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Analytical Integration of the Differential Equation for Water Storage*

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The integration of the storage differential equation at present is usually done mostly by graphical or numerical procedures. An approach to the analytical integration of that equation is the subject of this paper. A new method of fitting the given background curves by mathematically tractable expressions is introduced. The storage-outflow discharge relation is expressed in the form of a power function. A general differential equation for water storage $y'+cPy^2-cy^k=0$ is derived, with c and k constants for the given reservoir, outflow shape and type of flow, and P being the inflow hydrograph. The analytical solutions of this equation for $P=0$, $P=$constant, and certain $P=f(t)$ are given for the integrable cases (tables 1 to 3, eqs (12) to (29)). The application of the results obtained is discussed at the end of the paper.

1. Introduction

The known simple balance equation, with the inflow minus the outflow in a time interval equal to the storage change in that interval is

$$P-Q=\frac{dS}{dt} \tag{1}$$

which is normally derived from the definition itself; with P the inflow, Q the outflow, S the storage volume, and t the time.

The two basic differential equations which govern the unsteady water movement through lakes and channels are the continuity and the dynamic (or momentum) equations:

$$\frac{\partial(Av)}{\partial x}+\frac{\partial A}{\partial t}=0, \tag{2}$$

$$\frac{\partial v}{\partial t}+v\,\frac{\partial v}{\partial x}+g\,\frac{\partial h}{\partial x}=gS_f \tag{3}$$

where

A=cross-sectional flow area,
v=mean velocity of flow,
x=length along the lake or the channel,
t=time,
g=earth acceleration,
h=level of water surface in a cross section with reference to a fixed level, and
S_f=frictional slope.

The applications of eqs (2) and (3) must satisfy the basic assumptions made by De Saint Venant in their derivation, and specifically that the unsteady flow is gradually varied.

The eq (3) is omitted in many engineering problems, as in the routing of floods through great reservoirs and along channels of small slopes with unsteady flow very gradually varied, so that only eq (2) is used.

Two important problems which have not as yet been completely discussed, and are not considered here, are: (a) Under what conditions can eqs (2) and (3) be used, and (b) when can eq (3) be neglected without sensible error.

The first term of eq (2), $\partial(Av)/(\partial x)=(\partial Q)/(\partial x)$, can, for suitable small Δx, be replaced by $(\Delta Q)/(\Delta x)$; for a given time interval Δt and a reach of length Δx, can be replaced by $\Delta Q/\Delta x=(P-Q)/\Delta x$, where P is arrival or inflow discharge and Q is the departure or outflow discharge in m^3/s or in cfs. (Here P and Q are used instead of usual I and O.) The second term $(\partial A)/(\partial t)$ can be replaced for small Δt by $(\Delta A)/(\Delta t)=(\Delta A_1+\Delta A_2)/(2\Delta t)$, where $\Delta A_1=A_1''-A_1'$ and $\Delta A_2=A_2''-A_2'$, and ΔA_1 and ΔA_2 are changes of cross-sectional areas at the beginning, and at the end of the reach Δx during time Δt, and A_1' and A_2' are areas for the beginning, and A_1'' and A_2'' areas for the end of time interval Δt. Equation (2) can thus be expressed as

$$P-Q=\frac{\Delta A_1+\Delta A_2}{2}\,\frac{\Delta x}{\Delta t}=\frac{\Delta S}{\Delta t},$$

where ΔS is the volume change for the reach Δx during time interval Δt. If now $\Delta t\to 0$, this differ-

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ential equation becomes eq (1), in which S is the storage volume for body of water of the considered reach Δx, and dS/dt is the rate of change with time of this volume. In the case of a reservoir or lake, with small velocities along the lake and with inflow coming from different tributaries, the reach Δx is replaced by the water body of the lake. Strictly speaking, some end portions of lakes with shallow water and high inflow should be considered separately, but their influence on the outflow hydrograph can be neglected in comparison with the main influence of great fluctuating storage space. Equations (2) and (1) are identical, and the equation of continuity in its form of eq (1) will be called the *storage differential equation* for water in lakes and along channels of small slopes with unsteady flow very gradually varied with time.

The analytical integration of the storage differential equation would be useful in the solution of problems in cases where its terms can be approximated by integrable expressions. Some of these cases are: (a) Routing of very gradually varied floods along large channels of small bottom slope; (b) effect of unregulated lakes on flood waves; (c) routing of flood peaks through reservoirs for the inflow discharges higher than that discharge of hydrograph, which corresponds to the full capacity of open gates and valves; (d) study of the genesis of hydrograph and separation of water flow according to its origin (surface runoff, underground storage, lakes, channel storage, etc.); (e) outflow through partial openings in case of rupture of dam with no influence of tailrace levels on the outflow hydrograph, when the wave movement along the lake created by rupture can be neglected; (f) computation of the seepage water out of reservoirs, etc.

Equation (1) serves generally for the computation of relations between five functions:

(1) Inflow hydrograph, $P=f_1(t)$;
(2) outflow hydrograph, $Q=f_2(t)$;
(3) stage hydrograph, $H=f_3(t)$;
(4) outflow rating curve, $Q=f_4(H)$;
(5) storage function, $S=f_5(H)$; or area function $A=f_6(H)$, with five variables, Q, P, H, S, t.

When three of five functions with boundary conditions are known (three variables can be excluded), the eq (1) enables the computation of the relation between two remaining variables.

The usual integration procedures are numerical, graphical, combined numerical and graphical, and integration by special devices.

The analytical integration of eq (1) is not usually feasible in practice, mostly because of the difficulties of fitting easily three known curves out of five by tractable and integrable mathematical expressions.

The most common cases in the application of eq (1) have, as given, the following curves: (1) Storage or area function (obtained by survey); (2) outflow rating curve (obtained by gauging, hydraulic computation, model study, etc.); and (3) inflow (or outflow) hydrograph. Two other functions are normally to be computed; (1) outflow (or inflow) hydrograph, and (2) stage hydrograph.

The subject of this paper is the analytical integration of eq (1) for those cases for which inflow hydrograph, storage function and outflow rating curve can be entirely or partly fitted by simple expressions which make eq (1) integrable.

2. Fitting of Mathematical Expressions to Given Curves

2.1. Storage Function

The storage function, which relates lake or channel volume to its level referred to some datum, can be approximated either by the function (see references [1][1] to [5]) of the type

$$S=aH^m \tag{4}$$

or by the polynomial of the type

$$S=A_0+A_1H+A_2H^2+\ldots+A_mH^m \tag{4a}$$

where S=storage volume, H=depth of water above a reference level suitably selected, a and m, and A_0, A_1, A_2, . . ., A_m are coefficients to be determined from data of storage function.

The reference level of eq (4) is normally that of zero storage (lowest level of lake or reservoir, or river bed, etc.). The level of zero outflow does not coincide generally with the level of zero storage, and in that case another form of eq (4) is used:

$$S=a(H^m-H_0^m) \tag{4b}$$

where H_0=difference of the level of zero outflow and of the level of zero storage. The coefficients a and m in eq (4b) are the same as in eq (4). The use of eq (4), as will be shown later, gives a form to eq (1) which is integrable in many cases, but eq (4b) makes it less simple and more complex for analytical treatment.

Equation (4a) is well-suited to be used as the storage function, if enough terms $(m+1)$ are involved. It must be supposed that for the level $H=0$, the storage is A_0, so that eq (4a) can be used in case the level of zero outflow coincides with the level of storage A_0. If the level of zero outflow is changed, the coefficients A_0, A_1, A_2, . . ., A_m have also to be changed.

Instead of using eq (4b) repeatedly with various values of H_0 to cover the entire range of S versus H, a family of eqs (4) may be used with continuously changing a and m as function either of level of zero outflow or of H_c, the distance from level of zero outflow to a reference level. The coefficients a and m of eq (4) depend on level of zero outflow, or on difference of levels H_c, and are determined by the condition that $H=0$ of new zero storage coincides always with the level of zero outflow. This means that the coefficients a and m are to be determined for different levels of zero outflow (and therefore also of zero storage) as continuous functions. The

[1] Figures in brackets indicate the literature references at the end of this paper.

question arises whether eq (4) fits the thus conceived changing storage function as well as it fits the entire function from the lowest to the highest level of the body of water. In some cases eq (4), used in the manner described above, fits the upper parts of the storage curve better than the entire curve, but in other cases the reverse is true.

Figures 1 and 2 show the second case, and figures 3 and 4 the first case. Figures 1 and 3 show the storage functions $S=aH^m$ for the entire range of

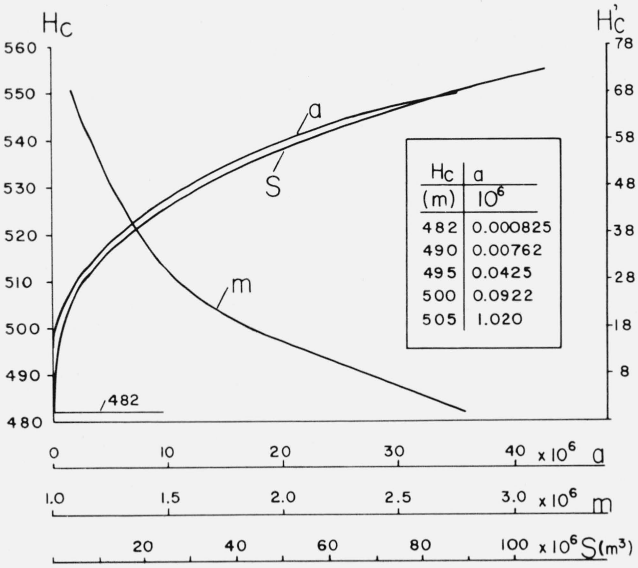

FIGURE 1. *Storage function $S=aH^m$, with a and m as functions of H_c, the levels of outflow zero (first example).*

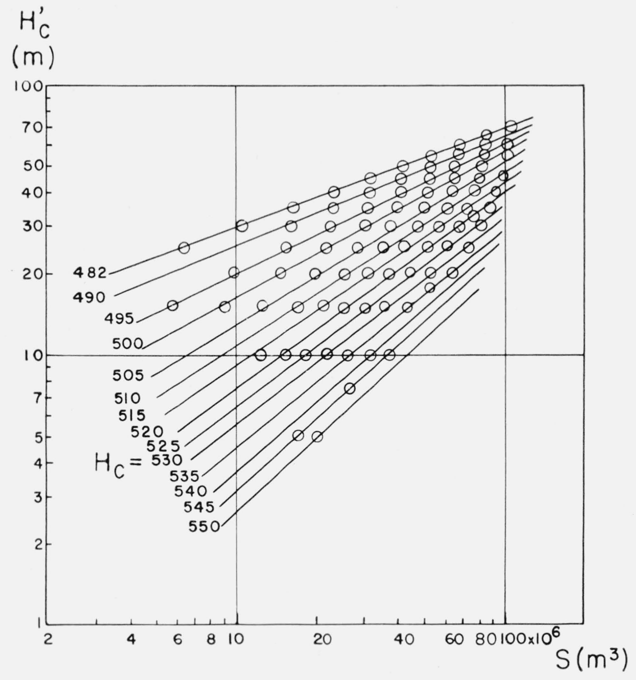

FIGURE 2. *Storage function of figure 1 in logarithmic scales for different values H_c of zero outflow, where the storage functions from H_c up to the highest level are fitted by the power functions $S=aH^m$ (first example).*

H=depth of water in reservoir at dam, and the functions $m=f_1(H_c)$ and $a=f_2(H_c)$ with H_c as distance of level of outflow zero to the sea level, or to the reservoir bottom at dam H_c'. Figures 2 and 4 show the fitting of eq (4) to the parts of storage curves for different values H_c. When H_c is expressed as H_c', or as distance to the bottom of reservoir, it can be used as dimensionless number $h=H_c'/H_m$, with H_m=maximum depth of reservoir in the normal operation.

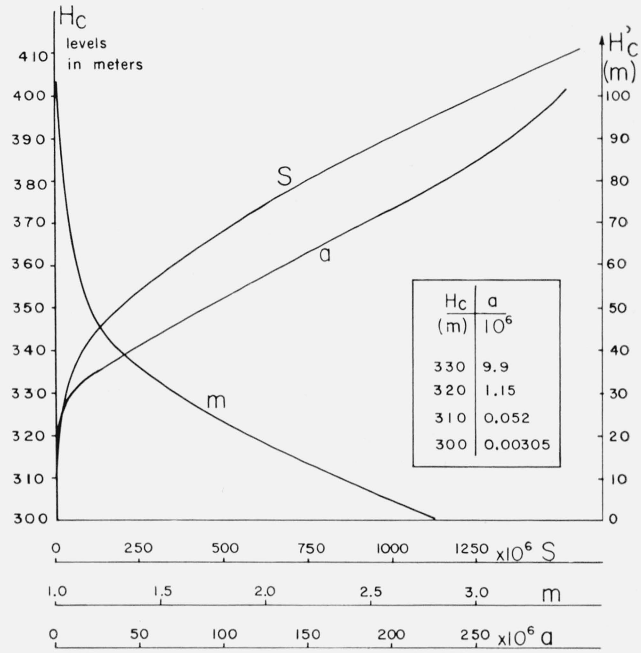

FIGURE 3. *Storage function $S=aH^m$, with a and m as functions of H_c, the levels of outflow zero (second example).*

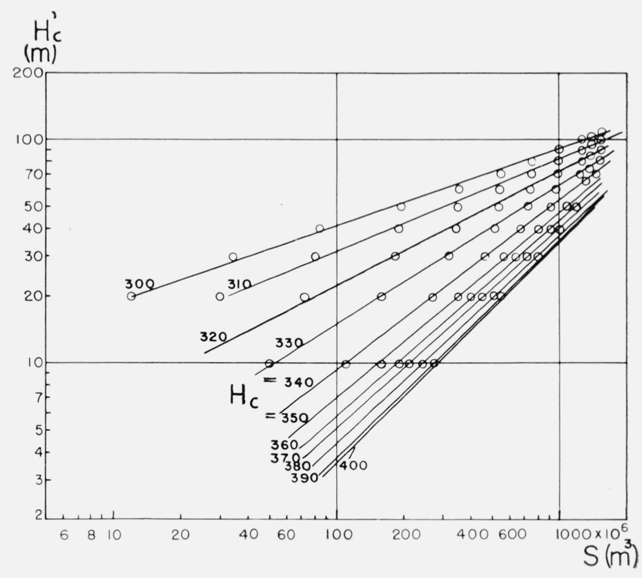

FIGURE 4. *Storage function of figure 3 in logarithmic scales for different values H_c of level of zero outflow, where the storage functions from H_c up to the highest level are fitted by the power functions $S=aH^m$ (second example).*

The storage functions in figures (1) and (3) are given from the reservoir bottom at dam up, and in figures 2 and 4 from the level of zero outflow up for each of given curves (as straight lines).

The function $m=f_1(H_c)$ decreases with higher value of H_c, i.e., higher level of zero outflow, and is approximately 1 for the highest levels and small range of level differences, where the storage function is nearly a straight line. For lower H_c and for higher range of storage levels, the value m is greater. The function $a=f_2(H_c)$ has the opposite characteristics.

If only one value of H_c is to be investigated, a and m are fixed coefficients, but if a range of H_c is to be investigated (dam-breaching problems with various breachings and different lowest outflow levels), the functions m and a of H_c can be easily obtained as in the case of figures 1 to 4.

When the sedimentation of reservoirs has to be taken into account, a and m are not only a function of level of zero outflow, but also of time (apart from sediment characteristics they depend on operation practice of reservoir, inflow hydrographs, etc.). As a and m make eq (4) less tractable in case they change with time (which is sometimes rather difficult to predict), they will be considered here as constants for a given reservoir, for given level of zero outflow, and for selected time period of reservoir life.

The coefficient m ranges within the limits 1 to 5, but for the majority of natural valleys the range is about 1 to 4. When H_c is high and the range of levels is very small, m is mostly 1.0 to 1.5 and rarely greater than 2; for highest range it is 2 to 5. The m depends not only on the range of levels, but also on the shape of reservoir cross sections. When those sections can be fitted by a power function for width of type $B=2pH^s$, m is function of s. The value a depends on H_c, but apart from that, a is higher for wider valleys, for lower river slopes, and for lower values of exponent s.

The property of the storage function to be well-fitted by an expression of the type of eq (4) can be used to compute this function in the case where only two points of the relation storage versus level are available. These two points can be found in the literature for practically any reservoir: capacities at highest and at lowest operational levels. Two pairs of values (S, H) are sufficient for the computation of values a and m in eq (4), thus allowing the determination of the curves of figures 1 and 3. Taking into account the errors in the survey of reservoirs and the fact that the storage function is being constantly changed by sedimentation and littoral erosion, the values a and m determined by eq (4) and the figures 1 to 4 are accurate enough for the analytical solution of practical problems.

2.2. Outflow Rating Curve

The outflow rating curves are relations of the departure discharge to the water level in the reservoir. They can be fitted also in many cases either by an expression of type

$$Q=bH^r, \tag{5}$$

or by a polynomial,

$$Q=b_0+b_1H+b_2H^2+b_3H^3+\ldots+b_rH^r, \tag{5a}$$

where Q=departure discharge, H=level above the depth for which $Q=b_0$, b and r or $b_0, b_1, b_2, b_3, \ldots b_r$ are coefficients depending only on boundary conditions (outlet shape) and on type of flow (closed or free outlets). The eq (5a) can be also conceived as a rating function expressed in power series form. The range of r in eq (5) for free surface outflow is 1.5—4.0 (usually 1.5 to 3.0) and for pressure type flow about $r=0.5$ in most cases. The range $r=0.5$—1.5 can be covered also in special cases for special shapes of outflows.

In case two types of flow, viz, flow under pressure and free surface flow, occur simultaneously, eq (5) takes the form

$$Q=b_1H_1^{r_1}+b_2H_2^{r_2}=b_1(H+H_a)^{r_1}+b_2H^{r_2}, \tag{5b}$$

where $H_2=H$ and $H_1=H+H_a$ are the hydraulic heads (level differences) to be taken for the former and latter types of flow respectively, b_1 and r_1 correspond to closed pressure flow and b_2 and r_2 to free surface flow. As r_1 and r_2 are not generally integers, eq (5b) is different from eq (5a).

The use of eqs (5a) and (5b) instead of eq (5) makes eq (1) integrable in a much smaller number of cases. Equation (5) will be used exclusively here.

2.3. Inflow Hydrograph

It is practically impossible to fit an entire natural inflow hydrograph by a single mathematical expression, but it is possible to fit some of their parts by tractable and simple functions.

The simplest forms of function $P=f(t)$ to be used for integration are: (a) $P=0$ (arrival discharge zero as dry period of year or as water fully stored in upstream reservoirs, etc.); (b) $P=P_0$=constant (or nearly constant, with low river flow slowly changing, regulated constant flow from upstream reservoirs, short-term operation of reservoirs, when flow could be placed by constant discharge); (c) $P=P_0-ft$ (gradually varied flow); $P=P_0t^{-s}$; $P=P_0e^{-ft}$; $P=P_0t^se^{-ft}$ (P_0, f, and s are different constants in each case).

2.4. Storage-Outflow Discharge Function

When the lowest outflow level coincides with the level of zero storage, or when reference levels for measuring H in both eqs (4) and (5) are the same (adjustment to this condition is always possible by taking the corresponding coefficients a and m from their functions, fig. 1 to 4), the elimination of H gives for the storage-outflow discharge function, or relation $S=f(Q)$,

$$S=\frac{a}{b^{m/r}}Q^{m/r}=\frac{1}{c}Q^{n}, \qquad (6)$$

where $c=b^{(m/r)}/a$ and $n=m/r$.

The coefficient n for free surface flow is in the range ¼ to 4, and for closed pressure flow 2 to 10, or a little greater.

The use of pairs of eqs (4) and (5a), (4) and (5b), (4a) and (5a), (4a) and (5b), (4b) and (5), (4b) and (5a), (4b) and (5b) gives a more complex relation $S=f(Q)$, in most cases in parametric form with H as the parameter, than the use of (4) and (5) or (4a) and (5). However, in this last case

$$S=a_0+\frac{1}{c_1}Q^{1/r}+\frac{1}{c_2}Q^{2/r}+\ldots+\frac{1}{c_m}Q^{m/r}, \qquad (6a)$$

which makes eq (1) less tractable than the use of eqs (4) and (5).

As eqs (4) and (5) fit well many storage and outflow functions, for different value of H_c, and as eq (6) allows the integration of eq (1) in many cases, eqs (4) and (5), and (6) will be used in this study.

3. General Type of Differential Equations

With the introduction of eq (6) into eq (1), one obtains $n\,Q^{n-1}\,dQ+c(Q-P)\,dt=0$.

The substitution $y=Q^{-n}$ and the replacement $k=(2n-1)/n=2-\frac{1}{n}=2-\frac{r}{m}$ then gives the following expression:

$$y'+cPy^2-cy^k=0. \qquad (7)$$

This is the general differential equation for storage reservoirs and storage channels with no artificial control of outlet flow, under conditions discussed previously in this paper. For free surface outflows the range of k is usually from about -2 for $n=¼$ to $k=7/4$ for $n=4$, and for the closed type outflows the range of k is from 3/2 to 19/10 or a little greater, but less than the limit value $k=2$. Practically, the range of k is from -2 to $+2$, where $k=+2$ is not possible.

The relations of k or n to m and r are given in figure 5. The most common range for both m (1 to 4) and r (0.5 and 1.5 to 3) is specially emphasized for reservoirs in river valleys. In most practical cases $k>0$, and usually $k>1/2$.

The eq (7) could also be expressed as

$$y'+cy^2\,(P-1)-cy^{r/m}=0. \qquad (8)$$

The general form of eqs (7) and (8) is

$$y'+My^h+Ny^g=0 \qquad (9)$$

where M and N are functions of t; h and g are any real numbers. By the substitution $y=Z^{(1/(h-1))}$ the general form is reduced to

$$Z'+(h-1)MZ^2+(h-1)NZ^{\frac{h+g-2}{h-1}}=0 \qquad (10)$$

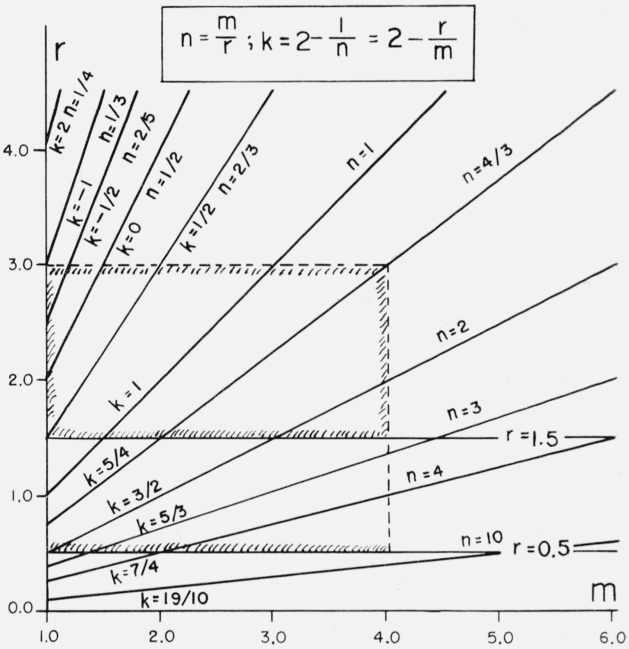

FIGURE 5. *Values n of storage-outflow discharge relationship $S=\frac{1}{c}Q^n$, and values $k=2-\frac{1}{n}$, in relation to m and r (exponents of storage and of rating curve, expressed in power form).*

which is eq (7) under conditions that $h-1=c$, $N=-1$, $M=P$, and $(h+g-2)/(h-1)=k$.

The general analytical solution of the ordinary differential equations of the type (7) or (10) for any k and for any expression of P does not exist. Chiellini [6] has shown that the solution is possible for $k=3$ if the expression $(M'N-MN')$ is different from M^3 by a constant factor. When k is a positive and integer number higher than 3, Luigi Conte [6, 7] has given the solution for special cases. As k is always less than two here, their procedures cannot be used for eq (7).

Both facts, (a) that k is a number derived from continuously changing natural conditions and is rarely an integer, and (b) that the function $P=f(t)$ is not simple and must be fitted by integrable expressions, make the analytical integration of eq (7) possible only in special cases. For $k=-2$, -1, 0, and 1 and some rational numbers the integration is simpler for some types of P, but the integration for intermediate values of k and for the majority of P functions has to be performed numerically or graphically. The analytical integration for special values of k and for selected functions $P=f(t)$ is useful, because the resulting functions give the type of curves which can be expected for intermediate values of k. Or, when k and outflow functions are known, some conclusions can be made about the inflow functions. The type of functions obtained in integrable cases can serve for control of curves obtained by other procedures.

4. Integration in Special Cases

Equation (7) will be analyzed for some values of k and some types of the function P. The values of k taken will be -2, -1, 0, 1 and, tentatively, $-3/2$, $-1/2$, $1/2$, $3/2$ along with some other rational numbers.

4.1. $P=0$

Equation (7) becomes

$$y'-cy^k=0, \tag{11}$$

with $y=y_0=Q_0^{-n}$ for $t=0$, where Q_0 is outflow discharge at the time $t=0$. The solution of eq (11) with $y=Q^{-n}$ is then

$$Q^{n-1}=Q_0^{n-1}-\left(1-\frac{1}{n}\right)ct. \tag{12}$$

This is the general expression for outflow hydrographs from storage reservoir with no inflow into the reservoir. The solution can be obtained from eq (12) for each value of n except for $n=1$ $(k=1)$, but eq (11) gives solution in this case directly (table 1). Therefore, for $P=0$, or when the constant inflow discharge is small and can be neglected in comparison to the outflow discharges, the analytical integration of eq (7) gives solutions for any $k(n)$, rational or irrational.

Table 1. $Q=Q_0\left(1-\frac{n-1}{n}\frac{ct}{Q_0^{n-1}}\right)^{\frac{1}{n-1}}$; $P=0$

k	n	Q
-2	$\frac{1}{4}$	$Q_0\left(1+3cQ_0^{\frac{3}{4}}t\right)^{-\frac{4}{3}}$
-1	$\frac{1}{3}$	$Q_0\left(1+2cQ_0^{\frac{2}{3}}t\right)^{-\frac{3}{2}}$
$-\frac{1}{2}$	$\frac{2}{5}$	$Q_0\left(1+\frac{3}{2}cQ_0^{\frac{3}{5}}t\right)^{-\frac{5}{3}}$
0	$\frac{1}{2}$	$Q_0\left(1+cQ_0^{\frac{1}{2}}t\right)^{-2}$
$\frac{1}{2}$	$\frac{2}{3}$	$Q_0\left(1+\frac{c}{2}Q_0^{\frac{1}{3}}t\right)^{-3}$
1	1	Q_0e^{-ct}
$\frac{4}{3}$	$\frac{3}{2}$	$Q_0\left(1-\frac{c}{3Q_0^{\frac{1}{2}}}t\right)^{2}$
$\frac{3}{2}$	2	$Q_0\left(1-\frac{c}{2Q_0}t\right)$
$\frac{5}{3}$	3	$Q_0\left(1-\frac{2c}{3Q_0^2}t\right)^{\frac{1}{2}}$
$\frac{7}{4}$	4	$Q_0\left(1-\frac{3c}{4Q_0^3}t\right)^{\frac{1}{3}}$

Equation (12) can be written in the form:

$$Q=Q_0\left(1-\frac{n-1}{n}\frac{ct}{Q_0^{n-1}}\right)^{\frac{1}{n-1}}. \tag{13}$$

This shows that the decrease of discharge ratio Q/Q_0 depends on both t and Q_0, except for $n=k=1$ (table 1), in which case it depends only on t.

The analytical expressions of eq (12) for some k (or n) values are given in table 1. Figure 6 gives curves for $n=\frac{1}{2}$, 1, $\frac{3}{2}$, 2 and 3, for given values of a and b, with $c=b^n/a$. The values a of eq (4) and b of eq (5) are taken constant for all five curves (for easier computation in examples), though they change when m and r (and so n) changes. For $n=2$ the outflow hydrograph is a straight line, which corresponds, among other cases, to the cylindric reservoir $(m=1)$ and to the outflow under pressure $(r=\frac{1}{2})$. The same type of linear hydrograph is obtained for $m=3$ and $r=\frac{3}{2}$, which is the case for some deep lakes with outflow function $Q=bH^{\frac{3}{2}}$. For river outlets from lakes, with $r\geq\frac{3}{2}$ and $m\leq 3$, $n\leq 2$, all hydrographs are concave upwards. For $n=1$, the solution of eq (11) is an exponential function. All functions for $n\leq 1$ are asymptotic with the time axis. Hydrographs for $n\geq 1$ do not have this asymptote, but cut the t—axis for finite values (for $n=2$, $t_0=2Q_0/c$). For the range $2>n>1$, the hydrographs are power functions and concave upward, but have finite values for $t=0$ (are not asymptotic to the t axis). For $n<1$ the curves are hyperbolic functions, asymptotic and concave upward.

All outflow hydrographs for very small water levels above the bottom of outflow orifice become asymptotic, because their outflow functions change due to the change of type of flow or to some secondary effects. In closed-type outflow the function changes from $r=\frac{1}{2}$ to about $r=\frac{3}{2}$ as soon as the water level drops to near the upper edge of the outflow orifice, and changes further for smaller outflow heights due to surface tension and other secondary effects. The exponent r increases so that $n=m/r\leq 1$.

The shapes of outflow hydrographs depend thus on the ratio of exponents of storage and outflow functions, and they can be: hyperbolic $(n<1)$, exponential $(n=1)$, power function with exponents higher than 1 $(2>n>1)$, straight line $(n=2)$ and power function with exponents lower than 1 $(n>2)$. They show that the natural conditions can produce different outflow hydrographs on the same river, depending on the lake and channel characteristics and outflow shapes.

The free spillways of reservoirs, or free outflows of shallow lakes, where the range of storage fluctuation is small, $m=1.0$ to 1.3, and $r\geq\frac{3}{2}$, have $n\leq 1$, and all outflow hydrographs are hyperbolic asymptotic functions (fig. 6). For the higher range of levels of outflow, m is greater, and the outflow hydrographs are power functions whose graphs are convex upward or downward. The smallest value of n for closed-type outflow is $n=\frac{1}{0.5}=2$, so that all hydrographs are power functions convex upward.

In case eq (6a) is used instead of eq (6), the following general solution of eq (7) for $P=0$ is obtained:

$$t=\sum_{m=1}^{m}\frac{m}{C_m(m-r)}\left(Q_0^{\frac{m-r}{r}}-Q^{\frac{m-r}{r}}\right), \tag{14}$$

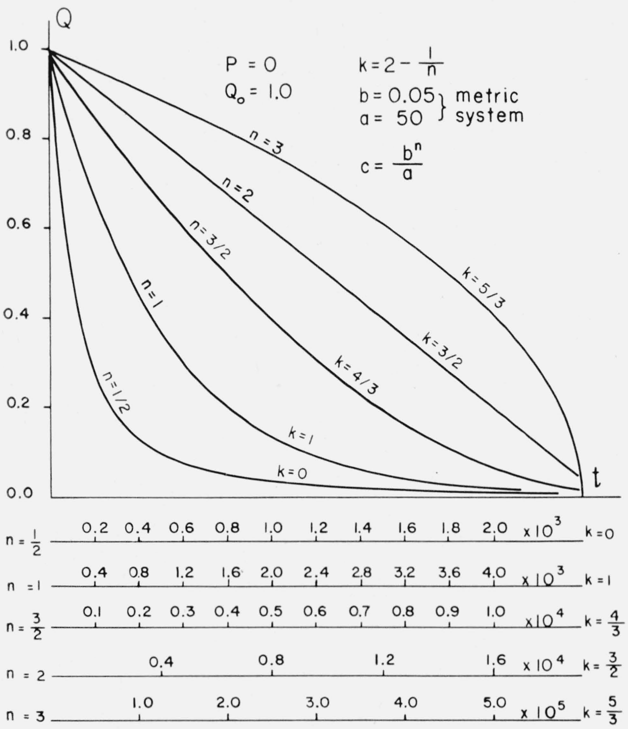

FIGURE 6. *Outflow hydrographs in their recession parts from Q_o to zero, related to different values k and n, in the case where inflow is zero.*

which for the case $m \geq 3$ is more complicated than eqs (12) or (13), particularly since eq (14) gives t as a function of Q rather than the more convenient inverse.

If both types of flow occur simultaneously, eq (5b) has analytical solutions in a much more limited number of cases.

4.2. $P=P_o=$Constant

In the case of constant or nearly constant inflow into the reservoir, with $P=P_o$, eq (7) becomes

$$y'+cP_0y^2-cy^k=0, \tag{15}$$

and the separation of variables gives

$$\int \frac{dy}{P_0y^2-y^k}+ct=K \tag{16}$$

with K=integration constant.

The indicated integration of eq (16) cannot be carried out analytically for arbitrary k, contrary to the case of eq (11). Equation (16) has, however, analytical solutions for many rational numbers, which is useful, since it is possible to approximate k by a rational number of type g/h, where g and h are integers. Equation (16) can be written in the form of the binomial integral

$$\int y^{-2}(P_0-y^{k-2})^{-1}dy+ct=K. \tag{17}$$

The binomial $y^{-2}(P_0-y^{k-2})^{-1}$ is of the form

$$y^s(ay^r+b)^p \tag{18}$$

which can be reduced to an algebraic function in case any one of the three numbers

$$p, \quad \frac{s+1}{r}, \quad \text{and} \quad \frac{s+1}{r}+p$$

is an integer. As $p=-1$ is an integer, when k is a rational number, eq (16) can always be reduced to an algebraic function.

The most general solutions of the binomial integral, if the obtained algebraic functions are integrable, are in the form of sum of logarithmic, algebraic and arctan terms. In some cases the arctan term does not appear in the solution, as is the case also with the algebraic term, but the logarithmic term is always present. The solutions of eq (16) will be analyzed here for some values of k.

By substitution of a new variable u in eq (16),

$$u=y^{1/h}\cdot P_0^{\frac{1}{2h-g}}, \tag{19}$$

the following general binomial expression of eq (16) for $k=g/h$ is obtained:

$$hP_0^{\frac{g-h}{2h-g}}\int u^{h-g-1}(u^{2h-g}-1)^{-1}du+ct=K \tag{20}$$

with $h-g-1$ and $2h-g$ integers.

For

$$f(u)=\int u^{h-g-1}(u^{2h-g}-1)^{-1}du \tag{21}$$

the constant K for $t=0$ $(u=u_0)$ is $K=hP_0^{(g-h)/(2h-g)}\times f(u_0)$, and the general solution has the form

$$t=R[f(u_0)-f(u)] \tag{22}$$

with $R=\frac{h}{c}P_0^{(g-h)/(2h-g)}$.

The following values and ranges of g and h are analyzed: (1) $g=2h$; (2) $h \leq g < 2h$; (3) $g<h$; (4) $g=0$, $h \neq 0$.

(1) $g=2h$. For $k=2$, eq (20) has no practical significance.

(2) $h \leq g < 2h$. The value $s=2h-g$ is always a positive integer number, and $j=h-g-1$ is always a negative integer number or zero. Equation (20) is integrable if $s=2h-g \leq 4$ or $s=6$, when the rational function can be separated in integrable terms, whatever the value of $j=h-g-1$. The solution is

$$t=\frac{h}{c}P_0^{(h/s)-1}[f(u_0)-f(u)]. \tag{22a}$$

(a) $s=2h-g=1,\ n=h.$

$$f(u)=\ln\frac{u-1}{u}+\frac{1}{u}+\frac{1}{2u^2}+\ .\ .\ .\ +\frac{1}{(n-1)u^{n-1}} \qquad \text{(21a)}$$

with $u=P_0Q^{-1}$.

This case covers the following values of n and k:

n: 1, 2, 3, 4, 5, 6, 7, etc.
k: 1, 3/2, 5/3, 7/4, 9/5, 11/6, 13/7, etc.

Some of these cases are given in table 2.

For $n=1$ $(k=1)$ only the logarithmic part of eq (21a) is present and eq (22a) gives:

$$Q=P_0+(Q_0-P_0)e^{-ct} \qquad \text{(21b)}$$

(b) $s=2h-g=2,\ n=h/2.$

$$f(u)=\ln\frac{u-1}{u+1}+\frac{2}{u}+\frac{2}{2u^3}+\ .\ .\ .\ +\frac{2}{2(n-1)u^{2(n-1)}} \qquad \text{(21c)}$$

with $u=P_0^{1/2}Q^{-1/2}$. This case covers the values:

n: 3/2, 5/2, 7/2, 9/2, etc.
k: 4/3, 8/5, 12/7, 16/9, etc.

Some of these are given in table 2.

TABLE 2. $P=P_0=Constant;\quad t=R\,[f(u_0)-f(u)]$

n	n	k	R	u	$f(u)$
$\frac{1}{4}$	0.25	-2	$\frac{1}{c}P_0^{-\frac{3}{4}}$	$P_0^{\frac{1}{4}}Q^{-\frac{1}{4}}$	$\ln\frac{u-1}{u+1}+2\arctan u$
$\frac{1}{3}$	0.33	-1	$\frac{1}{c}P_0^{-\frac{2}{3}}$	$P_0^{\frac{1}{3}}Q^{-\frac{1}{3}}$	$\ln\frac{(u-1)^2}{u^2+u+1}+4\arctan\left(\frac{2u+1}{3}\right)$
$\frac{1}{2}$	0.50	0	$\frac{1}{c}P_0^{-\frac{1}{2}}$	$P_0^{\frac{1}{2}}Q^{-\frac{1}{2}}$	$\ln\frac{u-1}{u+1}$
$\frac{2}{3}$	0.67	$\frac{1}{2}$	$\frac{2}{c}P_0^{-\frac{1}{3}}$	$P_0^{\frac{1}{3}}Q^{-\frac{1}{3}}$	$\ln\frac{(u-1)^2}{u^2+u+1}-4\arctan\left(\frac{2u+1}{3}\right)$
$\frac{3}{4}$	0.75	$\frac{2}{3}$	$\frac{3}{c}P_0^{-\frac{1}{4}}$	$P_0^{\frac{1}{4}}Q^{-\frac{1}{4}}$	$\ln\frac{u-1}{u+1}-2\arctan u$
1	1.00	1	$\frac{1}{c}$	P_0Q^{-1}	$\ln\frac{u-1}{u}$; $Q=P_0+(Q_0-P_0)e^{-ct}$
$\frac{5}{4}$	1.25	$\frac{6}{5}$	$\frac{5}{c}P_0^{\frac{1}{4}}$	$P_0^{\frac{1}{4}}Q^{-\frac{1}{4}}$	$\ln\frac{u+1}{u-1}+\frac{4}{u}+2\arctan u$
$\frac{4}{3}$	1.33	$\frac{5}{4}$	$\frac{4}{c}P_0^{\frac{1}{3}}$	$P_0^{\frac{1}{3}}Q^{-\frac{1}{3}}$	$\frac{1}{2}\ln\frac{(u-1)^2}{u^2+u+1}+\frac{3}{4}+2\arctan\left(\frac{2u+1}{3}\right)$
$\frac{3}{2}$	1.50	$\frac{4}{3}$	$\frac{3}{c}P_0^{\frac{1}{2}}$	$P_0^{\frac{1}{2}}Q^{-\frac{1}{2}}$	$\ln\frac{u-1}{u+1}+\frac{2}{u}$
$\frac{5}{3}$	1.67	$\frac{7}{5}$	$\frac{5}{c}P_0^{\frac{2}{3}}$	$P_0^{\frac{1}{3}}Q^{-\frac{1}{3}}$	$\frac{1}{2}\ln\frac{(u-1)^2}{u^2+u+1}+\frac{3}{2u^2}-2\arctan\left(\frac{2u+1}{3}\right)$
2	2.00	$\frac{3}{2}$	$\frac{2}{c}P_0$	P_0Q^{-1}	$\ln\frac{u-1}{u}+\frac{1}{u}$
$\frac{7}{3}$	2.33	$\frac{11}{7}$	$\frac{7}{c}P_0^{\frac{4}{3}}$	$P_0^{\frac{1}{3}}Q^{-\frac{1}{3}}$	$\ln\frac{(u^3-1)^3}{u}+\frac{1}{3u^3}$
$\frac{5}{2}$	2.50	$\frac{8}{5}$	$\frac{5}{c}P_0^{\frac{3}{2}}$	$P_0^{\frac{1}{2}}Q^{-\frac{1}{2}}$	$\ln\frac{u-1}{u+1}+\frac{2}{u}+\frac{2}{3u^3}$
3	3.00	$\frac{5}{3}$	$\frac{3}{c}P_0^2$	P_0Q^{-1}	$\ln\frac{u-1}{u}+\frac{1}{u}+\frac{1}{2u^2}$
$\frac{7}{2}$	3.50	$\frac{12}{7}$	$\frac{7}{c}P_0^{\frac{5}{2}}$	$P_0^{\frac{1}{2}}Q^{-\frac{1}{2}}$	$\ln\frac{u-1}{u+1}+\frac{2}{u}+\frac{2}{3u^3}+\frac{2}{5u^5}$
4	4.00	$\frac{7}{4}$	$\frac{4}{c}P_0^3$	P_0Q^{-1}	$\ln\frac{u-1}{u}+\frac{1}{u}+\frac{1}{2u^2}+\frac{1}{3u^3}$

(c) $s=2h-g=3,\ n=h/3$. The solutions for three values of h and n ($h=4$, $n=4/3$; $h=5$, $n=5/3$; $h=7$, $n=7/3$) are given in table 2.

(d) $s=2h-g=4,\ n=h/4$. For $h=5$, $n=5/4$ the solution is given in table 2.

(3) $g<h$. The analytical integration can be carried out when $-4\leq 2h-g=s\leq 4$, or $g/2-2\leq h\leq g/2+2$. For $h>g/2+2$ and $h<g/2-2$ the exponent $s=2h-g$ in eq (20) is higher than 4, so that the algebraic function $u^{g-h+1}(u^{2h-g}-1)$ can not be separated into integrable components except in special cases. The solutions for $n=3/4$ $(k=2/3)$, $n=2/3$ $(k=1/2)$, $n=1/3$ $(k=-1)$ and $n=1/4$ $(k=-2)$ are given in table 2.

(4) $g=0$, $h=1$; $n=1/2$ $(k=0)$. The solution is given in table 2. Figure 7 gives five curves for the values n: 1/2, 1, 3/2, 2 and 3, as in figure 6. All five are asymptotic to the value $P_0=0.20$, but for t nearer to zero they approach the curves of figure 6.

4.3. $P=f(t)$

The solution of eq (3) for four types of $P=f(t)$ can be obtained for special values of k, but the general analytical solution such as was obtained in the case of eq (13), or in many special cases of eq (22), is not possible.

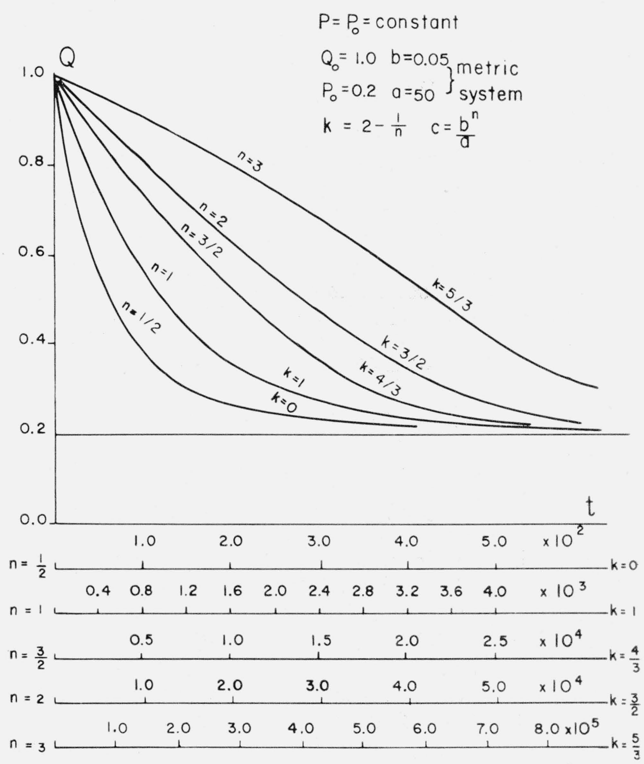

FIGURE 7. *Outflow hydrographs in their recession parts from Q_o to inflow $P=P_o$=constant, related to different values k and n.*

a. $k=1 \quad (n=1)$

Equation (7) becomes

$$y'+cPy^2-cy=0, \tag{23}$$

which is a Bernoulli equation, and has the general solution for $y=Q^{-1}$:

$$Q=e^{-ct}\left(K+c\int Pe^{ct}dt\right). \tag{24}$$

The solutions for four types of function $P=f(t)$ are given in table 3.

TABLE 3. $P=f(t)$

$k=1, \quad n=1, \quad s=integer$

$P=f(t)$	$Q=f_2(t)$
$P=P_o-ft$	$\left(P_o-ft+\frac{f}{c}\right)+\left(Q_o-P_o-\frac{f}{c}\right)e^{-ct}$
$P=P_o t^{-s}$	$\left[Q_o-\frac{P_o}{c^s}(-1)^s s!\right]e^{-ct}+\frac{P_o}{c^s}s!(-1)^s\left[1-\frac{ct}{1!}+\frac{c^2t^2}{2!}-\ldots+(-1)^s\frac{c^st^s}{s!}\right]$
$P=P_o \cdot e^{-ft}$	$\left(Q_o-\frac{cP_o}{c-f}\right)e^{-ct}+\left(\frac{cP_o}{c-f}\right)e^{-ft}$
$P_o=P_o \cdot t^s e^{-ft}$	$\left[Q_o-(-1)^{s+1}\frac{s!cP_o}{(c-f)^{s+1}}\right]e^{-ct}-(-1)^{s+1}\frac{cP_o s!}{(c-f)^{s+1}}$ $[1-1!(c-f)t+2!(c-f)^2t^2-\ \ldots\ +s!(c-f)^st^s]e^{-ft}$

Such analytical solutions can be obtained under condition that the integration of

$$\int Pe^{ct}dt \tag{25}$$

in eq (23) can be carried out in closed form.

b. $k=0 \quad \left(n=\frac{1}{2}\right)$

Equation (7) becomes

$$y'+cPy^2-c=0 \tag{26}$$

which is a special type of the Riccati equation, with $y=Q^{-1/2}$. As two particular integrals of eq (26) are known, viz, $y_0=Q_0^{-1/2}$ for $t=0$, and $Q=0$ or $Q=P_0$ for t infinite, the solution of eq (26) is possible. The substitution $Z=y-y_0$ gives

$$Z'+cPZ^2+2cPy_0Z+(y_0'+cPy_0^2-c)=0$$

and eq (26) becomes the Bernoulli equation

$$Z'+cPZ^2+2cPy_0Z=0 \tag{27}$$

with the solution

$$\frac{1}{y-y_0}=e^{2cy_0\int Pdt}\,(K+c\int Pe^{-2cy_0\int Pdt}\,dt) \tag{28}$$

The constant K is to be determined from the known value of Q for t infinite.

The solution for $P=P_0e^{-ft}$ is

$$Q=Q_0\left(\frac{e^{ge^{-ft}}-1}{e^{ge^{-ft}}+1}\right)^2 \tag{29}$$

where

$$g=2cP_0/f\sqrt{Q_0}.$$

The analytical integration is not possible for the three other P functions of table 3, because the integral $\int e^{at^s}dt$ can be obtained, in general, only in power series form.

This analysis shows that the analytical integration for simple expressions $P=f(t)$ can be done for $k=1$ and for $k=0$ in some cases.

5. Discussion

The analysis of the feasibility of fitting the given curves, as parts of differential equation for water storage, by mathematical expressions and its integration by analytical procedures in special cases, show some characteristics of outflow hydrographs.

The shape of outflow hydrographs from lakes and channel reaches is highly influenced by the ratio of exponents of storage function and outflow rating curve, both of the power type $y=sx^p$.

The analytical integration of the storage differential equation can be performed for any ratio of exponents in case of zero inflow, and for many ratios of exponents in case of constant inflow. The integration is, however, limited to a small number of ratios and inflow functions, when the inflow changes with time.

The usual procedure, to plot the recession part of river-flow hydrographs on semi-log paper and to fit it by a straight line, shows that in the majority of cases the fitting of a straight line is a rough approximation. The results of above analytical integration show that the straight line can be used for accurate fitting only if the ratio of exponents of storage function and outflow rating curve has a mean value near unity ($n=m/r=1$) and for zero or constant inflow. This can be considered a special case.

Though the discharge hydrographs of underground water in a river basin can be (theoretically) fitted in many cases during the recession period of no water supply to the underground by an exponential function of type Q_oe^{-ft}, the water storage either in lakes or along river channels influences highly the shape of river hydrographs, so that it becomes less and less of the pure exponential type. The higher the ratio between the effective storage (storage which influences the outflow hydrograph) of lakes and channels to the total river flow during the period of recession flow, and the higher the departure of the ratio of two exponents from unity, the greater is their influence on the shape of river hydrographs and the larger is the departure from the original hydrograph of the recession flow from the underground.

As the ratio $n=m/r$ changes along the river channel, the accurate flood routing procedures must take that fact into consideration.

The described analytical procedure is useful for the study of outflow hydrographs in case of rapid openings in water bodies (breaches of dams, rapid openings of gates and valves, breaches of channel walls or levees, etc.). The assumption of simple shape of openings and of simple inflow hydrographs (zero, constant inflow) is usual in case of dam breaches. As the study of dam breaches has to be made for many openings of quite different dimensions and shapes for different inflow discharges, the analytical integration as given above can have some advantages in the computation of hydrographs in comparison to the standard graphical or numerical procedures of integration.

The difficulties in fitting the background curves by tractable mathematical expressions and the difficulties of integrating analytically the storage differential equation, and especially in case the inflow changes with time, explain why the usual graphical and numerical procedures have taken so common a place in the engineering practice. Nevertheless, the analytical treatment of the storage differential equation can be useful for some problems, can serve for better understanding of relations, and can save efforts in problems, as, for instance, in case of outflow hydrographs out of dam breaches.

The assumptions usually taken for some breaches and for inflow discharge, and the accuracy of basic data in the routing of waves created downstream, justify the use of the procedure given for the fitting of mathematical expressions to the basic and given curves and the use also of the analytical integration of the storage differential equation for the computation of outflow hydrographs.

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