Additional Abscissas and Weights for Gaussian Quadratures of High Order: Values for n=64, 80, and 96

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Abscissas and weights for Gaussian quadratures of orders $n=64$, 80, and 96 are given to twenty decimal places.

In a previous paper,[2] tables of zeros of the Legendre polynomials $P_n(x)$ were published for various values of n up to $n=48$. It was also stated there that values were available for $n=64$, 80, and 96. Despite a brisk demand for these values, they were not published with the others because doubt was cast on their accuracy. It was feared that the round-off error accumulated in using the recursion formula for Legendre polynomials

$$P_n(x)=\left(2-\frac{1}{n}\right)xP_{n-1}(x)-\left(\frac{n-1}{n}\right)P_{n-2}(x) \qquad (1)$$

$$P_0(x)=1, \quad 1\ P_1(x)=x$$

would propagate to such an extent that for high values of n a significant loss of accuracy would result.

These values are now published since we have been able to show that these fears were unfounded. The way this was accomplished was as follows: $P_n(x_{kn}+5.10^{-22})$ $(k=1, \ldots, \frac{n}{2})$ was evaluated on the WEIZAC (Weizmann Automatic Computer), using triple-precision operations, with the thought that if $P_n(x_{kn}+5.10^{-22})$ would be of different sign than $P_n(x_{kn}-5.10^{-22})$, then x_{kn} would be a correct 21-place zero of $P_n(x)$, The only triple-precision operations needed were addition, subtraction, and multiplication. In addition, the triple-precision quotient of two single-precision numbers was needed. An interpretive routine simplified the program for evaluating $P_n(x)$.

The results of this computation were that the SEAC (Standards Electronic Automatic Computer) values were all correct to 21 places with one exception; one value was in error by 1 in the 21st place and has been corrected in the present table.

Because the weights were computed by the formula

$$a_{kn}=\frac{2(1-x_{kn}^2)}{[nP_{n-1}(x_{kn})]^2}\ k=1, 2, \ldots, n/2, \qquad (2)$$

and because $P_n(x_{kn})$ was found to be correct to within 1 unit in the 21st place, it was felt that $P_{n-1}(x_{kn})$ was also correct to 21 places and hence a_{kn} can be considered correct to the 20 places given in the table.

In the tables, only the abscissas lying between 0 and 1 have been listed.

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[2] P. Davis and P. Rabinowitz, Abscissas and weights for Gaussian quadraof high order, J. Research NBS **56**, 35 (1956) RP2645.

n = 64

Abscissas		Weights	
0.9993050417	35772139457	0.0017832807	2169643294
0.9963401167	71955279347	0.0041470332	6056246763
0.9910133714	76744320739	0.0065044579	6897836285
0.9833362538	84625956931	0.0088467598	2636394772
0.9733268277	89910963742	0.0111681394	6013112881
0.9610087996	52053718919	0.0134630478	9671864259
0.9464113748	58402816062	0.0157260304	7602471932
0.9295691721	31939575821	0.0179517157	7569734308
0.9105221370	78502805756	0.0201348231	5353020937
0.8893154459	95114105853	0.0222701738	0838325415
0.8659993981	54092819761	0.0243527025	6871087333
0.8406292962	52580362752	0.0263774697	1505465867
0.8132653151	22797559742	0.0283396726	1425948322
0.7839723589	43341407610	0.0302346570	7240247886
0.7528199072	60531896612	0.0320579283	5485155358
0.7198818501	71610826849	0.0338051618	3714160939
0.6852363130	54233242564	0.0354722132	5688238381
0.6489654712	54657339858	0.0370551285	4024004604
0.6111553551	72393250249	0.0385501531	7861562912
0.5718956462	02634034284	0.0399537411	3272034138
0.5312794640	19894545658	0.0412625632	4262352861
0.4894031457	07052957479	0.0424735151	2365358900
0.4463660172	53464087985	0.0435837245	2932345337
0.4022701579	63991603696	0.0445905581	6375656306
0.3572201583	37668115950	0.0454916279	2741814448
0.3113228719	90210956158	0.0462847965	8131441729
0.2646871622	08767416374	0.0469681828	1621001732
0.2174236437	40007084150	0.0475401657	1483030866
0.1696444204	23992818037	0.0479993885	9645830772
0.1214628192	96120554470	0.0483447622	3480295717
0.0729931217	87799039450	0.0485754674	4150342693
0.0243502926	63424432509	0.0486909570	0913972038

n = 80

Abscissas		Weights	
0.9995538226	51630629880	0.0011449500	0318694153
0.9976498643	98237688900	0.0026635335	8951268166
0.9942275409	65688277892	0.0041803131	2469489523
0.9892913024	99755531027	0.0056909224	5140319864
0.9828485727	38629070418	0.0071929047	6811731275
0.9749091405	85727793386	0.0086839452	6926085842
0.9654850890	43799251452	0.0101617660	4110306452
0.9545907663	43634905493	0.0116241141	2079782691
0.9422427613	09872674752	0.0130687615	9240133929
0.9284598771	72445795953	0.0144935080	4050907611
0.9132631025	71757654165	0.0158961835	8372568804
0.8966755794	38770683194	0.0172746520	5626930635
0.8787225676	78213828704	0.0186268142	0829903142
0.8594314066	63111096977	0.0199506108	7814199892
0.8388314735	80255275617	0.0212440261	1578200638
0.8169541386	81463470371	0.0225050902	4633246192
0.7938327175	04605449949	0.0237318828	6593010129
0.7695024201	35041373866	0.0249225357	6411549110
0.7440002975	83597272317	0.0260752357	6756511790
0.7173651853	62099880254	0.0271882275	0048638067
0.6896376443	42027600771	0.0282598160	5727686239
0.6608598989	86119801736	0.0292883695	8326784769
0.6310757730	46871966248	0.0302723217	5955798066
0.6003306228	29751743155	0.0312101741	8811470164
0.5686712681	22709784725	0.0321004986	7348777314
0.5361459208	97131932020	0.0329419393	9764540138
0.5028041118	88784987594	0.0337332149	8461152281
0.4686966151	70544477036	0.0344731204	5175392879

n = 80, Continued

Abscissas		Weights	
0.4338753708	31756093062	0.0351605290	4474759349
0.3983934058	81969227024	0.0357943939	5341605460
0.3623047534	99487315619	0.0363737499	0583597804
0.3256643707	47701914619	0.0368977146	3827600883
0.2885280548	84511853109	0.0373654902	3873049002
0.2509523583	92272120493	0.0377763643	6200139749
0.2129945028	57666132572	0.0381297113	1447763834
0.1747122918	32646812559	0.0384249930	0695942318
0.1361640228	09143886559	0.0386617597	7407646332
0.0974083984	41584599063	0.0388396510	5905196893
0.0585044371	52420668629	0.0389583959	6276953119
0.0195113832	56793997654	0.0390178136	5630665481

n = 96

Abscissas		Weights	
0.9996895038	83230766828	0.0007967920	65552012429
0.9983643758	63181677724	0.0018539607	88946921732
0.9959818429	87209290650	0.0029107318	17934946408
0.9925439003	23762624572	0.0039645543	38444686674
0.9880541263	29623799481	0.0050142027	42927517693
0.9825172635	63014677447	0.0060585455	04235961683
0.9759391745	85136466453	0.0070964707	91153865269
0.9683268284	63264212174	0.0081268769	25698759217
0.9596882914	48742539300	0.0091486712	30783386633
0.9500327177	84437635756	0.0101607705	35008415758
0.9393703397	52755216932	0.0111621020	99838498591
0.9277124567	22308690965	0.0121516046	71088319635
0.9150714231	20898074206	0.0131282295	66961572637
0.9014606353	15852341319	0.0140909417	72314860916
0.8868945174	02420416057	0.0150387210	26994938006
0.8713885059	09296502874	0.0159705629	02562291381
0.8549590334	34601455463	0.0168854798	64245172450
0.8376235112	28187121494	0.0177825023	16045260838
0.8194003107	37931675539	0.0186606796	27411467385
0.8003087441	39140817229	0.0195190811	40145022410
0.7803690438	67433217604	0.0203567971	54333324595
0.7596023411	76647498703	0.0211729398	92191298988
0.7380306437	44400132851	0.0219666444	38744349195
0.7156768123	48967626225	0.0227370696	58329374001
0.6925645366	42171561344	0.0234833990	85926219842
0.6687183100	43916153953	0.0242048417	92364691282
0.6441634037	84967106798	0.0249006332	22483610288
0.6189258401	25468570386	0.0255700360	05349361499
0.5930323647	77572080684	0.0262123407	35672413913
0.5665104185	61397168404	0.0268268667	25591762198
0.5393881083	24357436227	0.0274129627	26029242823
0.5116941771	54667673586	0.0279700076	16848334440
0.4834579739	20596359768	0.0284974110	65085385646
0.4547094221	67743008636	0.0289946141	50555236543
0.4254789884	07300545365	0.0294610899	58167905970
0.3957976498	28908603285	0.0298963441	36328385984
0.3656968614	72313635031	0.0302999154	20827593794
0.3352085228	92625422616	0.0306713761	23669149014
0.3043649443	54496353024	0.0310103325	86313837423
0.2731988125	91049141487	0.0313164255	96861355813
0.2417431561	63840012328	0.0315893307	70727168558
0.2100313104	60567203603	0.0318287588	94411006535
0.1780968823	67618602759	0.0320344562	31992663218
0.1459737146	54896941989	0.0322062047	94030250669
0.1136958501	10665920911	0.0323438225	68575928429
0.0812974954	64425558994	0.0324471637	14064269364
0.0488129851	36049731112	0.0325161187	13868835987
0.0162767448	49602969579	0.0325506144	92363166242

WASHINGTON, January 27, 1958.