# Partially Balanced Incomplete Block Designs With Two Associate Classes and Two Treatments Per Block 

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#### Abstract

In physical science experimentation the experimental material may divide naturally into groups of two objects (treatments) each. Thus, the experimental arrangement will involve what is known in the design of experiments as an incomplete block design. In this paper an investigation is made of a special kind of incomplete block design known as the partially balanced incomplete block design with two associate classes. In these designs the various comparisons of pairs of objects involve two kinds of precision.

An enumeration is made of all partially balanced incomplete block designs with two associate classes and two treatments per block where $r$, the number of times each treatment appears in the arrangement, is less than or equal to 10. Simple instructions are given for writing the experimental arrangements. These designs have a definite known method for analysis of the data. No attempt has been made to state which of the various designs are best, since it is felt that the best design must be determined for the particular investigation.


## 1. Introduction

It happens rather frequently in physical science experimentation, and occasionally in biological investigations, that the experimental material divides naturally into groups of two. For example, the comparison of the lengths of meter bars is conveniently made two at a time, the testing of experimental rubbers for automobile tires might be performed with the two halves of a tire made of different experimental rubber, or nutritional experiments might involve twins. Designs with two treatments per block have been studied by Yates [11] ${ }^{1}$, Kempthorne [10], Youden and Connor [9], and Bose and Nair [5].

The object of this paper is to investigate the class of partially balanced incomplete block (PBIB) designs with two associate classes having
(a) two treatments per block, and
(b) 10 or fewer replications of each treatment.

A complete enumeration has been made of all sets of parameter values for all designs of this class that satisfy the necessary conditions for PBIB designs with two associate classes set forth in $[3,4,6,7,8]$. Solutions (experimental arrangements) for all parameter sets given in the enumeration are contained in this paper with but four exceptions. In these four unsolved designs the number of treatment-pairs (blocks) is rather large, and an intensive search for solutions was not made.

Bose [1] has exhausted the class of PBIB designs with two associate classes and two replications of each treatment. Designs found in [1] that fall within the scope of this investigation are included here.

The great majority of the designs of the class under consideration may be classified according to four types discussed in [6], namely, group divisible designs, triangular designs, square designs, and cyclic designs.

## 2. Definition of a Partially Balanced Design With Two Associate Classes

A partially balanced incomplete block design with two associate classes is an arrangement of $v$ treatments in $b$ blocks, such that:

1. Each of the $v$ treatments occurs $r$ times in the arrangement, which consists of $b$ blocks each of which contains $k$ experimental units. No treatment appears more than once in any block.
2. Every pair among the $v$ treatments occurs together in either $\lambda_{1}$ or $\lambda_{2}$ blocks (and are said to be $i$ th associates, if they occur together in $\lambda_{i}$ blocks, $i=1,2$ ).

[^0]3. There exists a relationship of association between every pair of the $v$ treatments satisfying the following conditions:
a. Any two treatments are either first or second associates.
b. Each treatment has $n_{1}$ first and $n_{2}$ second associates.
c. Given any two treatments that are $i$ th associates, the number of treatments common to the $j$ th associates of the first and the $k$ th associates of the second is $p_{i k}^{i}$, and this number is independent of the pair of treatments with which we start. Furthermore, $p_{i k}^{i}=p_{k j}^{i} \quad(i, j, k=1,2)$.
For partially balanced incomplete block designs with two associate classes, it is wellknown that the following relationships are necessary:
\[

$$
\begin{gather*}
v r=b k,  \tag{1}\\
v=n_{1}+n_{2}+1,  \tag{2}\\
\lambda_{1} n_{1}+\lambda_{2} n_{2}=r(k-1),  \tag{3}\\
p_{11}^{1}+p_{12}^{1}+1=p_{11}^{2}+p_{12}^{2}=n_{1},  \tag{4}\\
p_{21}^{1}+p_{22}^{1}=p_{21}^{2}+p_{22}^{2}+1=n_{2},  \tag{5}\\
n_{1} p_{12}^{1}=n_{2} p_{11}^{1}, \quad n_{1} p_{22}^{1}=n_{2} p_{12}^{2} . \tag{6}
\end{gather*}
$$
\]

The eight parameters $v, b, r, k, \lambda_{1}, \lambda_{2}, n_{1}$, and $n_{2}$ are known as the parameters of the first kind, and the parameters $p_{i k}^{i}(i, j, k=1,2)$ are called the parameters of the second kind. The parameters of the second kind may be displayed as elements of two symmetric matrices.

$$
P_{1}=\left(\begin{array}{ll}
p_{11}^{1} & p_{12}^{1}  \tag{7}\\
p_{21}^{1} & p_{22}^{1}
\end{array}\right) \quad \text { and } \quad P_{2}=\left(\begin{array}{ll}
p_{11}^{2} & p_{12}^{2} \\
p_{21}^{2} & p_{22}^{2}
\end{array}\right) .
$$

For a general definition of a partially balanced incomplete block design with $m(\geq 1)$ associate classes and for a discussion of the classification, properties, and analysis of designs having $m=2$, the reader is referred to the excellent paper by Bose and Shimamoto [6]. The notation used in the present paper follows rather closely that in [6].

A design is said to be connected if for every two treatments, two blocks, or a block and a treatment, it is possible to pass from one to the other by means of a chain consisting alternately of blocks and treatments, such that every treatment of the chain occurs in each of the adjoining blocks. In this paper only designs possessing the property of connectedness are considered.

## 3. Group Divisible Designs With Two Treatments Per Block

In the group divisible designs the $v=m n$ treatments can be divided into $m$ groups of $n$ treatments each, such that any two treatments belonging to the same group are first associates, and any two treatments belonging to different groups are second associates. Bose and Shimamoto [6], who discovered the group divisible designs, showed that their association scheme is an $m \times n$ rectangular array of the $v$ treatments in which the rows constitute groups. For these arrays they showed that the following relations hold:

$$
\begin{equation*}
n_{1}=n-1, \quad n_{2}=n(m-1), \quad(n-1) \lambda_{1}+n(m-1) \lambda_{2}=r(k-1), \tag{8}
\end{equation*}
$$

and the parameters of the second kind are the elements of the symmetric matrices,

$$
P_{1}=\left(p_{i k}^{1}\right)=\left(\begin{array}{cc}
n-2 & 0  \tag{9}\\
0 & n(m-1)
\end{array}\right), \quad P_{2}=\left(p_{i k}^{2}\right)=\left(\begin{array}{cc}
0 & n-1 \\
n-1 & n(m-2)
\end{array}\right) .
$$

For the special case in which each block consists of two treatments ( $k=2$ ), each treatment of a
group is paired with each of the other $n-1$ treatments of the same group $\lambda_{1}$ times and with each of the treatments of the other $m-1$ groups $\lambda_{2}$ times to form the blocks of the design. Obviously, each of the $v=m n$ treatments occurs

$$
\begin{equation*}
r=(n-1) \lambda_{1}+n(m-1) \lambda_{2} \tag{10}
\end{equation*}
$$

times in the design that contains

$$
\begin{equation*}
b=m n\left[(n-1) \lambda_{1}+n(m-1) \lambda_{2}\right] / 2 \tag{11}
\end{equation*}
$$

blocks each of size $k=2$.
Note that for a given subclass of designs characterized by known values of $r, k=2, b, \lambda_{1}$, and $\lambda_{2}$, all designs of the subclass are determined by the sets of all integral values of $m$ and $n$ ( $m, n \geq 2$ ) satisfying (10) and (11).

For any PBIB design with two associate classes, Bose and Shimamoto define four computational constants $c_{1}, c_{2}, H$, and $\Delta$ by means of the following relations:

$$
\begin{gather*}
k^{2} \Delta=\left(r k-r+\lambda_{1}\right)\left(r k-r+\lambda_{2}\right)+\left(\lambda_{1}-\lambda_{2}\right)\left\{r(k-1)\left(p_{12}^{1}-p_{12}^{2}\right)+\lambda_{2} p_{12}^{1}-\lambda_{1} p_{12}^{2}\right\},  \tag{12}\\
k H=\left(2 r k-2 r+\lambda_{1}+\lambda_{2}\right)+\left(p_{12}^{1}-p_{12}^{2}\right)\left(\lambda_{1}-\lambda_{2}\right),  \tag{13}\\
k \Delta c_{1}=\lambda_{1}\left(r k-r+\lambda_{2}\right)+\left(\lambda_{1}-\lambda_{2}\right)\left(\lambda_{2} p_{12}^{1}-\lambda_{1} p_{12}^{2}\right),  \tag{14}\\
k \Delta c_{2}=\lambda_{2}\left(r k-r+\lambda_{1}\right)+\left(\lambda_{1}-\lambda_{2}\right)\left(\lambda_{2} p_{12}^{1}-\lambda_{1} p_{12}^{2}\right) . \tag{15}
\end{gather*}
$$

For group divisible designs with two treatments per block these expressions simplify to

$$
\begin{align*}
c_{1}=2 \lambda_{1} /\left(r+\lambda_{1}\right), \quad c_{2} & =2\left[\left(r+n \lambda_{1}\right) \lambda_{2}-(n-1) \lambda_{1}^{2}\right] / m n \lambda_{2}\left(r+\lambda_{1}\right),  \tag{16}\\
H & =n\left[\lambda_{1}+(2 m-1) \lambda_{2}\right] / 2, \tag{17}
\end{align*}
$$

and

$$
\begin{equation*}
\Delta=m n \lambda_{2}\left(r+\lambda_{1}\right) / 4 . \tag{18}
\end{equation*}
$$

It is known that $\lambda_{2}$, the "between group $\lambda$ " of group divisible designs, is never zero; hence there is no difficulty in evaluating $c_{2}$.

The efficiency, $E$, of group divisible designs with block size two, as compared with a randomized block design with the same number of treatments and replications, is given by

$$
\begin{equation*}
E=(n m-1) /\left[(n-1)\left(2-c_{1}\right)+n(m-1)\left(2-c_{2}\right)\right] . \tag{19}
\end{equation*}
$$

Table 1 contains the parameters of all group divisible designs with two treatments per block and $2 \leq r \leq 10$. The computational constants $c_{1}, c_{2}, H$, and $\Delta$, and also $E$ are given. Of course, there exist many other group divisible designs with $k \geq 3$, as well as designs with $k=2$ and $r>10$.

The construction of group divisible designs with two treatments per block by use of the groups (association schemes) is exceedingly simple. For example, let us construct design 2 of table 1. This design has parameters

$$
v=6, \quad b=9, \quad r=3, \quad k=2, \quad m=2, \quad n=3, \quad \lambda_{1}=0, \quad \lambda_{2}=1 .
$$

Let the six treatments be represented by the integers $1,2, \cdots, 6$, and let the $m=2$ groups be

$$
\begin{array}{lll}
1 & 2 & 3 \\
\hline 4 & 5 & 6 \\
\hline
\end{array}
$$

By the rule previously given for forming the blocks, no treatment of any group can occur in a block with any other treatment belonging to the same group (since $\lambda_{1}=0$ ), and each treatment

Table 1. Group divisible designs with two treatments per block and $2 \leq r \leq 10$

| Reference | $v$ | $b$ | $r$ | $m$ | $n$ | $\lambda_{1}$ | $\lambda_{2}$ | $c_{1}$ | $c_{2}$ | H | $\Delta$ | $E$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 1 | 4 | 4 | 2 | 2 | 2 | 0 | 1 | 0 | 1/2 | 3 | 2 | 3/5 |
| 2 | 6 | 9 | 3 | 2 | 3 | 0 | 1 | 0 | 1/3 | $9 / 2$ | 9/2 | 5/9 |
| 3 | 6 | 12 | 4 | 3 | 2 | 0 | 1 | 0 | 1/3 | 5 | 6 | 15/26 |
| 4 | 8 | 16 | 4 | 2 | 4 | 0 | 1 | 0 | 1/4 | 6 | 8 | 7/13 |
| 5 | 4 | 8 | 4 | 2 | 2 | 0 | 2 | 0 | 1/2 | 6 | 8 | 3/5 |
| 6 | 4 | 8 | 4 | 2 | 2 | 2 | 1 | 2/3 | 1/3 | 5 | 6 | 9/14 |
| 7 | 10 | 25 | 5 | 2 | 5 | 0 | 1 | 0 | 1/5 | 15/2 | 25/2 | 9/17 |
| 8 | 4 | 10 | 5 | 2 | 2 | 1 | 2 | 1/3 | 13/24 | 7 | 12 | 36/55 |
| 9 | 4 | 10 | 5 | 2 | 2 | 3 | 1 | $3 / 4$ | 1/8 | 6 | 8 | $3 / 5$ |
| 10 | 8 | 24 | 6 | 4 | 2 | 0 | 1 | 0 | 1/4 | 7 | 12 | 14/25 |
| 11 | 9 | 27 | 6 | 3 | 3 | 0 | , | 0 | 2/9 | 15/2 | 27/2 | 6/11 |
| 12 | 12 | 36 | 6 | 2 | 6 | 0 | 1 | 0 | 1/6 | 9 | 18 | 11/21 |
| 13 | 4 | 12 | 6 | 2 | 2 | 0 | 3 | 0 | 1/2 | 9 | 18 | 3/5 |
| 14 | 6 | 18 | 6 | 2 | 3 | 0 | 2 | 0 | 1/3 | 9 | 18 | 5/9 |
| 15 | 6 | 18 | 6 | 3 | 2 | 2 | 1 | 1/2 | 1/4 | 7 | 12 | 10/17 |
| 16 | 4 | 12 | 6 | 2 | 2 | 4 | 1 | $4 / 5$ | $-1 / 10$ | 7 | 10 | 5/9 |
| 17 | 14 | 49 | 7 | 2 | 7 | 0 | 1 | 0 | 1/7 | 21/2 | 49/2 | 13/25 |
| 18 | 6 | 21 | 7 | 2 | 3 | 2 | 1 | 4/9 | $5 / 27$ | 15/2 | 27/2 | 45/77 |
| 19 | 6 | 21 | 7 | 3 | 2 | 3 | 1 | $3 / 5$ | 2/15 | 8 | 15 | 75/133 |
| 20 | 4 | 14 | 7 | 2 | 2 | 1 | 3 | 1/4 | 13/24 | 10 | 24 | 9/14 |
| 21 | 4 | 14 | 7 | 2 | 2 | 3 | 2 | $3 / 5$ | 17/40 | 9 | 20 | 60/91 |
| 22 | 4 | 14 | 7 | 2 | 2 | 5 | 1 | 5/6 | $-1 / 3$ | 8 | 12 | 18/35 |
| 23 | 10 | 40 | 8 | 5 | 2 | 0 | 1 | 0 | 1/5 | 9 | 20 | 45/82 |
| 24 | 12 | 48 | 8 | 3 | 4 | 0 | 1 | 0 | 1/6 | 10 | 24 | 33/62 |
| 25 | 16 | 64 | 8 | 2 | 8 | 0 | 1 | 0 | 1/8 | 12 | 32 | 15/29 |
| 26 | 6 | 24 | 8 | 3 | 2 | 0 | 2 | 0 | 1/3 | 10 | 24 | 15/26 |
| 27 | 8 | 32 | 8 | 2 | 4 | 0 | 2 | 0 | 1/4 | 12 | 32 | 7/13 |
| 28 | 4 | 16 | 8 | 2 | 2 | 0 | 4 | 0 | 1/2 | 12 | 32 | 3/5 |
| 29 | 6 | 24 | 8 | 2 | 3 | 1 | 2 | 2/9 | 10/27 | 21/2 | 27 | 45/76 |
| 30 | 8 | 32 | 8 | 4 | 2 | 2 | 1 | 2/5 | $1 / 5$ | 9 | 20 | 35/62 |
| 31 | 4 | 16 | 8 | 2 | 2 | 2 | 3 | 2/5 | 8/15 | 11 | 30 | 45/68 |
| 32 | 6 | 24 | 8 | 3 | 2 | 4 | 1 | 2/3 | 0 | 9 | 18 | 15/28 |
| 33 | 4 | 16 | 8 | 2 | 2 | 4 | 2 | 2/3 | 1/3 | 10 | 24 | 9/14 |
| 34 | 4 | 16 | 8 | 2 | 2 | 6 | 1 | 6/7 | $-4 / 7$ | 9 | 14 | 21/44 |
| 35 | 12 | 54 | 9 | 4 | 3 | 0 | 1 | 0 | 1/6 | 21/2 | 27 | 22/41 |
| 36 | 18 | 81 | 9 | 2 | 9 | 0 | 3 | 0 |  | 27/2 | 81/2 | 17/33 |
| 37 | 6 | 27 | 9 | 2 | 3 | 0 | 3 | 0 | 1/3 | 27/2 | 81/2 | 5/9 |
| 38 | 6 | 27 | 9 | 3 | 2 | 1 | 2 | 1/5 | 7/20 | 11 | 30 | 25/42 |
| 39 | 6 | 27 | 9 | 2 | 3 | 3 | 1 | 1/2 | 0 | 9 | 18 | 5/9 |
| 40 | 8 | 36 | 9 | 4 | 2 | 3 | 1 | 1/2 | 1/8 | 10 | 24 | 28/51 |
| 41 | 4 | 18 | 9 | 2 | 2 | 1 | 4 | 1/5 | 43/80 | 13 | 40 | 40/63 |
| 42 | 6 | 27 | 9 | 3 | 2 | 5 | 1 | 5/7 | $-1 / 7$ | 10 | 21 | 35/69 |
| 43 | 4 | 18 | 9 | 2 | 2 | 5 | 2 | 5/7 | 13/56 | 11 | 28 | 28/45 |
| 44 | 12 | 60 | 10 | 6 | 2 | 0 | 1 | 0 | 1/6 | 11 | 30 | 33/61 |
| 45 | 15 | 75 | 10 | 3 | 5 | 0 | 1 | 0 | 2/15 | 25/2 | 75/2 | 21/40 |
| 46 | 20 | 100 | 10 | 2 | 10 | 0 | 1 | 0 | 1/10 | 15 | 50 | 19/37 |
| 47 | 10 | 50 | 10 | 2 | 5 | 0 | 2 | 0 | 1/5 | 15 | 50 | 9/17 |
| 48 | 4 | 20 | 10 | 2 | 2 | 0 | 5 | 0 | 1/2 | 15 | 50 | 3/5 |
| 49 | 8 | 40 | 10 | 2 | 4 | 2 | 1 | $1 / 3$ | 1/8 | 10 | 24 | 14/25 |
| 50 | 9 | 45 | 10 | 3 | 3 | 2 | 1 | 1/3 | $4 / 27$ | 21/2 | 27 | 36/65 |
| 51 | 10 | 50 | 10 | 5 | 2 | 2 | 1 | 1/3 | 1/6 | 11 | 30 | 27/49 |
| 52 | 8 | 40 | 10 | 4 | 2 | 4 | 1 | 4/7 | 1/28 | 11 | 28 | 98/185 |
| 53 | 4 | 20 | 10 | 2 | 2 | 2 | 4 | 1/3 | 13/24 | 14 | 48 | $36 / 55$ |
| 54 | 4 | 20 | 10 | 2 | 2 | 4 | 3 | 4/7 | 19/42 | 13 | 42 | 63/95 |
| 55 | 6 | 30 | 10 | 3 | 2 | 6 | 1 | 3/4 | -7/24 | 11 | 24 | 12/25 |
| 56 | 4 | 20 | 10 | 2 | 2 | 6 | 2 | 3/4 | 1/8 | 12 | 32 | 3/5 |
| 57 | 4 | 20 | 10 | 2 | 2 | 8 | 1 | 8/9 | -19/18 | 11 | 18 | 27/65 |

must occur once in a block with each of the treatments not in its group (since $\lambda_{2}=1$ ). Thus, the $b=9$ blocks of the design are the treatment-pairs in the following columns:

| 1 | 1 | 1 | 2 | 2 | 2 | 3 | 3 | 3 |
| :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- |
| 4 | 5 | 6 | 4 | 5 | 6 | 4 | 5 | 6 |

## 4. Triangular Designs With Blocks of Size Two

In [6] triangular designs are defined and discussed. Triangular designs have $v=n(n-1) / 2$ treatments and an association scheme described as follows. The treatments are arranged in order and in a symmetrical fashion about the principal diagonal of an $n \times n$ array in which the principal diagonal is blanked out. For example, if $n=4$ and the $v=6$ treatments are denoted by the integers $1, \cdots, 6$, the association scheme is

| $*$ | 1 | 2 | 3 |
| :--- | :--- | :--- | :--- |
| 1 | $*$ | 4 | 5 |
| 2 | 4 | $*$ | 6 |
| 3 | 5 | 6 | $*$ |

Two treatments lying in the same column are first associates, whereas treatments that do not lie in the same column are second associates. Thus, in the above example, treatments $2,3,4$, and 5 are inrst associates of treatment 1, whereas treatment 6 is the only second associate of treatment 1. In [6] it is shown that

$$
\begin{align*}
& n_{1}=2(n-2), \quad n_{2}=(n-2)(n-3) / 2,  \tag{20}\\
& P_{1}=\left(p_{i k}^{1}\right)=\left(\begin{array}{cc}
n-2 & n-3 \\
n-3 & (n-3)(n-4) / 2
\end{array}\right), \tag{21}
\end{align*}
$$

and

$$
P_{2}=\left(p_{i k}^{2}\right)=\left(\begin{array}{cc}
4 & 2(n-4)  \tag{22}\\
2(n-4) & (n-4)(n-5) / 2
\end{array}\right)
$$

where $n \geqq 4$.
If, when the $v$ treatments are arranged in an $n \times n$ array as described above, we form all possible treatment-pairs (blocks in the case $k=2$ ) by pairing each treatment of a row (or column) with every other treatment in the same row (or column), we obtain a design with parameters

$$
\begin{array}{llll}
v=n(n-1) / 2, & r=2(n-2), & \lambda_{1}=1, & n_{1}=2(n-2), \\
b=n(n-1)(n-2) / 2, & k=2, & \lambda_{2}=0, & n_{2}=(n-2)(n-3) / 2, \tag{23}
\end{array}
$$

and parameters of the second kind given by (21) and (22).
For triangular designs the computational constants $c_{1}, c_{2}, H$, and $\Delta$ of (12), (13), (14), and (15) simplify to

$$
\begin{array}{ll}
c_{1}=4 / n(n-1), & c_{2}=-2(n-4) / n(n-1), \\
H=(3 n-2) / 2, & \text { and } \quad \Delta=n(n-1) / 2 . \tag{25}
\end{array}
$$

The efficiency of a triangular design, as compared with a randomized block design with the same number of treatments and replications, is given by

$$
\begin{equation*}
E=n(n-1)(n+1) / 2(n-2)\left(n^{2}+3 n-2\right) . \tag{26}
\end{equation*}
$$

If, instead of forming the treatment-pairs for the blocks as described previously, we pair off each treatment with each of the treatments that does not lie in the same row or column
of the association scheme, we obtain a triangular design whose parameters are

$$
\begin{array}{lll}
v=n(n-1) / 2, & r=(n-2)(n-3) / 2, & \lambda_{1}=0 \\
b=n(n-1)(n-2)(n-3) / 8, & k=2, & \lambda_{1}=1 \tag{27}
\end{array}
$$

and (20), (21), and (22), where $n \geq 5$.
The constants $c_{1}, c_{2}, H$, and $\Delta$ for the latter triangular designs are given by

$$
\begin{gather*}
c_{1}=-8 / n(n-1)(n-4), \quad c_{2}=4 / n(n-1)  \tag{28}\\
H=\left(n^{2}-4 n+2\right) / 2, \quad \text { and } \quad \triangle=n(n-1)(n-2)(n-3)(n-4) / 16 \tag{29}
\end{gather*}
$$

and their efficiency is given by

$$
\begin{equation*}
E=\left(n_{1}+n_{2}\right) /\left[n_{1}\left(2-c_{1}\right)+n_{2}\left(2-c_{2}\right)\right] \tag{30}
\end{equation*}
$$

Table 2 gives the parameters of all triangular designs with $k=2$ and $2 \leq r \leq 10$, along with the corresponding values of $c_{1}, c_{2}, H, \Delta$, and $E$. Obviously, triangular designs with $k=2$ and $r>10$ can be found, as can designs with $k \geq 3$.

Table 2. Triangular designs with two treatments per block and $2 \leq r \leq 10$

| Reference | $v$ | $b$ | $r$ | $n$ | $\lambda_{1}$ | $\lambda_{2}$ | $c_{1}$ | $c_{2}$ | H | $\Delta$ | E |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 1 | 6 | 12 | 4 | 4 | 1 | 0 | 1/3 | 0 | 5 | 6 | 15/26 |
| 2 | 10 | 30 | 6 | 5 | 1 | 0 | 1/5 | $-1 / 10$ | 13/2 | 10 | 10/19 |
| 3 | 15 | 60 | 8 | 6 | 1 | 0 | 2/15 | -2/15 | 8 | 15 | 105/208 |
| 4 | 21 | 105 | 10 | 7 | 1 | 0 | 2/21 | $-1 / 7$ | 19/2 | 21 | 42/85 |
| 5 | 10 | 15 | 3 | 5 | 0 | 1 | $-2 / 5$ | 1/5 | 7/2 | 5/2 | 5/11 |
| 6 | 15 | 45 | 6 | 6 | 0 | 1 | $-2 / 15$ | 2/15 | 7 | 45/4 | 105/212 |
| 7 | 21 | 105 | 10 | 7 | 0 | 1 | -4/63 | 2/21 | 23/2 | 63/2 | 126/250 |
| 8 | 6 | 24 | 8 | 4 | 2 | 0 | 1/3 | 0 | 10 | 24 | 15/26 |
| 9 | 10 | 30 | 6 | 5 | 0 | 2 | $-2 / 5$ | 1/5 | 7 | 10 | 5/11 |

Construction of triangular designs with two treatments per block is easily done by use of the association scheme for the design and the rules for formation of the blocks stated earlier. For example, the association scheme for design 2 of table 2 is

| $*$ | 1 | 2 | 3 | 4 |
| ---: | ---: | ---: | ---: | ---: |
| 1 | $*$ | 5 | 6 | 7 |
| 2 | 5 | $*$ | 8 | 9 |
| 3 | 6 | 8 | $*$ | 10 |
| 4 | 7 | 9 | 10 | $*$ |

Since $\lambda_{1}=1$ and $\lambda_{2}=0$, the $b=30$ blocks of the design are formed by writing all pairs of numbers lying in the same row (or column) of the association scheme:

| 12 | 15 | 25 |  | 6 | 4 | 7 |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 13 | 16 | 28 | 3 | 8 | 4 | 9 |
| 14 | 17 | 29 | 3 | 10 | 4 | 10 |
| 23 | 56 | 58 | 6 | 8 | 7 | 9 |
| 24 | 57 | 59 | 6 | 10 | 7 | 10 |
| 34 | 67 | 89 | 8 | 10 | 9 | 10 |

## 5. Designs With Square Association Schemes

Consider $v=s^{2}$ treatments arranged in a square array of side $s$. Let this $s \times s$ array be the association scheme of the design. We shall form blocks of size $k$ by forming all possible distinct $k$-plets from the treatments lying in each of the rows and in each of the columns of the array. From one row (or column) it is possible to form $C_{k}^{s}$ distinct $k$-plets, where $C_{k}^{s}$ represents the number of combinations of $s$ things taken $k$ at a time, $k$ and $s$ being restricted in this application by $2 \leq k \leq s$. In this manner it is possible to form $b=2 s C_{k}^{s}$ blocks, each of size $k$, and each of the $v=s^{2}$ treatments will appear once in each of $r=2 C_{k-1}^{s-1}$ blocks. Two treatments lying in the same row or in the same column of the association scheme are first associates, while two treatments not lying in the same row or in the same column are second associates. Thus,

$$
\begin{equation*}
n_{1}=2(s-1) \quad \text { and } \quad n_{2}=(s-1)^{2} . \tag{31}
\end{equation*}
$$

Two treatments that are first associates occur together in $C_{k-2}^{s-2}$ blocks (when $s=k=2$, we define $C_{0}^{0}$ to be unity), while two treatments that are second associates do not occur together in any block. Hence,

$$
\begin{equation*}
\lambda_{1}=C_{k-2}^{s-2} \quad \text { and } \quad \lambda_{2}=0 . \tag{32}
\end{equation*}
$$

It is easy to verify that the matrices $P_{1}$ and $P_{2}$ of (33) give the correct values for the parameters of the second kind. Thus, we have a two-parameter family of designs with

$$
\begin{align*}
& v=s^{2}, \quad r=2 C_{k-1}^{s-1}, \quad \lambda_{1}=C_{k-2}^{s-2}, \quad n_{1}=2(s-1), \quad P_{1}=\left(\begin{array}{cc}
s-2 & s-1 \\
s-1 & (s-1)(s-2)
\end{array}\right), \\
& b=2 s C_{k}^{s}, \quad k=k, \quad \lambda_{2}=0, \quad n_{2}=(s-1)^{2}, \quad P_{2}=\left(\begin{array}{cc}
2 & 2(s-2) \\
2(s-2) & (s-2)^{2}
\end{array}\right), \tag{33}
\end{align*}
$$

where $s \geq k \geq 2$.
The expressions for $c_{1}, c_{2}, H$, and $\Delta$ simplify, in this type of design, to

$$
\begin{gather*}
c_{1}=k / s^{2}, \quad c_{2}=-k(s-2) / s^{2},  \tag{34}\\
H=\frac{3 s}{2} C_{h-2}^{s-2}, \quad \text { and } \quad \Delta=2(s / k)^{2} . \tag{35}
\end{gather*}
$$

The efficiency of these square designs, as compared with randomized block designs with the same number of treatments and replications, is

$$
\begin{equation*}
E=s(s+1)(k-1) / k(s-1)(s+3) . \tag{36}
\end{equation*}
$$

In the special case $s=k$, the set of parameters (33) becomes

$$
\begin{array}{lll}
v=s^{2}, & r=2, & \lambda_{1}=1,
\end{array} \quad n_{1}=2(s-1), \quad P_{1}=\left(\begin{array}{cc}
s-2 & s-1 \\
s-1 & (s-1)(s-2)
\end{array}\right), ~\left(\begin{array}{cc}
2 & 2(s-2)  \tag{37}\\
2(s-2) & (s-2)^{2}
\end{array}\right), ~ \$
$$

which is the family of familiar simple square lattice designs with $r=2$ replications.
Putting $k=2$ in (33), we obtain the family of square designs with two treatments per block, the parameters of which are

$$
\begin{array}{llll}
v=s^{2}, & r=2(s-1), & \lambda_{1}=1, & n_{1}=2(s-1), \\
b=s^{2}(s-1), & k=2, & \lambda_{2}=0, & n_{2}=(s-1)^{2}, \tag{38}
\end{array}
$$

where $s \geq 2$, and the parameters of the second kind are given by (33).
All of the designs discussed in this section are of the type known as Latin square designs with two constraints [6].

Setting $s=2,3$, $\cdot \cdot, 6$ in (38) we obtain the first five designs of table 3. Designs $6,7,8$, and 9 are $n$-plicates of design 1, i. e., their solutions are obtained by writing the solution of design $1, n=2,3,4,5$ times, respectively. Design 11 is obtained by pairing each treatment with the treatments not lying in the same row or column of the square association scheme. If the associate classes of design 11 are renamed, so that $\lambda_{1}=1$ and $\lambda_{2}=0$, it is seen that it is a Latinsquare type with $i=3$ constraints.

Table 3. Square designs with two treatments per block and $2 \leq r \leq 10$

| Reference | $v$ | $b$ | $r$ | $s$ | $\lambda_{1}$ | $\lambda_{2}$ | $c_{1}$ | $c_{2}$ | H | $\Delta$ | E |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 1 | 4 | 4 | 2 | 2 | 1 | 0 | 1/2 | 0 | 3 | 2 | 3/5 |
| 2 | 9 | 18 | 4 | 3 | 1 | 0 | 2/9 | $-2 / 9$ | 9/2 | $9 / 2$ | 1/2 |
| 3 | 16 | 48 | 6 | 4 | 1 | 0 | 1/8 | $-1 / 4$ | 6 | 8 | 10/21 |
| 4 | 25 | 100 | 8 | 5 | 1 | 0 | 2/25 | -6/25 | 15/2 | 25/2 | 15/32 |
| 5 | 36 | 180 | 10 | 6 | 1 | 0 | 1/18 | $-2 / 9$ | 9 | 18 | 7/15 |
| 6 | 4 | 8 | 4 | 2 | 2 | 0 | 1/2 | 0 | 6 | 8 | 3/5 |
| 7 | 4 | 12 | 6 | 2 | 3 | 0 | 1/2 | 0 | 9 | 18 | 3/5 |
| 8 | 4 | 16 | 8 | 2 | 4 | 0 | 1/2 | 0 | 12 | 32 | $3 / 5$ |
| 9 | 4 | 20 | 10 | 2 | 5 | 0 | 1/2 | 0 | 15 | 50 | 3/5 |
| 10 | 9 | 36 | 8 | 3 | 2 | 0 | 2/9 | $-2 / 9$ | 9 | 18 | $1 / 2$ |
| 11 | 16 | 72 | 9 | 4 | 0 | 1 | $-1 / 16$ | 1/8 | 10 | 24 | 80/123 |

Again, the association scheme and the rule for formation of the blocks provide the method of construction that holds for square designs with $k=2$ and $r>10$, as well as for the designs of table 3. To illustrate the construction of square designs, the association scheme for design 2 of table 3 may be represented by the $3 \times 3$

$$
\begin{array}{lll}
1 & 2 & 3 \\
4 & 5 & 6 \\
7 & 8 & 9
\end{array}
$$

array of integers $1,2, \cdots, 9$. Since $\lambda_{1}=1$ and $\lambda_{2}=0$, the $b=18$ blocks of the design are formed by pairing every number with every other number lying in the same row or column. Thus, the 18 blocks are

| 12 | 45 | 78 | 14 | 25 | 36 |
| :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- |
| 13 | 46 | 79 | 17 | 28 | $\underline{39}$ |
| $\underline{23}$ | $\underline{56}$ | $\underline{89}$ | $\underline{47}$ | $\underline{58}$ | $\underline{69}$ |

## 6. Cyclic Designs With Two Treatments Per Block

Another type of partially balanced design defined and discussed in [6] is the cyclic type. Let the $v$ treatments be represented by the integers $1,2, \cdots, v$. Then the first associates of treatment $i(i=1,2, \cdots, v)$ are

$$
\begin{equation*}
i+d_{1}, i+d_{2}, \cdots, i+d_{n_{1}} \quad(\bmod v) \tag{39}
\end{equation*}
$$

where the $d$ 's satisfy the conditions:

1. The $d$ 's are all different and $0<d_{j}<v \quad\left(j=1,2, \cdots, n_{1}\right) ;$
2. Among the $n_{1}\left(n_{1}-1\right)$ differences $\mathrm{d}_{j}-d_{k}\left(j \neq k ; j, k=1,2, \cdots, n_{1}\right)$ reduced mod $v$, each of the numbers $d_{1}, d_{2}, \cdots, d_{n_{1}}$ occurs $\alpha$ times and each of the numbers $e_{1}, e_{2}, \cdots, e_{n_{2}}$ occurs $\beta$ times, where $d_{1}, \cdots, d_{n_{1}}, e_{1}, \cdots, e_{n_{2}}$ are all the numbers $1,2, \cdots, v-1, \alpha=p_{11}^{1}$, and $\beta=p_{11}^{2}$.

In the special case of PBIB designs with two associate classes and $k=2$, determination of a difference set $d_{1}, d_{2}, \cdots, d_{n_{1}}$ satisfying 1 and 2 above is equivalent to construction of the design, since each of the treatment-pairs $\left(i, i+d_{j}\right)$ reduced $\bmod v\left(i=1,2, \cdots, v ; j=1,2, \cdots, n_{1}\right.$ is a block occurring $\lambda_{1}$ times, while each of the treatment-pairs ( $i, i+e_{k}$ ) reduced mod $\left(i=1,2, \cdots, v ; k=1,2, \cdots, n_{2}\right)$ is a block occuring $\lambda_{2}$ times.

In table 4 are listed 11 designs having cyclic solutions.
Table 4. Cyclic type designs with two treatments per block and $2 \leq r \leq 10$

| Reference | $v$ | $b$ | $r$ | $\lambda_{1}$ | $\lambda_{2}$ | $n_{1}$ | $n_{2}$ | $\alpha$ | $\beta$ | $c_{1}$ | $c_{2}$ | H | $\Delta$ | $E$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 1 | 5 | 5 | 2 | 1 | 0 | 2 | 2 | 0 | 1 | 2/5 | $-2 / 5$ | 5/2 | 5/4 | 1/2 |
| 2 | 5 | 10 | 4 | 2 | 0 | 2 | 2 | 0 | 1 | 2/5 | $-2 / 5$ | 5 | 5 | 1/2 |
| 3 | 5 | 15 | 6 | 3 | 0 | 2 | 2 | 0 | 1 | 2/5 | $-2 / 5$ | 15/5 | 45/4 | 1/2 |
| 4 | 5 | 15 | 6 | 2 | 1 | 2 | 2 | 0 | 1 | 26/55 | $2 / 5$ | 15/2 | 55/4 | 55/86 |
| 5 | 5 | 20 | 8 | 4 | 0 | 2 | 2 | 0 | 1 | 2/5 | $-2 / 5$ | 10 | 20 | 1/2 |
| 6 | 5 | 20 | 8 | 3 | 1 | 2 | 2 | 0 | 1 | 46/95 | 14/95 | 10 | 95/4 | 19/32 |
| 7 | 5 | 25 | 10 | 5 | 0 | 2 | 2 | 0 | 1 | 2/5 | -2/5 | 25/2 | 125/4 | 1/2 |
| 8 | 5 | 25 | 10 | 3 | 2 | 2 | 2 | 0 | 1 | 14/29 | 2/29 | 25/2 | 145/4 | 29/50 |
| 9 | 5 | 25 | 10 | 3 | 2 | 2 | 2 | 0 | 1 | 14/31 | 10/31 | 25/2 | 155/4 | 31/50 |
| 10 | 13 | 39 | 6 | 1 | 0 | 6 | 6 | 2 | 3 | 2/13 | $-2 / 13$ | 13/2 | 39/4 | 1/2 |
| 11 | 17 | 68 | 8 | 1 | 0 | 8 | 8 | 3 | 4 | 2/17 | $-2 / 17$ | 17/2 | 17 | 1/2 |

The designs 1 to 9 of table 4 have the difference set $d_{1}=2$ and $d_{2}=3(\bmod 5)$. These designs belong to the family having parameters

$$
\begin{array}{llll}
v=5, & r=2\left(\lambda_{1}+\lambda_{2}\right), & \lambda_{1}, & n_{1}=2,
\end{array} P_{1}=\left(\begin{array}{ll}
0 & 1 \\
1 & 1
\end{array}\right), ~ 子 \begin{array}{lll}
1 & 1  \tag{40}\\
b=5\left(\lambda_{1}+\lambda_{2}\right), & k=2, & \lambda_{2}, \\
n_{2}=2, & P_{2}=\left(\begin{array}{ll}
1 \\
1 & 0
\end{array}\right) .
\end{array}
$$

Designs of (40) have a geometrical representation that provides an association scheme. Number the vertices of the five-pointed star 1, 2, 3, 4, 5, in clockwise order. Each line of the

star connecting two vertices is a block occuring $\lambda_{1}$ times, while two vertices not connected by a line is a block occurring $\lambda_{2}$ times, where $\lambda_{1}$ and $\lambda_{2}$ are nonnegative integers satisfying (39).

Design 10 of table 4 has the difference set,

$$
d_{1}=2, d_{2}=5, d_{3}=6, d_{4}=7, d_{5}=8, d_{6}=11 \quad(\bmod 13) .
$$

Since $\lambda_{1}=1$ and $\lambda_{2}=0$, each of the treatment-pairs $\left(i, i+d_{j}\right)$ reduced $\bmod 13(i=1,2, \cdots, 13$; $j=1,2, \cdots, 6)$ occurs once to give the $b=39$ blocks:


Design 11 of table 4 has the difference set

$$
d_{1}=3, d_{2}=5, d_{3}=6, d_{4}=7, d_{5}=10, d_{6}=11, d_{7}=12, d_{8}=14,
$$

which, when developed mod 17 , will yield a table of first associates for the treatments 1 to 17 , where 0 is replaced with 17 . Since $\lambda_{1}=1$ and $\lambda_{2}=0$, pairing each of the treatments with each of its first associates will yield 68 distinct treatment-pairs, which are the blocks of the design.

## 7. Other Designs With Two Treatments Per Block

In table 5 are listed 8 sets of parameter values, 4 sets of which are unsolved.
Table 5. Other designs with two treatments per block and $2 \leq r \leq 10$

| Reference | $v$ | $b$ | $r$ | $\lambda_{1}$ | $\lambda_{2}$ | $n_{1}$ | $n_{2}$ | $p_{11}^{1}$ | $p_{11}^{2}$ | $c_{1}$ | $c_{2}$ | H | $\Delta$ | $E$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 1 | 16 | 40 | 5 | 1 | 0 | 5 | 10 | 0 | 2 | 1/8 | $-3 / 16$ | 6 | 8 | 12/25 |
| 2 | 16 | 80 | 10 | 2 | 0 | 5 | 10 | 0 | 2 | 1/8 | -3/16 | 12 | 32 | 12/25 |
| 3 | 16 | 80 | 10 | 1 | 0 | 10 | 5 | 6 | 6 | 1/8 | $-1 / 12$ | 10 | 24 | 18/35 |
| *4 | 21 | 105 | 10 | 1 | 0 | 10 | 10 | 4 | 5 |  |  |  |  |  |
| *5 | 26 | 130 | 10 | 1 | 0 | 10 | 15 | 3 | 4 |  |  |  |  |  |
| 6 | 27 | 135 | 10 | 1 | 0 | 10 | 16 | 1 | 5 | 2/27 | -2/27 | 12 | 135/4 | 117/236 |
| *7 | 50 | 175 | 7 | 1 | 0 | 7 | 42 | 0 | 1 |  |  |  |  |  |
| *8 | 56 | 280 | 10 | 1 | 0 | 10 | 45 | 0 | 2 |  |  |  |  |  |

We shall now give a construction for design 1. Let $\theta$ represent any treatment of the design, let the $n_{1}=5$ first associates of $\theta$ be $a, b, \cdots, e$, and let the $n_{2}=10$ second associates of $\theta$ be the integers $1,2, \cdots, 10$. Since $\lambda_{1}=1, \lambda_{2}=0$, and $p_{11}^{1}=0$, the five treatments that appear in the blocks containing a common treatment must be second associates of each other, and since $p_{11}^{2}=2$, any pair of these treatments must have just two common first associates. It is easily seen that the following 25 blocks satisfy these requirements:

$$
\begin{aligned}
& \theta \text { a a } 1 \text { b } 1 \text { c } 2 \text { d } 3 \text { e } 4 \\
& \theta \text { b a } 2 \text { b } 5 \text { c } 5 \text { d } 6 \text { e } 7 \\
& \underline{\theta \text { c a } 3} \text { b } 6 \text { c } 8 \text { d } 8 \text { e } 9 \\
& \underline{\theta \text { d a } 4} \text { b } 7 \text { c } 9 \text { d } 10 \text { e } 10 \\
& \theta \text { e }
\end{aligned}
$$

Reapplying the above requirement, it is seen that no treatment-pairs can be formed from treatments within the following groups:

$$
(1,2,3,4), \quad(1,5,6,7), \quad(2,5,8,9), \quad(3,6,8,10), \quad(4,7,9,10) .
$$

Hence, the three remaining blocks containing 1 must contain 8,9 , and 10 ; the three remaining blocks containing 2 must contain 6, 7, and 10; the remaining blocks containing 3 must contain 5, 7, and 9 ; the remaining blocks containing 4 must contain 5, 6, and 8 . This gives the following blocks:

$$
\begin{array}{lllllll}
\frac{1}{4} & 8 & 2 & 6 & 3 & 4 & 4 \\
\hline 1 & 9 & 2 & 7 & 3 & 7 & 46 \\
\hline 1 & 10 & 2 & 10 & 3 & 9 & 48 \\
\hline
\end{array}
$$

The remaining three blocks must contain 5, 6, . ., 10. Reasoning as before, it is clear
that these blocks are

$$
\underline{510}, 69, \text { and } 78 .
$$

This gives the $b=40$ blocks of design 1. The reader can check the constancy of the $p_{i k}^{i}(i, j, k=1,2)$ for all treatment-pairs of the design.

The solution for design 2 of table 5 consists of the blocks of design 1 written twice.
Let us now develop some theory that will enable us to construct design 3 of table 5 .
Consider any PBIB design with 2 associate classes and 2 treatments per block, and let its parameters be denoted by $v, b, r, k=2, \lambda_{1}, \lambda_{2}, n_{1}, n_{2}$, and $p_{i k}^{i}(i, j, k=1,2)$. Let us adopt the convention $\lambda_{1}>\lambda_{2} \geq 0$. The blocks of this design are some $b$ treatment-pairs from among the $v(v-1) / 2$ distinct treatment-pairs that can be formed from the $v$ treatments. The $b$ blocks will not all be different unless $\lambda_{1}=1$ and $\lambda_{2}=0$. Let $D$ represent this design.

Let us form a design $D^{\prime}$ from $D$ as follows: If a treatment-pair $(\alpha, \beta)$ occurs in $\lambda_{1}$ blocks of $D$, let $(\alpha, \beta)$ occur in $\lambda_{2}$ blocks of $D^{\prime}$; if $(\alpha, \beta)$ occurs in $\lambda_{2}$ blocks of $D$, let $(\alpha, \beta)$ occur in $\lambda_{1}$ blocks of $D^{\prime}$. We shall call design $D^{\prime}$ the pair-complement of design $D$.

Let the parameters of design $D^{\prime}$ be represented by the usual letters carrying primes. Any treatment $\alpha$ will occur in $\lambda_{1}$ blocks with each of its $n_{1}$ first associates in design $D$ and will consequently occur in $\lambda_{2}$ blocks with each of these $n_{1}$ treatments in design $D^{\prime}$. Similarly, $\alpha$ will occur in $\lambda_{1}$ blocks in $D^{\prime}$ with each of the $n_{2}$ second associates of $\alpha$ in $D$. If we put $\lambda_{2}^{\prime}=\lambda_{2}$, then clearly,

$$
\begin{equation*}
v^{\prime}=v, \quad k^{\prime}=2, \quad \lambda_{1}^{\prime}=\lambda_{1}, \quad \lambda_{2}^{\prime}=\lambda_{2}, \quad n_{1}^{\prime}=n_{2}, \quad n_{2}^{\prime}=n_{1} . \tag{41}
\end{equation*}
$$

Consequently, $\alpha$ occurs in

$$
\begin{equation*}
r^{\prime}=\lambda_{2} n_{1}+\lambda_{1} n_{2} \tag{42}
\end{equation*}
$$

blocks in $D^{\prime}$, and $D^{\prime}$ will contain

$$
\begin{equation*}
b^{\prime}=v\left(\lambda_{2} n_{1}+\lambda_{1} n_{2}\right) / 2 \tag{43}
\end{equation*}
$$

blocks. A bit of reflection on the parameters of the second kind for $D^{\prime}$ reveals that

$$
P_{1}^{\prime}=\left(p_{i k}^{\prime \prime}\right)=\left(\begin{array}{ll}
p_{22}^{2} & p_{21}^{2}  \tag{44}\\
p_{12}^{2} & p_{11}^{2}
\end{array}\right)
$$

and

$$
P_{2}^{\prime}=\left(p_{i k}^{2 \prime}\right)=\left(\begin{array}{ll}
p_{22}^{1} & p_{21}^{1}  \tag{45}\\
p_{12}^{1} & p_{11}^{1}
\end{array}\right) .
$$

It is easy to verify that the parameters of $D^{\prime}$ satisfy the necessary relations among the parameters set forth in [4].

Consideration of the association schemes of group divisible, triangular, Latin square, and cyclic designs with two treatments per block shows that if design $D$ is one of these types, then its pair-complement $D^{\prime}$ is of the same type. It should also be noted that the existence of $D$ as a connected design does not guarantee that $D^{\prime}$ will be a connected design. For instance, if $p_{21}^{2}=\lambda_{2}=0$, then $D^{\prime}$ will be a disconnected design of the group divisible type.

The parameters of design 1 are

$$
\begin{align*}
& v=16, \quad r=5, \quad \lambda_{1}=1, \quad n_{1}=5, \quad P_{1}=\left(\begin{array}{ll}
0 & 4 \\
4 & 6
\end{array}\right) \\
& b=40, \quad k=2, \quad \lambda_{2}=0, \quad n_{2}=10, \quad P_{2}=\left(\begin{array}{ll}
2 & 3 \\
3 & 6
\end{array}\right) \tag{46}
\end{align*}
$$

If design $1^{\top}$ is design $D$, then its pair-complement $D^{\prime}$ will have parameters

$$
\begin{array}{llll}
v^{\prime}=16, & r^{\prime}=10, & \lambda_{1}^{\prime}=1, & n_{1}^{\prime}=10,
\end{array} P_{1}^{\prime}=\left(\begin{array}{ll}
6 & 3 \\
3 & 2
\end{array}\right), ~ 子\left(\begin{array}{ll}
6 & 4  \tag{47}\\
4 & 0
\end{array}\right), ~ . ~ \lambda_{2}^{\prime}=0, \quad n_{2}^{\prime}=5, \quad P_{2}^{\prime}=\left(\begin{array}{ll} 
\\
b^{\prime}=80, & k^{\prime}=2,
\end{array}\right.
$$

But (47) is design 3 of table 5. Hence, design 3 is the pair-complement of design 1.
After a change of notation the $b=80$ blocks of design 3 may be written as follows:


An examination of the pair-complements of the designs discussed in this paper will show that they are either
a. Disconnected designs in which we have no interest,
b. Designs with $r>10$,
c. Self pair-complementary, i. e., have the same parameters as $D$, or
d. Other designs that are included in this paper.

We shall next prove a theorem that will enable us to construct design 6 of table 5 .
Suppose a construction exists for a design $D$ that has parameters $v, b, r, k=3, \lambda_{1}=1$, $\lambda_{2}=0, n_{1}, n_{2}$, and $p_{i k}^{i}(i, j, k=1,2)$. We shall form a design $D^{*}$ from design $D$ as follows: If $B_{j}=(\alpha, \beta, \gamma)$ is a block of $D$, then the treatment-pairs $(\alpha, \beta),(\alpha, \gamma)$, and $(\beta, \gamma)$ shall be blocks of $D^{*}$. Obviously, any treatment-pair $(\theta, \pi)$ that occurs in $\lambda_{i}$ blocks of $D$ will also occur in $\lambda_{i}^{*}=\lambda_{i}$ blocks of $D^{*},(i=1,2)$. Any treatment of $D^{*}$ will consequently have the same $i$ th associates as it has in $D$. Hence, the parameters of the second kind $p_{i k}^{i *}(i, j, k=1,2)$ of $D^{*}$ will be the same as the corresponding parameters of $D$. Thus, $D^{*}$ is a two-associate class PBIB design with parameters,

$$
\begin{array}{llll}
v^{*}=v, & r^{*}=2 r, & \lambda_{1}^{*}=1, & n_{1}^{*}=n_{1},
\end{array} P_{1}^{*}=\left(p_{i k}^{1}\right), ~ 子, ~ \lambda_{2}^{*}=0, \quad n_{2}^{*}=n_{2}, \quad P_{2}^{*}=\left(p_{i k}^{2}\right) . ~ \$ k^{*}=2, \quad \lambda_{2}^{*}=3 b, \quad .
$$

This proves the theorem:
Theorem. If a solution for design $D$ having parameters $v, b, r, k=3, \lambda_{1}=1, \lambda_{2}=0, n_{1}, n_{2}$, and $p_{i k}^{i}(i, j, k=1,2)$ exists, then a solution for design $D^{*}$, whose parameters are given by (48), also exists and consists of all distinct treatment-pairs that can be formed from the treatments within the blocks of $D$.

Now Bose and Clatworthy [2] have solved the design whose parameters are

$$
\begin{align*}
& v=27, \quad r=5, \quad \lambda_{1}=1, \quad n_{1}=10, \quad P_{1}=\left(\begin{array}{ll}
1 & 8 \\
8 & 8
\end{array}\right) \\
& b=45, \quad k=3, \quad \lambda_{2}=0, \quad n_{2}=16, \quad P_{2}=\left(\begin{array}{ll}
5 & 5 \\
5 & 10
\end{array}\right) \tag{49}
\end{align*}
$$

The blocks of design (49) are:


Thus, by the theorem, design 6 of table 5, having parameters

$$
\begin{aligned}
& v^{*}=27, \quad r^{*}=10, \quad \lambda_{1}^{*}=1, \quad n_{1}^{*}=10, \quad P_{1}^{*}=P_{1}, \\
& b^{*}=135, \quad k^{*}=2, \quad \lambda_{2}^{*}=0, \quad n_{2}^{*}=16, \quad P_{2}^{*}=P_{2},
\end{aligned}
$$

has a solution that may be obtained by forming the three distinct treatment-pairs from the treatments in each of the blocks of design (49).

## 8. The Enumeration Problem

There exist many sets of parameters for PBIB designs with two associate classes, $k=2$ and $2 \leq r \leq 10$, that satisfy the necessary relations set forth in [4] and restated in [6]. However, not all such sets of parameters have solutions (constructions). The writer has obtained an exhaustive enumeration of the sets of parameters for the class of designs investigated in this paper by use of theory developed in reference [7]. Those parameter sets that fail to satisfy the necessary conditions for the existence of solutions given in [3, 6, and 8] were discarded. The remaining sets of parameters appear in this paper. Constructions are given for all designs for which parameters are listed in tables 1 to 5 , with four exceptions. These four designs are the designs of table 5 marked with asterisks. For these designs, $b$ is rather large, and consequently an intensive search for solutions was not made.

For suggestions leading to the construction of designs 3 and 6 of table 5, the writer is indebted to W. J. Youden.

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[^0]:    ${ }^{1}$ Figures in brackets indicate the literature references at the end of this paper.

