RP262

PRISM SIZE AND ORIENTATION IN MINIMUM-DEVIATION REFRACTOMETRY

By L. W. Tilton

ABSTRACT

An investigation of the optical error in the pointing of telescopes, and of variations therein with width of objective aperture, indicates that the effect of marginal aberration is often sufficiently deleterious to seriously impair the advantages which should result from the use of the larger apertures. The utility of larger prisms for more accurate index measurement is, therefore, subject to question. The experienced ratios of the probable error in pointing to the limit of angular resolution are, moreover, smaller than was expected for the narrower effective widths of telescope apertures and frequently they do not exceed 1/30. For sixth decimal place determinations of refractive index by minimum-deviation measurements when using a 60° prism, it is shown that a suitable and sufficient length of prism surface is approximately 2 cm, a constant value for all transparent media.

The systematic errors in measured index which result from improper orientation of a prism for minimum deviation are discussed. It is found that the residuals in prism azimuth as determined by simple hand and eye adjustment have an approximately normal distribution, and that their average values do not vary greatly with wave length. The requisite precision in azimuthal orientation is quantitatively established and the corresponding tolerances for prisms having various indices and refracting angles are expressed by contours. These curves may be used for evaluating prism-orientation corrections to index measurements. It is concluded that large telescope apertures and special procedures for securing accuracy in prism orientation are unnecessary in minimum-deviation measurements leading to an approximate precision of P. E. $= \pm 1 \times 10^{-6}$ in the computed index of refraction.

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I. INTRODUCTION

In a previous paper[1] on certain aspects of prism refractometry it was assumed that the errors of telescope pointings do not constitute a serious problem even when seeking to attain a precision extending to the sixth decimal place in index of refraction. In the design of spectrometers for high accuracy in index measurement, however, the desirability of increasing the pointing accuracy by providing rather wide apertures in the telescopic system is quite generally implied and sometimes it is definitely stated that the greater widths are necessary for obtaining sufficient precision in angular measurements. Since effective aperture is dependent on the length or horizontal extent of the polished vertical surface of the prism, it therefore becomes of considerable importance to determine the requisite dimensions of a prism for index measurements of a stipulated accuracy by means of a given number of minimum-deviation observations. In the investigation of prism size, and also in the related problem of establishing the necessary precision in the azimuthal orientation of a prism, the most important factors to be considered are the accuracy with which telescopes may be pointed and the degree of homogeneity of the refracting media.

The first of these factors, the metrological power of telescopes, has not received adequate attention in spectometry, perhaps because it has usually been second in importance to the question of homogeneity. It appears, however, that several types of optical glass may be improved in the degree of their index uniformity [2] by a modified heat-treatment procedure, a carefully executed isothermic annealing,[3] and consequently it now seems appropriate to consider the questions of prism size and orientation from the standpoint of the metrological accuracy of telescope pointings.

Although the accuracy and precision which are discussed in this paper are based on optical considerations only, it should be noticed that no general assumption concerning an equivalent excellence in the mechanical design of a spectrometer is directly implied. For example, the maximum prism size which it is proposed to establish is not a value necessarily required for optimum work on a given instrument. It should be viewed simply as an upper limiting size which need not be exceeded when seeking a specified accuracy of index measurement. Moreover, the required systematic correction to index measurements which involve ordinary procedures in prism orientation is, for a given observer, approximately independent of errors other than those which are optical in nature and affect telescope pointing.

The symbols used are explained in the text, but for convenience of reference the definitions of the various characters are summarized here as follows:

$A \equiv$ the refracting angle of a prism;

$D \equiv$ the angle of minimum deviation produced by a prism;

$\Delta D \equiv$ a small departure from minimum deviation which is caused by an error in prism orientation;

$i \equiv$ the angle which an incident ray makes with the normal to the face of a prism;

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[1] L. W. Tilton, B. S. Jour. Research, **2** (RP64), p. 923; 1929.

[2] L. W. Tilton, Variations in the Optical Density of Glass, Meeting, Philosophical Soc., Washington, May, 1929. Abstract in J. Wash. Acad. Sci., **20**, pp. 12–13; 1930.

[3] L. W. Tilton, A. N. Finn, and A. Q. Tool, B. S. Sci. Papers (No. 572), **22**, pp. 719–736; 1927–28. See also J. Am. Ceram. Soc., **11**, pp. 292–295; 1928.

$\Delta i \equiv$ a small error in the orientation of a prism for minimum deviation;

$K \equiv$ the ratio between limiting precision of angular measurement and limit of angular resolution;

$L \equiv$ the length or horizontal dimension of a prism face or surface;

$L_D \equiv$ the maximum prism surface length which is required for (double) minimum-deviation index measurements having an accuracy of $\pm 1 \times 10^{-6}$;

$\lambda \equiv$ wave length;

$m \equiv$ the limiting precision of angular (telescopic) measurement expressed in terms of (the absolute value of) the probable error of a single pointing;

$m' \equiv$ the probable error of the mean of N observations of an elemental angular magnitude in minimum-deviation refractometry;

$N \equiv$ the number of complete determinations of an angular magnitude;

$N_A \equiv$ the number of measurements of the refractive angle of a prism from the mean of which there may be computed an index having a probable error of $\pm 1 \times 10^{-6}$;

$N_D \equiv$ the number of determinations of (double) minimum deviation from the mean of which there may be computed an index having a probable error of $\pm 1 \times 10^{-6}$;

$n \equiv$ the relative index of refraction of a medium;

$r \equiv$ the limit of angular resolution with a telescope;

$T_A \equiv$ the tolerance, in measuring refracting angle, which corresponds to an error of $\pm 1 \times 10^{-6}$ in the measurement of refractive index;

$T_{2D} \equiv$ the tolerance, in measuring double minimum deviation, which corresponds to an error of $\pm 1 \times 10^{-6}$ in the measurement of refractive index;

$T'_{\Delta i} \equiv$ the tolerance in each prism orientation which corresponds to an error of $+1 \times 10^{-6}$ in the index computed from a single observation of double deviation; and

$T_{P.E.i} \equiv$ the tolerance in precision of a series of prism orientations which corresponds to an error of $+1 \times 10^{-6}$ in the index computed from the average observed minimum deviation.

II. ON THE PRECISION OF TELESCOPE POINTINGS

Both Michelson [4] and Wadsworth [5] have discussed the limiting precision of angular measurement or metrological limit, m, which may be attained with a telescope, and have related it to the limit of angular resolution, r, in the form of an equation

$$m = Kr \tag{1}$$

where K is a factor to which Michelson assigns a value of 1/4 or 1/5 while Wadsworth cites experimental evidence showing that certain errors in telescope pointings correspond to a value as small as 1/15 for this parameter.

[4] A. A. Michelson, Am. J. Sci., **39**, p. 115; 1890. Nature, **49**, pp. 56–60; 1893–94.
[5] F. L. O. Wadsworth, Phil. Mag., **44**, pp. 83–97; 1897. Astrophys, J., **16**, pp. 267–270; 1902.

It is well known that the value of K is greatly influenced by certain conditions of measurement, particularly by the degree of magnification which is used; also such matters as contrast and intensity of illumination are sometimes mentioned as important. The effect of spherical aberration in causing variations [6] in K is another matter to be considered. This subject first engaged the writer's attention because of certain favorable results which were obtained in measuring angles and indices of refraction of small prisms. Later, when using small apertures under relatively unfavorable conditions of illumination, it was found that the probable errors of a single micrometric pointing corresponded to values of K as small as 1/40, whereas with the full apertures of the same lens systems the values ranged from 1/11 to 1/17. Incidentally, these last-named values of K are in good agreement with the above-mentioned limits which were named by Wadsworth and which, presumably, were also obtained at full apertures.

1. RATIO OF METROLOGICAL AND RESOLUTION LIMITS

The spectrometer objectives which were used for taking the data of Figure 1 have clear apertures of 47 mm and focal lengths of 40 cm. They give good optical definition and have "full theoretical" resolving power, even at full apertures. The eyepiece has a focal length of 11 mm so that the total magnification of the telescope is approximately 37; a value which has hitherto been considered sufficiently high for a telescope of this size. In most instances a vertical spider hair was mounted in the plane of a wide collimator slit and 60° crossed spider hairs were used as the movable reference marks in the focal plane of the telescope. The precision of pointings made with a narrow bright-line slit as the object does not, however, differ very significantly from that obtained when using the spider hair object, as is shown by a comparison of curves A and B of Figure 1.

In view of the possible effects of the intensity of illumination,[7] curve C was obtained for an electric light source which was very bright as compared with the monochromatic radiation ($\lambda = 5461$A) used for the other curves. It seemed, therefore, that variations in intensity were not of importance [8] in this case but the various apertures were, nevertheless, made of equal area (see fig. 1) so that equal intensities were employed when varying the resolving power by these apertures of different widths.

The effect of eyestrain and fatigue of the observer in observations of this character is thought to be so important that but little significance should be attached to the differences in absolute value as depicted by the different levels of these three curves of Figure 1. Moreover, it should be mentioned that the curves are drawn directly through the plotted points, not because it is thought that the slopes are determined with a corresponding accuracy, but merely to avoid the personal element which would affect a smoothing process.

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[6] Since K is merely a ratio between certain angular limits, its variations with aperture do not in themselves give any information concerning variations in the absolute values of the pointing errors. It may, however, be assumed that the degree of its constancy is an indication of the type of aberration correction which has been achieved in an objective.

[7] A. Pelzer, Zeitschr. für Instrumentenkunde, 46, p. 374; 1926.

[8] The plotted points for curve C depend on only 6 sets of 10 observations each as compared with 8 and 10 similar sets for the other curves.

2. EFFECTS OF ABERRATION AND MAGNIFICATION

The curves of Figure 1 indicate that, with the objectives used, an appreciable increase in pointing accuracy can not be gained by an increase in prism size. Although the use of higher eyepiece magnification was found to be advantageous at the medium and larger apertures, values of K more favorable than 1/17 were not reached for the full objective width. The data thus seem to call in question the validity of the assumption that metrological power is directly related to resolution. The only explanations herein offered, however, are (1) that the values of K which are optically attainable in angular measure-

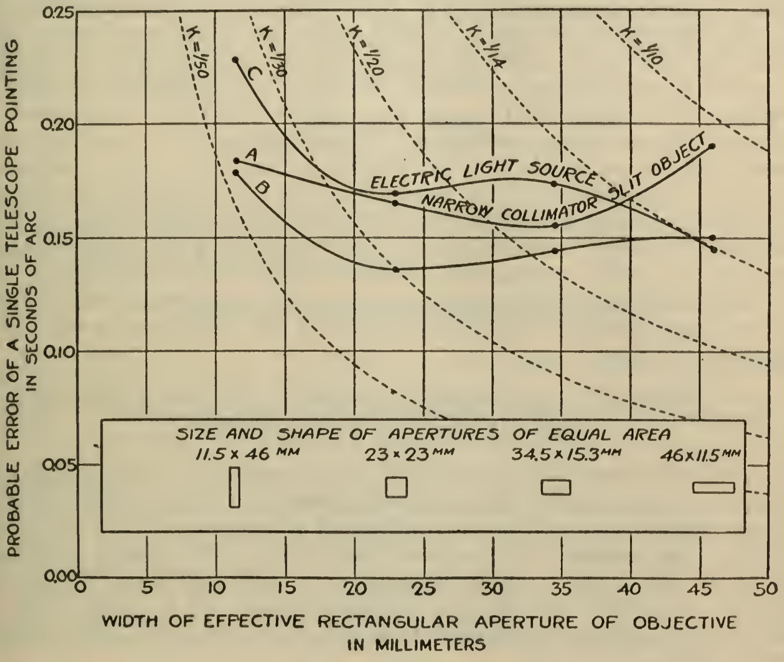

FIGURE 1.—*Nonconstancy in the ratio, K, of probable error in telescope pointing to Dawes' limit of resolution*

These data were obtained with spectrometer objectives having focal lengths of 40 cm. The focal length of the eyepiece was 11 mm. With the exceptions noted, monochromatic light (λ=5,461A) was employed and the object sighted was a vertical spider hair in the collimator slit. Objective apertures of equal area were used, as shown in the inset.

ments with a telescope are much more favorable than have hitherto been explicitly reported in the literature, and (2) that aberration of the outer zones of the lenses plays a conspicuous part in the metrological performance of telescope objectives.[9]

In fact, a confirmation of these views seems to be contained in the data which Noetzli [10] published when determining the variation of pointing accuracy with magnification. By reducing his mean square

[9] In this connection it is of interest to refer to the experiments reported by Guild (Trans. Opt. Soc., **31**, pp. 1–14; 1929–30) on the precision of optical settings made by means of the unaided eye. The fact that his results are approximately independent of pupillary aperture may, perhaps, also be due to a masking effect of increased aberration when using the wider apertures.

[10] A. Noetzli, Zeitschr. für Instrumentenkunde, **35**, pp. 65–73; 1915.

errors to probable errors and using his experimentally determined limits of resolution (which seem unaccountably small) the values of K may be computed and they are found to range from 1/30 to 1/60 for the reduced apertures which were used in pointings with the 41 cm objective, but they are almost as large as 1/20 for the full apertures of the 51 cm focal length objective.

Moreover, further investigations in progress in this laboratory indicate that 1/25 or 1/30 are values of K which are approximately constant and, from an optical standpoint, are easily attainable when using typical [11] spectrometer objectives of 40 or 50 cm focal length for measurements on prisms of any reasonable size provided they do not utilize more than approximately two-thirds of the clear apertures of the lens systems. In some of this work two observers have been employed and their average results are in satisfactory agreement. It should be emphasized, however, that it is advantageous to employ eyepieces of shorter focal length than those which are considered as sufficient for optimum resolution. When using apertures of approximately 2.5 cm in width and the typical objectives above mentioned (40 or 50 cm focal length) it seems that, for obtaining the maximum accuracy in optical pointings, the total telescope magnification should be at least as high as 40 or 50. The optimum total magnification does not, however, necessarily increase with width of aperture. In fact, these optimum values appear, in general, to be greater when using intermediate apertures than for the full or for very narrow widths of effective openings.

III. NUMBER OF OBSERVATIONS AS RELATED TO PRECISION OF POINTINGS

In order to determine the appropriate number, N, of observations which should be used in making, with a prism of given surface length, L, and refracting angle, A, a number of measurements of refractive index, n, the average of which is expected to have a given precision, say P. E. $= \pm 1 \times 10^{-6}$, it is necessary to consider both refractive-angle and minimum-deviation measurements. It is convenient to recall [12] appropriate elemental tolerances in refractive-index goniometry, such as

$$T_A = \pm 0.413 \frac{\sin^2 A/2}{\sin D/2} \text{ seconds} \tag{2}$$

for measurements on the refracting angle, and

$$T_{2D} = \pm 0.825 \frac{\tan\left(\frac{A+D}{2}\right)}{n} \text{ seconds} \tag{3}$$

[11] The adjective "typical" is used to designate objectives which have an approximately constant value of K, at least over two-thirds or three-fourths of their apertures. Such lens systems permit a real increase in pointing accuracy as effective width of aperture is increased, whereas it seems possible that objectives of special or unusual design may favor the use of small apertures at the expense of the accuracy which can be attained with the larger objective widths. The 40-cm objectives, used in the observations represented in Figure 1, appear to be examples of the special class, and it is suggested that the 41-cm objective used by Noetzli may also be of the same character. The metrological performance of such lens systems may be summarized as unusually good at small apertures and somewhat worse than normal at full apertures. The defective performance at the larger apertures may be aggravated by the use of insufficient eyepiece magnification. This is known to be the case in Figure 1 because the total magnification of 37 was afterward found inadequate except at the 11.5-mm aperture. Optical systems having this special character of correction are especially desirable in spectrometry for work on small prisms, but for larger prisms they may decrease the actual precision of pointings.

[12] See p. 921 of paper cited in footnote 1.

for observations of double deviation. These tolerances correspond to an accuracy of $\pm 1 \times 10^{-6}$ in index measurements. Moreover, it will be assumed that the telescope system is sufficiently free from aberration to permit a resolution increasing at the theoretical rate with the widths of the rectangular apertures actually employed; that is, a limit of resolution expressible as

$$r = \frac{\lambda}{L \cos i} \tag{4}$$

where i is the angle which the incident rays make with the normal to the face of the prism. Then from equations (1) and (4) the probable error, arising solely from the optical pointings made during N complete determinations of an angular magnitude, such as that of double minimum deviation or of prism angle, is

$$m' = \pm \frac{206 \times 10^3 \, K\lambda\sqrt{2}}{L \cos i\sqrt{N}} \text{ seconds} \tag{5}$$

where $\sqrt{2}$ appears in the numerator because both positive and negative errors in telescope pointings are equally probable at the right and at the left-hand observational positions.

1. REFRACTIVE-ANGLE MEASUREMENTS

It is found that prism size is not a troublesome factor in prism-angle measurements because repetition is in this case quite useful in the reduction of the errors. It is desirable, however, to express the relationship which should exist between the number of observations, N, and the prism length, L, for a requisite precision of refractive angle measurement in minimum-deviation goniometry.

The elemental tolerance equation (2) is applicable in this case, and equation (5) applies to the measurement of prism angles by the method of autocollimation provided the appropriate value of 0° is assigned to the angle i. By imposing suitable tolerance conditions one may equate the right-hand members of equations (2) and (5) and obtain

$$N_A = \left(\frac{7 \times 10^5 \, \mathrm{K} \, \lambda \sin \mathrm{D}/2}{L \sin^2 A/2} \right)^2 \tag{6}$$

which is a general statement of the appropriate relationship between N_A and L. The subscript, A, is used to denote that this particular value of N is determined by the precision required in the refractive-angle measurements. The presence of D in equation (6) shows that the index, as well as the angle of the prism, is involved in determining, from the standpoint of angle-measuring accuracy, the necessary number of observations for use in minimum-deviation refractometry. Equation (6) is optically consistent with probable errors in the mean value of the refractive angle which, in turn, are equivalent to $\pm 1 \times 10^{-6}$ in the computed index of refraction. If it be assumed that the magnification, the contrast conditions in the image, and the aberration of the prism are consistent with a normal value of K, then the proper N_A and L relationship for use in sixth decimal place refractive-index measurements by the minimum-deviation method is numerically determinable. Considering 0.55μ as the effective wave length of white light (which is used in prism-angle measurements), and using for K the value 1/25 which, according to the author's experience (see Section II), is fairly conservative, equation (6) becomes

$$N_A=\left(\frac{6.2 \sin D/2}{L}\right)^2 \tag{7}$$

for 60° prisms, if L is expressed in centimeters. When using indices of 1.4, 1.6, and 1.8, respectively, it will, therefore, be found that prism surface lengths (or horizontal dimensions of the polished vertical faces) of 1.5, 2.4, and 3.5 cm are ample, even if but one measurement of prism angle is to be made.

2. MINIMUM-DEVIATION MEASUREMENTS

The proper relationship between number of observations, prism size, and error in minimum-deviation measurement is dependent on the use or nonuse of special procedures for prism orientation in azimuth and these two cases should be considered separately. The general case in which no special procedures are used is of more importance than the special case because the usefulness of repeated observations is rigorously limited and thus more severe requirements are imposed on prism size. Since, however, the general case involves a consideration of the systematic errors in prism orientation (see Section IV), its discussion is deferred and appears in Section V.

In the special case, correct prism azimuth being assured, the only optical errors involved are the accidental errors of the two telescope pointings. By imposing suitable tolerance conditions one may equate the right-hand members of equations (3) and (5) and obtain a result which may be written as

$$N_D=\left(\frac{35\times 10^4 K\lambda}{L \sin A/2}\right)^2 \tag{8}$$

for the particular conditions which exist at the minimum-deviation position of a prism. The subscript, D, is used to denote that this value of N is determined by considerations of the requisite precision in deviation measurements. Equation (8) expresses a relationship between N and L which is optically consistent with probable errors in the mean angle of double deviation which, in turn, are equivalent to $\pm 1\times 10^{-6}$ in the computed index of refraction. Obviously there is under these conditions no severe requirement concerning prism size because of the important effect of a few repetitions in measurement. Considering, however, only the unfavorable case of long wave length radiation ($\lambda=0.7\mu$), and again using for K the value 1/25, equation (8) becomes approximately

$$N_D=\left(\frac{4}{L^2}\right) \tag{9}$$

for 60° prisms, provided L is expressed in centimeters. Consequently, when making but a single observation of double deviation, it is found that a 2-cm length of prism surface is all that is required. It should be noticed that this result is independent of the value of the deviation.

IV. PRISM ORIENTATION IN AZIMUTH AND ITS RELATION TO DEPARTURES FROM MINIMUM DEVIATION

The precise orientation of a prism for minimum deviation is a rather troublesome procedure, and the matter is sometimes referred to as a disadvantage of this method of index measurement. The difficulty arises because the errors in orientation, irrespective of their sign, give rise to corresponding departures from minimum deviation which are always positive. Furthermore, upon repeating the observations, the average of all of these errors in the deviation tends toward some definite value which is not zero.

In order to overcome this difficulty Hastings [13] computed the proper prism orientations and set the table accordingly after determining one reference position experimentally. Macé de Lépinay [14] noted two slightly different orientations which produced identical total deviation and then used the mean azimuth of these two orientations. Schönrock [15] computed the correct angles of incidence from preliminary data and then used reflections from the prism surfaces in making precise orientations of the prism.

Most of these special procedures require, or are facilitated by, a circle for measuring the azimuth of the prism table; a feature which is often lacking. When required to make this prism adjustment without the measurement of table orientation, one may follow Carvallo [16] and Gifford,[17] each of whom utilized reflections from the polished third side of a prism, the former even making corrections for nonisoscelism of the prism and for refocusing. This artifice, however, will not always work when the prism is placed on the table in such manner as to permit the symmetrical use of all apertures. For example, the intensity of the reflected image may be sufficiently great to interfere with precise pointings on the refracted image; or, on the other hand, for large prisms, the reflected image may be entirely absent because the necessary rays do not enter the objective. Consequently, it is desirable to quantitatively investigate the magnitude of the systematic errors which occur in practice and also to determine the requisite precision in prism table orientation in order to determine the exact circumstances in which special methods of prism orientation are necessary.

1. EXPERIENCED PRECISION OF SIMPLE ORIENTATION

Considering, then, the attainable degree of precision in the orientation of a prism, by simple hand adjustment and by observation of the deviated image, it appeared, as stated by Wadsworth,[18] quite possible to attain such precision that the error in minimum deviation does not exceed the approximate limiting metrological accuracy of the effective telescope aperture. This limit, as stated in Section II, 2, is more favorable than has hitherto been recognized; and, in the experience of two observers in this laboratory, is almost always less than ±0.2 seconds as measured by the probable error of a single pointing.

[13] C. S. Hastings, Am. J. Sci., **15**, p. 272; 1878.
[14] J. Macé de Lépinay, Annales de Chimie et de Physique (7), **5**, p. 227; 1895.
[15] O. Schönrock, Zeitschr. für Instrumentenkunde, **40**, p. 95; 1920.
[16] E. Carvallo, Annales de l'Ecole Normale Superieure (3), **7**, supplement, pp. 75 and 93; 1890.
[17] J. W. Gifford, Proc. Royal Soc., London, **70**, p. 330; 1902.
[18] F. L. O. Wadsworth, Astrophys. J., **16**, p. 269; 1902; **17**, p. 11; 1903.

It thus seemed improbable,[19] in the practice of prism refractometry, that the errors resulting from prism-orientation inaccuracies would exceed $+1\times10^{-6}$ in index. In fact, Schönrock [20] reports for certain data an average error of $+0.25''$ in D, due to prism orientation in the usual way, that is, a value which is equivalent to less than $+1\times10^{-6}$ in refractive index.

An extended series of prism-orientation error determinations has been made by the writer in order to investigate the above-expressed views. These experiments were made on a prism of 45° refracting angle with an index of 1.52 for sodium light and included the use of various wave lengths and several different aperture widths of the

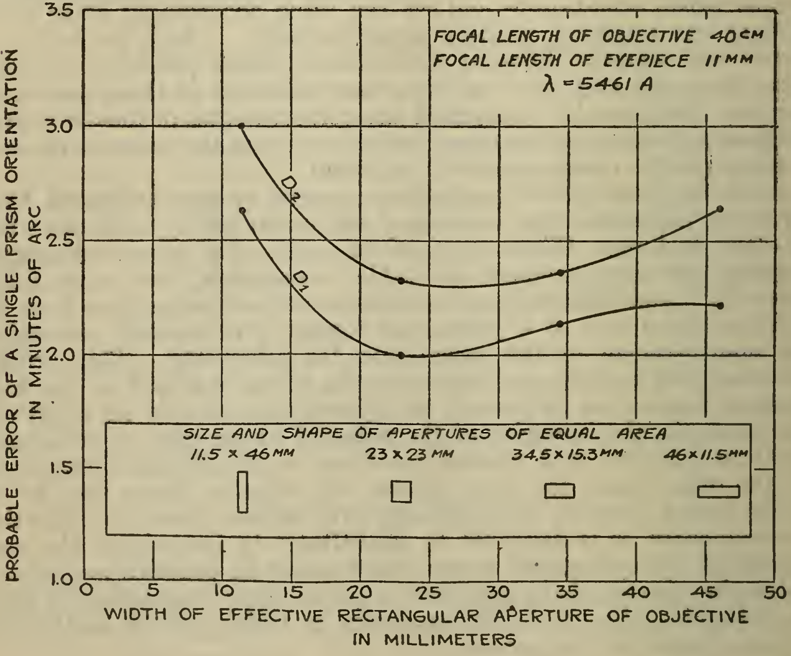

FIGURE 2.—*Experimental precision of prism orientation for various apertures*

The data for these curves were taken under conditions identical with those which were employed when obtaining curve B of Figure 1. The observations for curve D_2 were taken somewhat more rapidly than those for curve D_1 and also with greater friction of the prism-table axis. Each plotted point depends on 8 sets of observations having 10 values in each. The prism has a refractive angle of 45° and an index of 1.52 for sodium light.

telescope objective. A plane parallel plate was mounted on the prism table, approximately normal to the rays emerging from the prism. An autocollimating device and a scale in the focal plane of the (stationary) telescope permitted rapid and accurate readings which corresponded to the several azimuths in which the prism was successively adjusted.

(a) VARIATION WITH APERTURE

The apertures used in these orientation experiments are the ones used in the pointing error investigation which is discussed in Section II. Some results of this work are exhibited in Figure 2 which

[19] See figure 2, p. 922 of paper cited in footnote 1.

[20] See footnote 15, p. 67.

shows the probable error in orientation, *P. E. i*, as a function of objective aperture. The observations for curve D_1 of this figure and for curve *B* of Figure 1 were made alternately in order to secure strictly comparable working conditions, including such factors as eye-strain and fatigue of the observer. A marked similarity in the slopes of these curves (*B* and D_1) suggests an intimate connection between prism orientation and pointing precision. This relationship is investigated and quantitatively evaluated in Section V, 1. Some preliminary data, taken rapidly and with an unnecessarily large amount of friction of the prism-table axis, yielded somewhat less precision and are represented by curve D_2.

(b) VARIATION WITH WAVE LENGTH

A knowledge of the manner in which orientation error varies with wave length is necessary in order to reach a decision concerning the

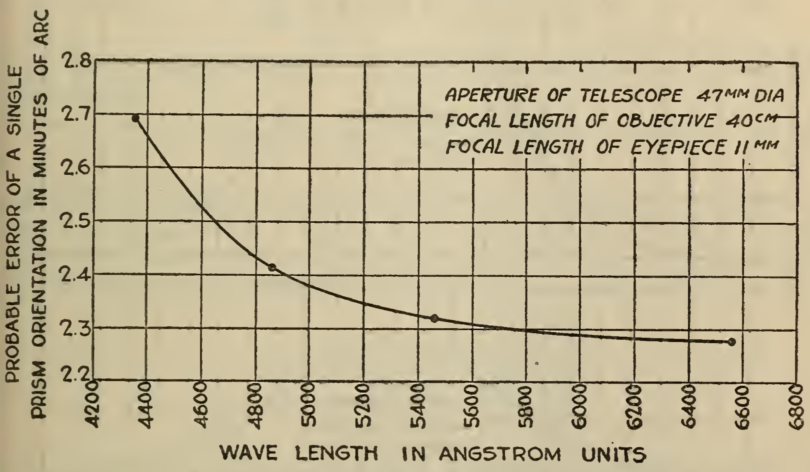

FIGURE 3.—*Experimental precision of prism orientation for various wave lengths*

The advantage of increased metrological power which should result from the use of shorter wave lengths is more than balanced by the deleterious influences of aberration, low visibility, and chromatic parallax.

usefulness of employing special procedures for obtaining accurate orientation. The curve of Figure 3 shows some values of *P. E. i* which were obtained with the previously mentioned 45° prism of 1.52 index of refraction. For each of the 4 wave lengths used these data result from 8 sets consisting of 10 observations each. The increased error corresponding to the use of shorter wave lengths is contrary to expectations. In agreement with this result, however, the writer has also found that errors in micrometric telescope pointing are larger when using radiation of the higher frequencies. It is suggested that relatively greater aberration and lower visibility, together with certain difficulties due to chromatic parallax,[21] may be somewhat more than sufficient to offset the advantages of increased resolution resulting from the shorter wave lengths.

[21] J. Guild, Proc. Phys. Soc., London, 29, pp. 311–337; 1916–17; or Nat. Phys. Lab., Collected Researches, 4, pp. 251–270; 1920.

(c) NORMALITY IN DISTRIBUTION OF RESIDUALS

The probable errors shown in Figures 2 and 3 were computed from the average residuals by Peters' formula and thus involve the assumption that the errors in orientation have a normal distribution. Moreover, in part 2 of this section it is found that if such an error distribution exists, the evaluation of a tolerance in the precision of prism orientation is facilitated. Consequently, in lieu of an assumption concerning this distribution, there is presented in Figure 4 some experimental data on the character of the error curve which is obtained when making prism orientations by the ordinary methods.

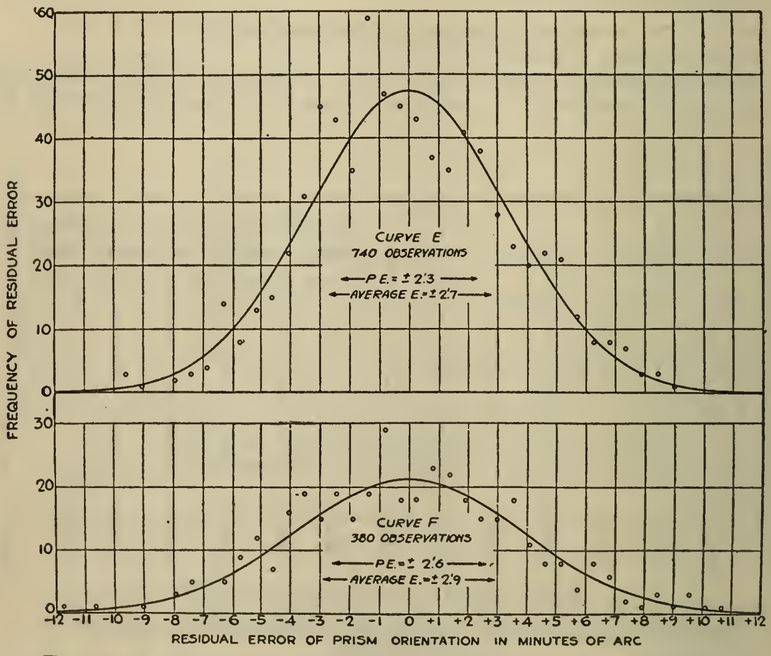

FIGURE 4.—*Normality in distribution of prism orientation residuals*

The plotted points show the number of residuals which were experienced upon grouping the data to the nearest half turn of a mircrometer screw for which one turn represents 1.09 minutes of arc. Thus each group extends over an interval of 0.54 of a minute in azimuth. The curves show the theoretical distribution of errors according to the probability equation. The probable errors, as originally obtained by determining the widths of the 50 per cent zones for the individual residuals agree satisfactorily with values computed from the second moments of the grouped data.

A total of 1,120 observations was available for use in this test of the normality in the distribution of the errors in prism orientation. Four different wave lengths had been used and five sizes of aperture were represented. Using the method of grouping, these data were first treated as one composite whole and afterwards separated into the two main divisions, one of a higher and the other of a lower precision, which give rise to the curves E and F of Figure 4. On both of these curves, and also for the composite curve which is not reproduced, the total number of negative residuals is slightly in excess of that for the positive residuals and the residual group in which the largest number of residuals is found is also of negative sign. Aside from this slight and negligible[22] skewness, which indicates the writer's tendency to favor small excess clockwise orientations, and also to obtain moderately large positive residuals, the agreement of the experimentally determined points with the normal curves is very satisfactory.[23]

2. TOLERANCE IN PRECISION OF ORIENTATION

The deviation produced by a prism is necessarily rather insensitive to changes in angle of incidence, i, under the conditions which obtain when the prism is at or near the position of minimum deviation. This fact is quantitatively shown[24] by the equations

$$\frac{dD}{di}=0$$

and

$$\frac{d^2D}{di^2}=2\tan i\left(1-\frac{\tan^2 A/2}{\tan^2 i}\right) \tag{10}$$

from which, using Taylor's theorem, it is possible to express[25] the relationship

$$\Delta D=\tan i\left(1-\frac{\tan^2 A/2}{\tan^2 i}\right)\Delta i^2 \tag{11}$$

which exists between small departures from minimum deviation and errors in orientation. Upon substituting for ΔD of equation (11) the appropriate elemental tolerance (in radians), namely, one-half of that expressed (in seconds) by equation (3), there results

$$\Delta i^2=\frac{2\times 10^{-6}\tan\left(\frac{A+D}{2}\right)}{n\tan i\left(1-\frac{\tan^2 A/2}{\tan^2 i}\right)}$$

and, since at minimum deviation i is identical with $(A+D)/2$ and $\sin A/2=n^{-1}\sin i$, one readily finds a particular value of Δi which may be expressed as the tolerance

$$T'_{\Delta i}=\pm 4.86\cos A/2\left(\frac{n}{n^2-1}\right)^{1/2}\text{ minutes} \tag{12}$$

which applies to each of the two prism orientations involved in a single observation of 2 D.

It is seldom possible, however, to obtain desirably precise results from a single observation of minimum deviation, and the errors which

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[22] For both exhibits of Figure 4 the median and the mode are slightly negative with respect to the mean but the quantitative measures of the asymmetry which were calculated from the third moments, although positive in both cases, scarcely exceed their standard errors. For curves E and F, respectively, the values are +0.10±0.09 S. D. and +0.12±0.13 S. D.

[23] The quantitative measures of the symmetrical departure from the normal form, as calculated from the fourth moments, are negative in each case, but for curves E and F, respectively, the values are only −0.31±0.18 S. D. and −0.09±0.25 S. D. Moreover, the use of Elderton's Table of Goodness of Fit gives probabilities of 0.23 and 0.73, respectively, for these curves and thus indicates that the data conform to a normal distribution about as well as could be expected if the samples were taken from a perfectly normal universe.

[24] See, for example, J. P. C. Southall, Geometrical Optics, p. 82; 1910 (New York). There, however, in the last equation on that page, 2 tan $\beta/2$ should appear instead of sin β.

[25] Assuming sufficiently small departures from the minimum-deviation prism position, equation (11) agrees with Cornu's expression for excess deviation (Annales de l'Ecole Normale Superieure (2), **9**, p. 100; 1880).

are decreased directly as the square root of the number of observations are not the errors in deviation, but are those errors which are made in average azimuth of the prism. This average azimuth rapidly approaches the correct azimuth but it does not, however, have any comparable corresponding value of the deviation. Consequently the useful tolerance to establish, when investigating this error in minimum-deviation refractometry, is not one for a single error in prism orientation or for an average orientation but a tolerance which relates properly to one-half of the average departure from double minimum deviation,[26] namely, to $\Sigma\Delta D/N$. Since the values of Δi correspond to a normal distribution of residuals, a suitable tolerance is easily found. This becomes apparent when the process of summation is applied to equation (11) in evaluating the average departure

$$\frac{\Sigma\Delta D}{N}=\tan i\left(1-\frac{\tan^2 A/2}{\tan^2 i}\right)\cdot(1.483\ P.\ E.\ i)^2 \tag{13}$$

in minimum deviation. The quantity $\Sigma\Delta i^2/N$ is, for a normal distribution, the second power[27] of the "standard deviation" or "error of mean square" and thus is expressed as $(1.483\ P.\ E.\ i)^2$, where $P.\ E.\ i$ is the probable error of a single orientation of the prism. Then from equations (3) and (13) one finds the tolerance

$$T_{P.\,E.\,i}=\pm 3.28\cos A/2\left(\frac{n}{n^2-1}\right)^{1/2}\text{ minutes} \tag{14}$$

which is required in the precision of a series of 2 N prism orientations, namely, a tolerance which corresponds to an absolute error of $+1\times10^{-6}$ in the index resulting from one-half the average of all of the N measured values of double minimum deviation. From equation (14) there have been computed the values which are plotted in Figure 5 to show, for a wide range of refractive angle, the variation in this tolerance for work on media of various indices.

3. CORRECTION FOR PRISM-ORIENTATION ERROR

Because the investigations of prism-orientation error which are reported in this paper were made by a single observer; and, because of the variety of possible instrumental and observational conditions under which minimum-deviation measurements may be undertaken, the results are not sufficiently extensive to permit of conclusive, wide generalization. As has been mentioned, however, these orientation errors are simply related to pointing errors and certain unpublished data show that a second observer's pointing errors are at least as small as those on which this paper is based. Therefore, it appears that one may easily so arrange the working conditions that special orientation procedures are unnecessary in sixth decimal place refractometry.

Consequently, before adopting a special program for increased accuracy in orientation under any given set of experimental conditions, it is advisable to determine by N observations the probable error of a

[26] The deviation observed is 2 D, but the systematic departure therefrom, due to the two prism orientations, is $2\Delta D$ and not $\sqrt{2}\Delta D$ (where ΔD is the departure from minimum deviation for a single prism orientation), because, as previously mentioned, every ΔD has a positive sign. Hence, one-half of the average departure from double minimum deviation is simply $\Sigma\Delta D/N$.

[27] See a text on error theory; for example, E. T. Whittaker and G. Robinson, The Calculus of Observations, p. 182; 1924 (London).

single prism orientation by ordinary hand adjustment and then to use Figure 5 for determining the sufficiency of the precision which is thus attainable. In fact, an observer who has initially determined, with a given spectrometer, his probable error in prism orientation for various spectral lines and prism sizes may thereafter consult the curves of Figure 5 and apply the proper index corrections. For the orientation precision shown by the curves, the correction is -1×10^{-6} in index and for other values of the precision the correction varies directly as $(P.\ E.\ i)^2$.

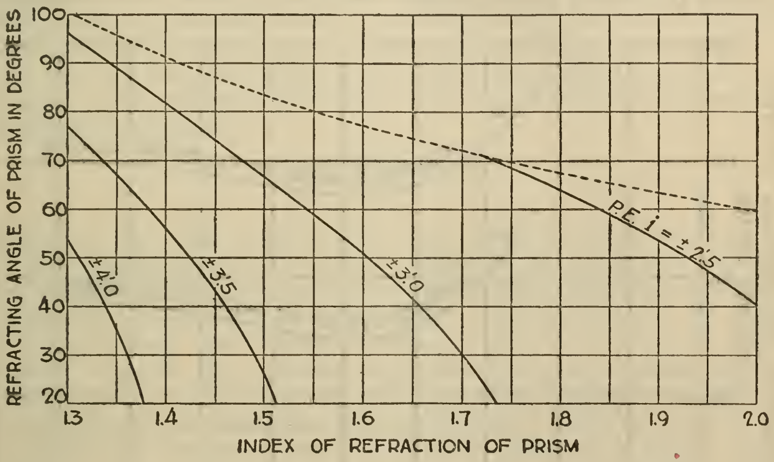

FIGURE 5.—*Iso-tolerance curves, in minutes of arc, for precision in orientation of prism*

In order that the systematic prism orientation error in the index of refraction, as computed from one-half the average of N observed double minimum deviations, shall not exceed $+1\times10^{-6}$, it is necessary that at least one-half of all individual errors in orientation, Δi, shall not exceed the tolerance which may be read from this figure. The contours for these probable errors of single orientations were computed from equation (14). The dotted line is that of the limiting condition, $A=2$ arcs in n^{-1}, at which both incident and emergent light grazes the prism surfaces. When seeking a certain degree of accuracy in index measurement under any given experimental conditions the probable error in orientation by simple hand adjustment should be determined and the curves of this figure may then be used either for determining the absolute value of the negative correction which should be applied to the index or for reaching a decision concerning the necessity of using more precise methods for orientation adjustment.

Although the repetition of observations has no systematic effect on the error in index which results from incorrect orientations, it does permit increased reliance on the computation of the absolute value of such errors; that is, the accuracy of index corrections, made in the manner suggested, is a function of the number of independent observations.

V. PRISM SIZE AND SIMPLE ORIENTATION PROCEDURE

It is suggested in Section IV, 1, (*a*) that there may be a definite connection between the precision of prism orientation and the precision of telescope pointing. Such a relationship, if known, can be used in determining what size of prism should be adequate for use in sixth decimal place refractometry without resorting to the correction discussed in the preceding section.

1. RELATION OF POINTING ERROR TO SYSTEMATIC DEPARTURES FROM MINIMUM DEVIATION

The quantitative evaluation of this relationship is, however, complicated by the fact that the systematically positive departures from minimum deviation (as caused by errors in orientation) can not [28] follow a normal error distribution curve such as may safely be assumed for the accidental errors of telescope pointing. The departures from minimum deviation which correspond to experienced values of the probable error in orientation might, of course, be computed by equation (11) and a curve, similar to D_1' of Figure 6, be so determined.

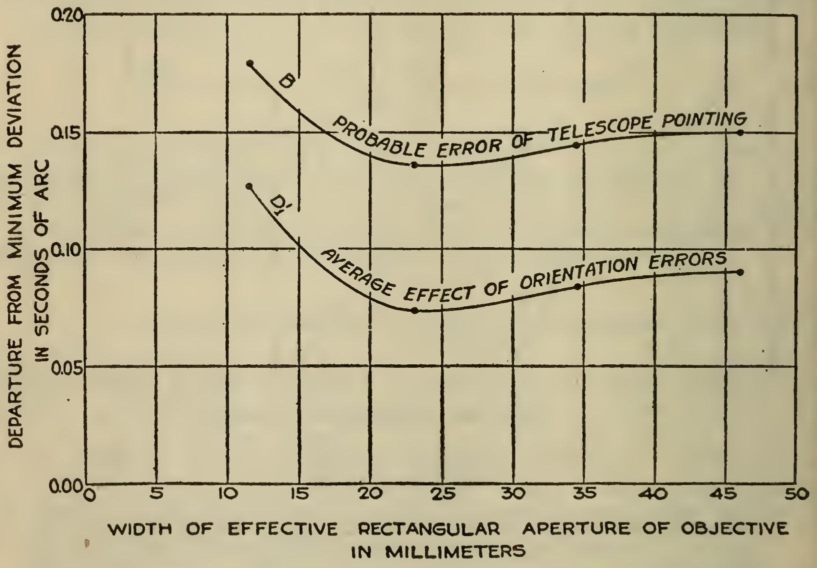

FIGURE 6.—*Comparative effects of pointing and orientation errors on minimum deviation*

Curve B is reproduced from Figure 1, and D_1' is computed from curve D_1 of Figure 2 by means of equation (13). From a comparison of the ordinates of these curves it appears that the average systematic effect of prism orientation error is equal to 0.6 of the absolute value of the probable error of a single telescope pointing.

Its ordinates would average less than one-third of those of curve B, as drawn in Figure 1 and reproduced in Figure 6. This result signifies that 50 per cent of all individual departures from minimum deviation are less than one-third the probable error of a single telescope pointing. If, however, the root mean square error in orientation is taken as the basis of comparison, the relative heights of the resulting curves would be different, the ordinates of the "D" curve being in this case about one-half of those for the comparable "B" curve. It seems, in fact, from further similar analysis of the experimental data that approximately 99 per cent of all errors in simple prism orientation do not cause departures from minimum deviation in excess of a limiting angular value which also includes 99 per cent of the optical errors in telescope pointing.

[28] Equation (11) considered in connection with the equation for normal error distribution shows that ΔD and Δi can not both be normally distributed, and it has been shown in Section IV, 1, (c) that the values of Δi follow a normal curve.

Under these circumstances it appears that no single comparison of the levels of curves, such as B and D_1' is of general utility. Allowing for experimental error, however, the last mentioned or 99 per cent comparison may be considered as definite quantitative evidence of a significant relationship. It shows that the limiting errors in the (optical) pointing of a telescope are also the limiting departures from minimum deviation which are caused by simple orientation procedures. In the present investigation a more directly useful relationship to establish is that between the probable error in telescope pointing and the arithmetical average of the departures from minimum deviation, since the latter is an actual error which affects an index measurement resulting from a number of independent orientations for minimum-deviation observations. Accordingly, equation (13) has been used in the computations for curve D_1' of Figure 6, and a comparison of its ordinates with those of curve B yields the approximate empirical relationship

$$\frac{\Sigma\Delta D}{N}=0.6\ m \tag{15}$$

which, for the various apertures, exists between (the absolute value of) the probable error of a single telescope pointing and one-half of the average (positive) departure from double minimum deviation as caused by errors in prism orientation. Incidentally, the distribution of the pointing errors being normal, it may be remarked that the average error in the pointing of a telescope is approximately 1.2 m. Thus equation (15) gives some idea of that favorable nature of the distribution of the departures from minimum deviation which makes it possible by simple procedures to statistically select prism azimuths that are intermediate between those positive and negative limiting positions which are barely perceivable as erroneous.

2. SUITABLE AND SUFFICIENT PRISM SIZE

For the purpose of determining prism size, the most readily useful measure of the precision of a series of prism orientations is the simple arithmetical average of the resulting departures from minimum deviation and this average value, $\Sigma\Delta D/N$, has been related to pointing error by equation (15).

Consequently, from equations (1), (4), and (15) the average departure from double minimum deviation, arising solely from N prism orientations at each of the two observational positions involved in N determinations of double deviation, is

$$\frac{2\Sigma\Delta D}{N}=\frac{25\times10^4 K\lambda}{L\cos i}\ \text{seconds.} \tag{16}$$

In obtaining equation (16) the radical sign is removed from the factor $\sqrt{2}$, as used in the numerator of equation (5), because the departures from minimum deviation are of positive sign at the orientations for both right and left observational positions. Moreover, this average departure, unlike the error expressed by equation (5), is independent of the number of observations. This is a strictly fundamental limitation to which reference has already been made in Sections III, 2 and IV, 3. It may be noticed that this empirical result, equation (16), when compared with equation (5), indicates that, in general, the systematic prism-orientation error in double deviation is not very different in absolute value from the combined probable error of the two (optical) pointings in a single determination of double deviation.

By again imposing suitable tolerance conditions, as was done in treating the special case expressed by equation (8), the right-hand members of equations (3) and (16) may be equated to give

$$L_D = \frac{30 \times 10^4 K\lambda}{\sin A/2} \tag{17}$$

as the general expression for an appropriate and sufficient value for length of prism surface. Incidentally, it may be remarked that equation (17), considered together with Figures 1 and 2 of the paper first above cited,[29] shows a further advantage of using prism angles of more than 60° when working on media of low refractive index.

If, as in Section III, it be again assumed that the magnification, the contrast conditions in the image, and the aberration of the prism, are consistent with a normal value of K, then the proper value of L for use in sixth decimal place index measurements by the usual practices of the minimum-deviation method is determinable. Using for K the value of 1/25 and again considering the longer wave lengths of the visible spectrum, it is evident that an approximate value is

$$L_D = \text{cosec } A/2 \text{ centimeters.} \tag{18}$$

For 60° prisms of any refracting medium it is thus unnecessary, when making deviation measurements, to use prisms with a length of surface exceeding 2 cm unless it is desired to limit all of the various individual[30] errors to something less than approximately $\pm 1 \times 10^{-6}$ in their effect on the computed index of refraction.

WASHINGTON, October, 1930.

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[29] See footnote 1, p. 60.

[30] In this investigation the various limits and tolerances are not considered collectively but are established individually at values consistent with errors of $\pm 1 \times 10^{-6}$ in index of refraction. It is, of course, recognized that this procedure is not adequate for accurate sixth decimal place refractometry; but, with a few repetitions in the observations, the approximation is very good.