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Abscissas and Weights for Gaussian Quadratures of High Order

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Weights and abscissas are presented for the Gaussian quadrature rules of order $n=16$, 20, 24, 32, 40, 48. These constants were computed on Standards Automatic Computer by the method described and have passed a number of checks with about 20 places of decimals. Values of the weights and abscissas are also available for $n=64$, 80, and 96.

1. Introduction

Gaussian quadrature rules have hitherto been computed up to the case $n=16$ with 15 place accuracy [1].[1] In the work in the Numerical Analysis Section of the National Bureau of Standards, frequent use has been made of the rule $n=16$ and even of halving and quartering the interval for increased accuracy. For this reason, it is felt that the constants for rules of higher order will prove to be of use in working with electronic digital computers.

For further evidence of the practical utility of high-order rules, the reader may consult Hartree [2], Henrici [3], and Reiz [4]. Exact values of these quantities are also interesting in view of certain unsettled theoretical conjectures that have been made about distribution of the weights and abscissas [7].

2. Method of Computation

We deal with the Gaussian quadrature rule of order n on the interval $[-1,1]$:

$$\int_{-1}^{+1} f(x)dx=\sum_{k=1}^{n} a_{kn} f(x_{kn}). \tag{1}$$

The rule (1) holds exactly whenever f is a polynomial of degree $\leq 2n-1$. The abscissas $x_{kn} (k=1,2, \ldots, n)$ are the n zeros of the Legendre polynomial of order n: $P_n(x_{kn})=0$, whereas the weights are given by the expression

$$a_{kn}=\frac{k_{n+1}}{k_n} \frac{-1}{p_{n+1}(x_{kn})p'_n(x_{kn})}, \tag{2}$$

where $p_n x_n = k_n x_n + \ldots$ are the normalized Legendre polynomials. See, e. g., Szegö [5, p. 47]. Making use of the relationship

$$(1-x^2)P'_n(x)=n[P_{n-1}(x)-xP_n(x)], \tag{3}$$

we are able to derive from (2) the following alternate expression for the weights a_{kn}, which is useful for computation:

$$a_{kn}=\frac{2(1-x_{kn}^2)}{[nP_{n-1}(x_{kn})]^2} \qquad (k=1,2, \ldots, n). \tag{4}$$

[1] Figures in brackets indicate the literature references at the end of this paper.

To obtain a first approximation to the zeros of the Legendre polynomials, we make use of the following inequality derived by Szegö [6]: Let

$$x_{kn}=\cos \theta_{kn} \qquad (k=1,2, \ldots, n). \tag{5}$$

Then

$$\frac{j_k}{[(n+1/2)^2+c/4]^{1/2}}<\theta_{kn}<\frac{j_k}{(n+1/2)} \qquad (k=1,2, \ldots), \tag{6}$$

where j_k $(k=1,2, \ldots)$ are the successive zeros of the Bessel function $J_0(x)$ and $c=1-(2/\pi)^2$.

The first 150 values of j_k may be found in [8]. A preliminary computation of (6) with $n=16$ showed that the value of θ_{kn} is closer to the left-hand bound, and that five or six decimal places can be secured intially by employing

$$x_{kn}^{(1)}=\cos \frac{j_k}{((n+1/2)^2+c/4)^{1/2}} \tag{7}$$

as a first approximation to x_{kn}. The value $x_{kn}^{(1)}$ was successively improved by using the Newton formula

$$x_{kn}^{(i+1)}=x_{kn}^{(i)}-\frac{P_n(x_{kn}^{(i)})}{P'_n(x_{kn}^{(i)})}. \tag{8}$$

The derivative in (8) was computed from (3), whereas the Legendre polynomials themselves were computed from the recursion

$$\left.\begin{array}{l} nP_n(x)=(2n-1)xP_{n-1}(x)-(n-1)P_{n-2}(x) \\ P_0(x)=1, \qquad P_1(x)=x. \end{array}\right\} \tag{9}$$

After the first approximation, the successive approximations were computed in double precision, and a shutoff value of $\epsilon=2^{-74}$ was employed in the iteration (8).

Although the abscissas and the weights are symmetric about $x=0$, all were computed independently. The starting values $x_{kn}(^1)$ for two symmetric points were different (cf. (7)). This served as one check of the accuracy of the computation. Additional checks were provided by computing the six quantities given in eq (10), page 37.

Abscissas		Weights	

n = 2

Abscissas		Weights	
0.5773502691	89625764509	1.0000000000	0000000000

n = 4

Abscissas		Weights	
0.8611363115	94052575224	0.3478548451	3745385737
0.3399810435	84856264803	0.6521451548	6254614262

n = 8

Abscissas		Weights	
0.9602898564	97536231684	0.1012285362	9037625915
0.7966664774	13626739592	0.2223810344	5337447054
0.5255324099	16328985818	0.3137066458	7788728733
0.1834346424	95649804939	0.3626837833	7836198296

n = 16

Abscissas		Weights	
0.9894009349	91649932596	0.0271524594	1175409485
0.9445750230	73232576078	0.0622535239	3864789286
0.8656312023	87831743880	0.0951585116	8249278481
0.7554044083	55003033895	0.1246289712	5553387205
0.6178762444	02643748447	0.1495959888	1657673208
0.4580167776	57227386342	0.1691565193	9500253818
0.2816035507	79258913230	0.1826034150	4492358886
0.0950125098	37637440185	0.1894506104	5506849628

n = 20

Abscissas		Weights	
0.9931285991	85094924786	0.0176140071	3915211831
0.9639719272	77913791268	0.0406014298	0038694133
0.9122344282	51325905868	0.0626720483	3410906357
0.8391169718	22218823395	0.0832767415	7670474872
0.7463319064	60150792614	0.1019301198	1724043503
0.6360536807	26515025453	0.1181945319	6151841731
0.5108670019	50827098004	0.1316886384	4917662689
0.3737060887	15419560673	0.1420961093	1838205132
0.2277858511	41645078080	0.1491729864	7260374678
0.0765265211	33497333755	0.1527533871	3072585069

n = 24

Abscissas		Weights	
0.9951872199	97021360180	0.0123412297	9998719954
0.9747285559	71309498198	0.0285313886	2893366318
0.9382745520	02732758524	0.0442774388	1741980616
0.8864155270	04401034213	0.0592985849	1543678074
0.8200019859	73902921954	0.0733464814	1108030573
0.7401241915	78554364244	0.0861901615	3195327591
0.6480936519	36975569252	0.0976186521	0411388827
0.5454214713	88839535658	0.1074442701	1596563478
0.4337935076	26045138487	0.1155056680	5372560135
0.3150426796	96163374387	0.1216704729	2780339120
0.1911188674	73616309159	0.1258374563	4682829612
0.0640568928	62605626085	0.1279381953	4675215697

n = 32

Abscissas		Weights	
0.9972638618	49481563545	0.0070186100	0947009660
0.9856115115	45268335400	0.0162743947	3090567060
0.9647622555	87506430774	0.0253920653	0926205945
0.9349060759	37739689171	0.0342738629	1302143310
0.8963211557	66052123965	0.0428358980	2222668065
0.8493676137	32569970134	0.0509980592	6237617619
0.7944837959	67942406963	0.0586840934	7853554714
0.7321821187	40289680387	0.0658222227	7636184683
0.6630442669	30215200975	0.0723457941	0884850622
0.5877157572	40762329041	0.0781938957	8707030647
0.5068999089	32229390024	0.0833119242	2694675522
0.4213512761	30635345364	0.0876520930	0440381114
0.3318686022	82127649780	0.0911738786	9576388471
0.2392873622	52137074545	0.0938443990	8080456563
0.1444719615	82796493485	0.0956387200	7927485941
0.0483076656	87738316235	0.0965400885	1472780056

n = 40

Abscissas		Weights	
0.9982377097	10559200350	0.0045212770	9853319125
0.9907262386	99457006453	0.0104982845	3115281361
0.9772599499	83774262663	0.0164210583	8190788871
0.9579168192	13791655805	0.0222458491	9416695726
0.9328128082	78676533361	0.0279370069	8002340109
0.9020988069	68874296728	0.0334601952	8254784739
0.8659595032	12259503821	0.0387821679	7447201764
0.8246122308	33311663196	0.0438709081	8567327199
0.7783056514	26519387695	0.0486958076	3507223206
0.7273182551	89927103281	0.0532278469	8393682435
0.6719566846	14179548379	0.0574397690	9939155136
0.6125538896	67980237953	0.0613062424	9292893916
0.5494671250	95128202076	0.0648040134	5660103807
0.4830758016	86178712909	0.0679120458	1523390382
0.4137792043	71605001525	0.0706116473	9128677969
0.3419940908	25758473007	0.0728865823	9580405906
0.2681521850	07253681141	0.0747231690	5796826420
0.1926975807	01371099716	0.0761103619	0062624237
0.1160840706	75255208483	0.0770398181	6424796558
0.0387724175	06050821933	0.0775059479	7842481126

n = 48

Abscissas		Weights	
0.9987710072	52426118601	0.0031533460	5230583863
0.9935301722	66350757548	0.0073275539	0127626210
0.9841245837	22826857745	0.0114772345	7923453949
0.9705915925	46247250461	0.0155793157	2294384872
0.9529877031	60430860723	0.0196161604	5735552781
0.9313866907	06554333114	0.0235707608	3932437914
0.9058791367	15569672822	0.0274265097	0835694820
0.8765720202	74247885906	0.0311672278	3279808890
0.8435882616	24393530711	0.0347772225	6477043889
0.8070662040	29442627083	0.0382413510	6583070631
0.7671590325	15740339254	0.0415450829	4346474921
0.7240341309	23814654674	0.0446745608	5669428041
0.6778723796	32663905212	0.0476166584	9249047482
0.6288673967	76513623995	0.0503590355	5385447495
0.5772247260	83972703818	0.0528901894	8519366709
0.5231609747	22233033678	0.0551995036	9998416286
0.4669029047	50958404545	0.0572772921	0040321570
0.4086864819	90716729916	0.0591148396	9839563574
0.3487558862	92160738160	0.0607044391	6589388005
0.2873624873	55455576736	0.0620394231	5989266390
0.2247637903	94689061225	0.0631141922	8625402565
0.1612223560	68891718056	0.0639242385	8464818662
0.0970046992	09462698930	0.0644661644	3595008220
0.0323801709	62869362033	0.0647376968	1268392250

$$\left.\begin{array}{lll}\sum_{k=1}^{n} a_{kn}=2, & \sum_{k=1}^{n} a_{kn}x_{kn}=0, & \sum_{k=1}^{n} a_{kn}x_{kn}^{2}=2/3 \\ \sum_{k=1}^{n} a_{kn}x_{n}^{3}=0, & \sum_{k=1}^{n} a_{kn}x_{n}^{4}=2/5, & \\ \sum_{k=1}^{n/2} x_{kn}^{2}=\frac{n(n-1)}{2(2n-1)}=-\frac{B_n}{A_n}, & & \end{array}\right\} \quad (10)$$

where

$$P_n(x)=A_nx^n+B_nx^{n-1}+\ldots, n \text{ even}.$$

These checks were all met to within 2 units in the 20th decimal place. The first four checks in (10) were carried out on SEAC at the time of the computation, and the last two were made directly from the final tabulation.

In the tables only the abscissas lying between 0 and 1 have been listed.

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3. References

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