident plane 
wave. The first antiresonance frequency has been 
given in the form of a universal curve for a very small 
uniform current loop with a core of an arbitrary com-
position of \(\mu_s\) and \(\varepsilon_s\), subject to the condition that the 
refraction coefficient \(N = \sqrt{\mu_s\varepsilon_s}\) is extremely large. 
It should be noted that the above results can be 
directly applied to the first magnetic-type resonance 
frequency of a sphere in an incident plane wave. To 
show quantitative impedance characteristics of a 
loop with an arbitrary composite core, the radiation 
impedances for two cases, namely \(\varepsilon = 100, \mu_s = 1\), 
and \(\mu_s = 100, \varepsilon_s = 1\), have been calculated and com-
pared with an air loop. It is concluded that high-\(\mu\) 
core is desirable for a comparatively lower frequency 
region, and high-\(\varepsilon\) core is rather desirable in an antiresonance frequency region.
The author thanks E. M. Kennough for suggesting this investigation as well as for guidance in the course of the work, and also Dr. R. G. Kouyoumjian for his valuable discussions.

4. References

[1] J. Herman, Thin wire loop and thin biconical antennas in finite spherical media, Ph. D. Dissertation, University of Maryland, Publication No. 25, 331 (1957); Further extensions of loop antenna theory, TR–756 (July 1957), Diamond Ordnance Fuze Laboratory.


Publication of the National Bureau of Standards*

Selected Abstracts


Recent experimental evidence favouring the $\rho K_1(\rho)$ correlation function for describing the turbulence of refractivity in the troposphere and stratosphere, K. A. Norton, *J. Atmospheric and Terrest. Phys.*, 15, 206 (1959).

$\rho K_1(\rho)$ is a normalized measurement for the refractivity of the refractive index in space at a fixed time, and $\rho = r/b$ is a normalized distance between two points in the atmosphere. The three kinds of evidence are (a) direct measurements of the variations of $\rho$ with time at a fixed location as made with a refractometer, (b) measurements of the variations with time of...