Bounds for Characteristic Roots of Matrices II

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This is a continuation of an earlier note (Duke Mathematical Journal, volume 15, pages 1043-44 (1948)). It deals with bounds for the characteristic roots of matrices with positive (nonnegative) elements, and with bounds for multiple roots.

This note is a continuation of an earlier one [1]. There the position of the characteristic roots of an \( n \times n \) matrix \( A = (a_{ik}) \) inside or on the boundary of the \( n \) circles \( C_i \) with centers \( a_{ii} \) and radii \( A_i = \sum_{k \neq i} |a_{ik}| \) was studied.

In particular, it was shown in [1] that for \( n = 2 \) the common part of the circles cannot contain a root unless it is a common boundary point. This fact is not true for \( n > 2 \) as is, for example, shown by the matrix

\[
\begin{pmatrix}
0 & 1 & 0 \\
0 & 0 & 1 \\
0 & -15 & -8
\end{pmatrix}
\]

which has the roots 0, -3, -5. The root 0 is contained in all three circles. It can however be shown that an analogue of the situation for \( n = 2 \) holds if further conditions are imposed on the elements \( a_{ii} \).

Theorem 1. The dominant root of a matrix of positive elements cannot be a common point of all \( n \) circles \( C_i \) unless it is a common boundary point of at least two of the circles.

Proof. It is known [2] that the dominant root \( \lambda \) of such a matrix is real and positive and that the corresponding characteristic vector \( x_1, \ldots, x_n \) can be chosen in such a way that all its components are positive. Consider then the equation

\[
a_{i1} x_1 + \cdots + (a_{ii} - \lambda) x_i + \cdots + a_{in} x_n = 0,
\]

or

\[
(\lambda - a_{ii}) x_i = \sum_{k \neq i} a_{ik} x_k.
\]

As the right-hand side of (2) is positive, it follows that

\[
\lambda - a_{ii} > 0.
\]

Equation (2) implies that

\[
(\lambda - a_{ii}) x_i \geq A_i \min_{k \neq i} x_k.
\]

Let \( \lambda \) be an inner point of all circles \( C_i \), that is,

\[
\lambda - a_{ii} < A_i, \quad i = 1, \ldots, n.
\]

It follows from (4) that

\[
x_i > \min_{k \neq i} x_k, \quad i = 1, \ldots, n.
\]
could only occur, if the matrix $A$ can be transformed
to the form
\[
\begin{pmatrix}
P & 0 \\
U & Q
\end{pmatrix}
\]
by the same permutation of the rows and columns.

Thus the following theorem holds:

**Theorem 2.** Let $A$ be a matrix of nonnegative elements which cannot be transformed to the form
\[
\begin{pmatrix}
P & 0 \\
U & Q
\end{pmatrix}
\]
by the same permutation of the rows and columns. Then the real dominant root of $A$ cannot be a common point of all $n$ circles $C_i$ unless it is a common boundary point of at least two of these circles.

Consider now matrices whose main diagonal elements are arbitrary real, but whose off-diagonal, elements are positive (nonnegative). It is known [3, 4] that such a matrix has as root with largest real part a real positive (nonnegative) number and the components of the corresponding vectors can be chosen to be all positive (nonnegative). It is clear that theorems 1 and 2 can be generalized to these matrices.

The remaining two theorems concern multiple roots of general matrices. It is easy to find examples of matrices with a multiple root that is contained as inner or boundary point in only one circle $C_i$. It can, however, be proved that in this case the rank of the matrix
\[
(a_{ik}-\lambda \delta_{ik})
\]
is $n-1$.

**Theorem 3.** A characteristic root $\lambda$ which is an inner or boundary point of only one circle $C_i$ cannot have two independent characteristic vectors corresponding to it.

**Proof.** Assume that $|a_{ii}-\lambda| \leq A_i$ and $|a_{ii}-\lambda| > A_i$
for $i \neq 1$. Consider any characteristic vector $x_1, \ldots, x_n$ which corresponds to $\lambda$. Then the relations
\[
|a_{ii}-\lambda| |x_i| \leq A_i \max_{k \neq i} |x_k|
\]
hold. For $i > 1$ they imply that
\[
|x_i| \leq \max_{k \neq i} |x_k|
\]
Hence
\[
|x_i| = \max_k |x_k| > |x_i| \text{ for } i > 1.
\]
This eliminates the possibility of a vector $y_1, \ldots, y_n$
that corresponds to $\lambda$ and is independent of $x_1, \ldots, x_n$,
because otherwise a linear combination of both vectors could be found in which the first component is 0 and which does not vanish identically.

**Theorem 4.** If $A$ has a characteristic root $\lambda$ of multiplicity $n-1$ and with $n-1$ independent characteristic vectors then $\lambda$ is contained in at least $n-1$ circles $C_i$.

**Proof.** The matrix $(a_{ik}-\lambda \delta_{ik})$ has the same vectors as $(a_{ik})$ and as roots the numbers $\mu - \lambda$ when $\mu$ runs through all roots of $(a_{ik})$. The circles that correspond to the matrix $(a_{ik}-\lambda \delta_{ik})$ have the same radii as the original ones, but their centers are moved from $a_{ii}$ to $a_{ii} - \lambda$. Hence, we may restrict ourselves to the case where $\lambda = 0$. In this case the matrix can be transformed to the form
\[
X = \begin{pmatrix}
0 \\
& \ddots \\
& & 0 \\
& & & \alpha
\end{pmatrix}
\]
by means of a nonsingular matrix $U = (u_{ik})$. Denote by $\Delta$ the determinant $|u_{ik}|$ and by $U_{ik}$ the cofactor of $u_{ik}$. It is then easy to see that the original matrix $(a_{ik}) = UXU^{-1}$ is of the form
\[
(a_{ik}) = \begin{pmatrix}
\alpha u_{1i} U_{1i1} & \alpha u_{1i} U_{2i1} & \ldots & \alpha u_{1i} U_{ni1} \\
\alpha u_{2i} U_{1i1} & \alpha u_{2i} U_{2i1} & \ldots & \alpha u_{2i} U_{ni1} \\
& \ldots & \ldots & \ldots \\
\alpha u_{ni} U_{1i1} & \alpha u_{ni} U_{2i1} & \ldots & \alpha u_{ni} U_{ni1}
\end{pmatrix}
\]
Assume now that 0 lies outside one of the circles, say $C_i$. This implies
\[
|U_{1i}| > \sum_{j \neq i} |U_{jn}|.
\]
This, however, implies that
\[
|U_{in}| < \sum_{j \neq i} |U_{jn}|, \quad i = 2, \ldots, n.
\]
From this it follows that 0 lies inside all circles $C_i,
\quad i = 2, \ldots, n$, which proves the theorem.


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