Solution of the Telegrapher's Equation With Boundary Conditions on Only One Characteristic

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I. Introduction

In one treatment of planetary atmospheric flow as horizontal, autobarotropic, nonviscous, and nondiverging in a plane, Rossby [9] considered the idealized case of a constant west-wind component, \( U \), and a south-wind component, \( v \), dependent on the west-to-east distance coordinate, \( \psi \), and time, \( t \), but independent of the south-to-north distance coordinate, \( \mu \). It was shown in [5] that this \( v \) satisfies the telegrapher's equation (eq 3) below, where \( x = \psi - U t, t = 4 \beta r \). The parameter \( \beta = 2 \Omega \cos \phi (d\phi/d\mu) \) is here considered constant; \( \Omega \) is the angular speed of the earth's rotation, and \( \phi \) is latitude. For this simple atmospheric model, the meteorological forecast problem is one of determining \( v(x, t) \) for future times \( t \), given only \( v(x, 0) \). But on a plane the specification of \( v(x, 0) \) is not sufficient to determine \( v(x, t) \) for many \( t > 0 \), because the line \( t = 0 \) is a characteristic of eq 3 (see p. 254 of [11]). Having its initial conditions on only one characteristic is an unusual feature of the present problem that does not seem to have arisen in other physical problems known to the author to lead to the telegrapher's equation.

The author shows that the forecast problem has a unique solution when it is assumed that the world is round, that is, when the solution is assumed to be periodic in \( x \). The problem is stated in section II and solved in section III. In section IV the solution is represented in terms of a Green's function. In section V a procedure is outlined for computing the Green's function by improving the convergence of its Fourier series. In section VI certain auxiliary polynomials, \( \sigma_k(x) \), used in section V are discussed and related to the Bernoulli polynomials. In section VII are reported without proof a few results on the approximate solution of the problem by a difference equation, taken from [6]. In section VIII is given a table of values of the Green's function, as computed in the Computation Unit of the Institute for Numerical Analysis.

The present author first reported this work in [7]. Independently of the research reported here, Charney, Eliassen, and Hunt of the Institute for Advanced Study considered the telegrapher's equation while investigating numerical weather prediction in general. Their research was reported in [1] and is written up in [2]. The work of these men includes much of what is reported here, and much more.

II. Statement of the Problem

Let \( C \) be the circumference of a unit circle; let us adopt an angle coordinate \( x \) for \( C : -\pi < x < \pi \). Let \( I \) be the set of time-instants \( t : 0 \leq t < \infty \). Let \( R \) be the closed two-dimensional region consisting of all points \( (x, t) \) with \( x \) in \( C \) and \( t \) in \( I \). Let \( f(x) \) be a real-valued function that satisfies the follow-
ing hypotheses, but which is otherwise arbitrary: 
\[ H_1: f(x) \text{ is sectionally smooth}^3 \text{ on } C. \] 
Moreover, 
\[ f(x) = \frac{1}{2}[f(x+0)+f(x-0)], \quad (\text{all } x). \quad (1) \]

\[ H_2: f(x) \text{ has the average value zero:} \]
\[ \int_{-\pi}^{\pi} f(x)dx = 0. \quad \text{ (2)} \]

The problem is to find a real-valued function, \( v(x, t) \), defined everywhere on \( R \), with the following four properties:

- \( P_1: v_t \text{ exists}^4 \text{ and is continuous throughout } R. \)
- \( P_2: v_x \text{ and } v_{tx} = \partial v_t/\partial x \text{ exist and are continuous everywhere in } R \) except, at most, for a finite number of values of \( x \).
- \( P_3: \text{Whenever } v_x \text{ is defined, the following hyperbolic partial differential equation (the telegrapher’s equation) is satisfied:} \]
\[ v_{xx} + \frac{1}{4}v = 0. \quad \text{ (3)} \]
- \( P_4: \text{For } t=0, v(x, t) \text{ reduces to } f(x): \]
\[ v(x, 0) = f(x). \]

III. Solution of the Problem, Uniqueness

One gets a formal solution by separation of variables and use of Fourier series. Assume a solution of eq 3 of form \( v(x, t) = X(x) T(t) \). Then \( v_x = X'(x) T'(t) \), and eq 3 takes the form
\[ \frac{X'(x)}{X(x)} \cdot \frac{T'(t)}{T(t)} = -\frac{1}{4}. \quad \text{ (4)} \]

The two factors in eq 4 must themselves be constant:
\[ \frac{X'(x)}{X(x)} = \lambda, \quad \text{ (5)} \]
\[ \frac{T'(t)}{T(t)} = -\frac{1}{4}\lambda. \quad \text{ (6)} \]

A solution of eq 5 for \(-\infty < x < \infty \) is \( X(x) = e^{\lambda x} \). For \( x \) on the circle \( C \), however, one must have \( X(-\pi) = X(\pi) \), or \( e^{-\lambda \pi} = e^{\lambda \pi} \). Taking logarithms, one sees that \(-\pi \lambda = \pi \lambda + 2n\pi i (n=0, \pm 1, \pm 2, \ldots )\). Hence \( \lambda = ni \) (\( n=0, \pm 1, \pm 2, \ldots )\). Since the value \( \lambda = 0 \) is incompatible with eq 4, there remain the following fundamental solutions of eq 5:
\[ X_n(x) = e^{ni x}, \quad (n=\pm 1, \pm 2, \pm 3, \ldots ). \]

The subscript \( n \) denotes partial derivatives.

Corresponding to \( X_n(x) \), the solution \( T_n(t) \) of eq 6 for \( \lambda = ni \) is \( T_n(t) = \exp \left(i \frac{t}{4n} \right) \). Hence for \( n=1, \pm 2, \ldots \) the functions \( X_n(x) T_n(t) = \exp \left[i(nx+ t/4n) \right] \) have properties \( P_1, P_2, \) and \( P_3 \). By taking linear combinations of the functions \( X_nT_n \) and \( X_{-n}T_{-n} \), one obtains the equivalent pair of functions \( \cos \left(nx+ t/4n \right) \) and \( \sin \left(nx+ t/4n \right) \). Both of the latter functions have properties \( P_1, P_2, \) and \( P_3 \).

In order to obtain a solution with enough degrees of freedom to satisfy \( P_4 \), consider the series,
\[ v(x, t) \sim \sum_{n=1}^{\infty} \left[ a_n \cos \left(nx+ \frac{t}{4n} \right) + b_n \sin \left(nx+ \frac{t}{4n} \right) \right], \quad \text{ (7)} \]
where \( a_n, b_n \) are undetermined constants. We postpone a discussion of the convergence of the series (eq 7) for \( t \neq 0 \) and consider it for \( t=0 \), where \( v(x, 0) \) is supposed to equal \( f(x) \):
\[ v(x, 0) \sim \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx). \quad \text{ (8)} \]

If the series in eq 8 actually does converge to \( f(x) \) for all \( x \), it is shown on p. 274 of [12] that the coefficients, \( a_n, b_n \), must be the Fourier coefficients of \( f \). Conversely, by p. 25 of [12], the hypothesis \( H_1 \) is sufficient to insure that the Fourier series of \( f \) actually converges to \( f(x) \); it even converges uniformly for \( x \) in any interval bounded away from a discontinuity of \( f \). Moreover, the hypothesis \( H_2 \) implies that in the Fourier series \( a_0 = 0 \). We henceforth stipulate that the series (eq 8) is the Fourier series of \( f \). It is important to note \(^6 \) that \( H_1 \) implies that \( a_n \) and \( b_n \) are \( O(1/n) \); that is, there exists a constant \( M < \infty \) such that
\[ |na_n| \leq M, \quad |nb_n| \leq M \quad \text{ (all } n). \quad \text{ (9)} \]

1. Proof of Convergence

There remains only a proof that the series in eq 7 actually does converge to a function \( v(x, t) \) with the required properties, \( P_1, P_2, P_3, P_4 \). It will be useful to have the following representations of \( \cos \left(t/4n \right) \) and \( \sin \left(t/4n \right) \). They are proved by Taylor’s formula and hold for all values of \( t/4n \):

\(^3 \) That is, both \( f(x) \) and \( f'(x) \) are continuous on \( C \) except for a finite number of jump discontinuities.

\(^4 \) The subscripts denote partial derivatives.

\(^6 \) See p. 18 of [12].
where $|\alpha_n| = |\alpha \left( \frac{t}{4n} \right)| \leq 1$;

\[
\begin{align*}
\cos \left( \frac{t}{4n} \right) &= 1 - \frac{\alpha t^2}{32n^2}, \\
\sin \left( \frac{t}{4n} \right) &= \frac{\beta n t}{4n},
\end{align*}
\]  
(10)

where $|\beta_n| = |\beta \left( \frac{t}{4n} \right)| \leq 1$;

\[
\begin{align*}
\sin \left( \frac{t}{4n} \right) &= \frac{\beta n t}{4n} - \gamma_n t^3, \\
\sin \left( \frac{t}{4n} \right) &= \frac{t}{4n} - \gamma_n t^3 - 384n^3,
\end{align*}
\]  
(12)

where $|\gamma_n| = |\gamma \left( \frac{t}{4n} \right)| \leq 1$.

Using eq 10 and 11, one sees that for any fixed $t$,

\[
\begin{align*}
&\sum_{n=1}^{\infty} \left[ a_n \cos \left( nx + \frac{t}{4n} \right) + b_n \sin \left( nx + \frac{t}{4n} \right) \right] = \\
&\sum_{n=1}^{\infty} \left[ (a_n \cos nx + b_n \sin nx) \cos \left( \frac{t}{4n} \right) + \\
&b_n \cos nx - a_n \sin nx \cos \left( \frac{t}{4n} \right) \right] - \\
&\sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) - \\
&\frac{t^2}{32} \sum_{n=1}^{\infty} a_n \left( \frac{a_n}{n^2} \cos nx + \frac{b_n}{n^2} \sin nx \right) + \\
&\frac{t^2}{4n} \sum_{n=1}^{\infty} \beta_n \left( \frac{b_n}{n^2} \cos nx - \frac{a_n}{n^2} \sin nx \right) - \\
&\Sigma_0 + \Sigma_1 + \Sigma_2.
\end{align*}
\]  
(13)

Representation as the sum of three series is permitted because each of the series $\Sigma_0$, $\Sigma_1$, $\Sigma_2$ converges. $\Sigma_0$ converges for all $x$ because it is the Fourier series of $f$; its convergence is uniform in any interval bounded away from a discontinuity of $f(x)$. Fix any positive number $t_1$, and restrict the consideration to $t$'s such that $0 \leq t \leq t_1$. Since $a_n$ and $b_n$ are $O(1/n)$, $\Sigma_1$ and $\Sigma_2$ are convergent uniformly in $x$ and $t$. For example, $\Sigma_2$ is dominated by $(t_1/4) \sum (|b_n|/n + |a_n|/n)$, a series convergent like $\sum (1/n^2)$. The series of eq 7 is thus convergent for all $x$, $t$ and defines by its limit a function $v(x, t)$:

\[
v(x, t) = \sum_{n=1}^{\infty} \left[ a_n \cos \left( nx + \frac{t}{4n} \right) + b_n \sin \left( nx + \frac{t}{4n} \right) \right].
\]  
(14)

Moreover, the series (eq 14) converges uniformly for $x$, $t$ such that $x$ is bounded away from a jump of $f(x)$.

Since $\Sigma_1$ and $\Sigma_2$ converge uniformly, they converge to continuous functions of $x$ and $t$. Thus the only discontinuities of $v(x, t)$ are those from $\Sigma_0$, that is, those of $f(x)$. This is the property of hyperbolic differential equations that discontinuities in their solutions are propagated along characteristics. Let the set of discontinuities of $f(x)$ be denoted by $E$.

For all $x$, $t$ one may obtain $v_t$ by termwise differenitiation of eq 14, because by eq 9 the resulting series is absolutely and uniformly convergent in both $x$ and $t$:

\[
v_t(x, t) = \frac{1}{4} \sum_{n=1}^{\infty} \left[ b_n \cos \left( nx + \frac{t}{4n} \right) - \frac{a_n}{n} \sin \left( nx + \frac{t}{4n} \right) \right].
\]  
(15)

Moreover, $v_t$ is continuous for all $x$, $t$, so that $P_1$ holds. Now one may not obtain $v_t$ by termwise differentiation of eq 14, because the resulting series will generally not converge. However, $v_x$ does exist and is a continuous function of $x$ and $t$ for all $t$ and for all $x$ not in $E$. To see this, one uses eq 10 and 12 to carry the Taylor formula (eq 13) to one higher power of $t$. It is found that

\[
v(x, t) = f(x) + \frac{t}{4} \sum_{n=1}^{\infty} \left( \frac{b_n}{n} \cos nx - \frac{a_n}{n} \sin nx \right) - \\
\frac{t^2}{32} \sum_{n=1}^{\infty} \alpha_n \left( \frac{a_n}{n^2} \cos nx + \frac{b_n}{n^2} \sin nx \right) - \\
\frac{t^3}{384} \sum_{n=1}^{\infty} \gamma_n \left( \frac{b_n}{n^3} \cos nx - \frac{a_n}{n^3} \sin nx \right).
\]  
(16)

By termwise differentiation of eq 16, it is found that

\[
v_t(x, t) = f'(x) - \frac{t}{4} \sum_{n=1}^{\infty} \left( a_n \cos nx + b_n \sin nx \right) - \\
\frac{t^2}{32} \sum_{n=1}^{\infty} \alpha_n \left( \frac{b_n}{n} \cos nx - \frac{a_n}{n} \sin nx \right) + \\
\frac{t^2}{384} \sum_{n=1}^{\infty} \gamma_n \left( \frac{a_n}{n^2} \cos nx + \frac{b_n}{n^2} \sin nx \right) = \\
f'(x) - \frac{t}{4} f(x) + \Sigma_3 + \Sigma_4.
\]  

Restricting attention to $t$ with $0 \leq t \leq t_1 < \infty$, one sees that the expressions $\Sigma_3$ and $\Sigma_4$ are uniformly convergent with respect to $x$ and $t$. The series leading to $f(x)$ is uniformly convergent for $x$ in any interval bounded away from a discontinuity.

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of \( f(x) \). Hence, for \( x \) not in \( E \), termwise differentiation leads to the correct value of \( v_x \). Moreover, \( v_t \) is continuous in \( x \) and \( t \) whenever \( x \) is a point of continuity of \( f(x) \). To get \( v_{tx} \) one may differentiate eq 15 termwise with respect to \( x \):

\[
v_{tx}(x,t) = \quad -\frac{1}{4} \sum_{n=1}^{\infty} \left[ a_n \cos \left( nx + \frac{t}{4n} \right) + b_n \sin \left( nx + \frac{t}{4n} \right) \right]
\] (17)

As remarked after eq 14, the series in eq 17 is uniformly convergent for \( x \), \( t \) such that \( x \) is bounded away from the discontinuities of \( f(x) \). Hence \( v_{tx} \) is continuous in \( x \) and \( t \) for all \( x \), \( t \) except for \( x \) in \( E \). Since \( v_1 \), \( v_x \), \( v_{tx} \) are all continuous, \( v_{tx} \) exists and equals \( v_{tx} \) except on the lines corresponding to the discontinuities of \( f(x) \). This shows that \( v(x, t) \) has property \( P_2 \). The eq 17 and 14 show that \( v \) satisfies the telegrapher's equation (eq 3). Finally, property \( P_4 \) was taken care of by the selection of \( \{a_n, b_n\} \). Thus the problem is solved completely.

2. Proof of Uniqueness

It will be shown that \( v(x, t) \) is the only function that solves the above problem. Suppose that \( v_1(x, t) \) were a second solution. Then the difference, \( w(x, t) = v - v_1 \), satisfies the same problem with \( f(x) = 0 \). For each \( x_1 \), by property \( P_1 \), \( w(x_1, t) \) and \( w_t(x_1, t) \) are continuous functions of \( t \) for \( 0 \leq t < \infty \), while \( w(x, 0) = 0 \) and \( w_x(x, 0) = 0 \) are, of course, continuous functions of \( x \). For each \( t \), let the value of \( w(x, t) \) be extended as a periodic function of \( x \) to all \( x \) in the interval \([-2\pi, 2\pi]\). Now the values \( w(-\pi, t) \) and \( w(x, 0) \) are given on two characteristics of eq 3. By pp. 21 to 22 of [10] they are therefore sufficient to determine \( w(x, t) \) uniquely for all \( x, t \). On the other hand, the values \( w(\pi, t) \) and \( w(x, 0) \) are also sufficient to determine \( w(x, t) \) for all \( x, t \). Since \( w(-\pi, t) = w(\pi, t) \) and \( w(x, 0) = w(-x, 0) = 0 \), it is seen by symmetry that \( w(x, t) = w(-x, t) \). Now since the values of \( x \) lie on a circle, there is nothing exceptional about the line \( x = \pi \). The above argument will also show that, for each value of \( x_1 \), \( w(x_1 + x, t) = w(x_1 - x, t) \). It follows that for each fixed \( t \), \( w(x, t) = constant \), whence \( w(x, t) = h(t) \). By eq 3, \( -\frac{1}{4} w = w_{tx} = (d/dx)h'(t) = 0 \). Hence, the constant value of \( w(x, t) \) must be everywhere zero. Then \( v = v_1 \), and the solution \( v(x, t) \) given by eq 14 is unique.

The results of section III may be summarized in the following theorem, phrased in the notation of section II.

**Theorem 1.** If the real-valued function \( f(x) \) defined on \( C \) satisfies hypotheses \( H_1 \) and \( H_2 \), then there exists a unique function, \( v(x, t) \), defined on \( R \) and possessing properties \( P_1 \), \( P_2 \), \( P_3 \), and \( P_4 \). If eq 8 is the Fourier series of \( f(x) \), then \( v(x, t) \) is defined explicitly by eq 14.

It is of mathematical interest to note that Theorem 1 can be extended to general functions \( f(x) \) of bounded variation. That is, one may replace \( H_1 \) by the weaker hypothesis

\( H'_1 \): \( f(x) \) is of bounded variation on \( C \). Moreover, \( f(x) = \frac{1}{2}[f(x+0) + f(x-0)] \) \((all x)\).

The solution \( v(x, t) \) is required to have property \( P_2 \), instead of \( P_5 \):

\( P'_2 \): \( v_x \) exists in \( R \) except for \( x \) in a set \( E \) \((E \subset C)\) of Lebesgue measure zero; for all \( t \) and for all \( x \) not in \( E \), \( v_{tx} \) exists and is a continuous function of \( x \) and \( t \).

The extension of Theorem 1 is stated as follows:

**Theorem 2.** If the real-valued function \( f(x) \) defined on \( C \) satisfies hypotheses \( H_1 \) and \( H'_1 \), then there exists a unique function \( v(x, t) \) defined on \( R \) and possessing properties \( P_1 \), \( P'_2 \), \( P_3 \), and \( P_4 \). If eq 8 is the Fourier series of \( f(x) \), then \( v(x, t) \) is defined explicitly by eq 14.

The convergence proof of section III, 2 requires only slight modification to serve as a proof of Theorem 2. For an arbitrary function \( f(x) \) of bounded variation, there need be no interval of continuity; one may therefore not expect the series (eq 14) to converge uniformly in any interval. The termwise differentiations of section III, 2 can, however, be justified for almost all \( x \) by the fact that the resulting series are Fourier series.

IV. Representation by a Green's Function

The formula (eq 14) for the solution of the problem of section II is directly adapted to numerical computation only when the Fourier coefficients \( a_n, b_n \) converge rapidly to zero. But some of the most important cases in meteorology are where \( f(x) \) has discontinuities (see footnote 7).

1 This extension seems to have no meteorological interest. However, it is of much importance in meteorology to deal with functions \( f(x) \) with some discontinuities; such discontinuities occur at fronts between air masses.

2 Same as eq 1.

3 It follows from \( P_1 \) and \( P'_2 \) that \( v_{tx} \) exists and equals \( v_{tx} \) for all \( t \) and for all \( x \) not in \( E \); see pp. 55 to 57 of [3].

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With such an \( f \) the Fourier coefficients are, roughly speaking, of the order \( O(1/n) \), and for those \( f \) the convergence of eq 14 is hopelessly slow.

It is possible, however, to improve the convergence of eq 14 to such a degree that computation of \( v(x, t) \) is reasonably possible. The procedure will be illustrated in section V for one particular choice of \( f(x) \):

\[
f(x) = \sigma_0(x) = \sum_{n=1}^{\infty} \sin nx = \begin{cases} \frac{1}{2}(\pi-x) & (0 < x \leq \pi) \\ (x=0) \\ -\frac{1}{2}(\pi+x) & (-\pi \leq x < 0). \end{cases}
\]  

(18)

It may be shown by direct computation that the Fourier series of \( \sigma_0(x) \) is the series of eq 18. It then follows that the series converges to \( \sigma_0(x) \) for all \( x \). The reason for choosing \( \sigma_0(x) \) is two-fold: (a) it is of meteorological interest to see how a simple discontinuity in \( v(x, 0) \) is propagated, as \( t \) increases; (b) for any \( f(x) \) that is sectionally smooth, it is possible to represent the corresponding \( v(x, t) \) in terms of the solution for the special initial condition \( v(x, 0) = \sigma_0(x) \).

The present section is devoted to proving the property (b). Suppose, therefore, that \( G(x, t) \) is the solution of the problem of section II with the initial condition \( \sigma_0(x) \); then \( G(x, 0) = \sigma_0(x) \). Let a sectionally smooth function \( f(x) \) be given that satisfies eq 1 and 2. Let \( f(x) \) have the jump \( J_k = f(x_k+0) - f(x_k-0) \) at the point \( x_k \) \((k = 1, 2, \ldots, K)\). Let \( \sigma_0(x) \) be continued periodically for \( x \in [-2\pi, 2\pi] \). Then \( (J_k/\pi) \sigma_0(x-x_k) \) also has the jump \( J_k \) at the point \( x_k \). Now

\[
\xi(x) = f(x) - \frac{1}{\pi} \sum_{k=1}^{K} J_k \sigma_0(x-x_k)
\]

is a continuous function, since all the jumps have been removed. Moreover, \( \xi(x) \) and the functions \( (J_k/\pi) \sigma_0(x-x_k) \) all satisfy eq 1 and 2. There is, therefore, a unique solution to the problem of section II for each of these functions. For the function \( (J_k/\pi) \sigma_0(x-x_k) \) the solution is \( (J_k/\pi) G(x-x_k, t) \). It will be shown below that the solution \( y(x, t) \) corresponding to the initial values \( \xi(x) \)

is given by

\[
y(x, t) = \frac{1}{\pi} \int_{-\pi}^{\pi} G(x-u, t) \xi'(u) du.
\]  

(19)

Since the problem of section II is linear in the initial condition \( f(x) \), and since

\[
f(x) = \xi(x) + \frac{1}{\pi} \sum_{k=1}^{K} J_k \sigma_0(x-x_k),
\]  

(20)

it follows that

\[
v(x, t) = \frac{1}{\pi} \sum_{k=1}^{K} J_k G(x-x_k, t) + \frac{1}{\pi} \int_{-\pi}^{\pi} G(x-u, t) \xi'(u) du.
\]  

(21)

The formula (eq 21) is the desired representation of \( v(x, t) \) in terms of the solution \( G(x, t) \) to the single problem where \( f(x) = \sigma_0(x) \). The nature of eq 21 indicates that \( G(x, t) \) may be called the Green’s function of the problem of section II. The representation (eq 21) is not only of theoretical importance, but it can also be used for approximating the solutions for general boundary values, \( f(x) \), once the Green’s function is tabulated. The practical problem then becomes one of approximating the integral in eq 21 by some numerical process. This latter problem is not treated here.

It remains to prove eq 19. First, it may be observed that, for each \( x \), since \( \sigma_0 \) and \( \xi \) are periodic,

\[
\frac{1}{\pi} \int_{-\pi}^{\pi} \sigma_0(x-u) \xi'(u) du = \frac{1}{\pi} \int_{-\pi}^{\pi} \sigma_0(x-u) d\xi(u) = \frac{1}{\pi} \int_{-\pi}^{\pi} \xi(u) d\sigma_0(x-u) - \frac{1}{\pi} \int_{-\pi}^{\pi} \sigma_0(x-u) \xi'(u) du = \xi(x) + \frac{1}{\pi} \int_{-\pi}^{\pi} \xi(u) \sigma_0'(x-u) du = \xi(x) - \frac{1}{2\pi} \int_{-\pi}^{\pi} \xi(u) d\xi(u) = \xi(x).
\]  

(22)

In eq 22 we have used two Riemann-Stieltjes integrals. The last step is true because \( \xi(x) \) satisfies eq 2. By eq 13,

\[
G(x, t) = \sigma_0(x) + \frac{t}{4} \sum_{n=1}^{\infty} \frac{\beta_n}{n^2} \cos nx - \frac{t^2}{32} \sum_{n=1}^{\infty} \frac{\alpha_n}{n^3} \sin nx.
\]

Hence

\[
\frac{1}{\pi} \int_{-\pi}^{\pi} G(x-u, t) \xi'(u) du = \frac{1}{\pi} \int_{-\pi}^{\pi} \sigma_0(x-u) \xi'(u) du + \frac{t}{4\pi} \int_{-\pi}^{\pi} \left[ \sum_{n=1}^{\infty} \frac{\beta_n}{n^2} \cos nx \cos nu + \sin nx \sin nu \right] \xi'(u) du - \frac{t^2}{32\pi} \int_{-\pi}^{\pi} \left[ \sum_{n=1}^{\infty} \frac{\alpha_n}{n^3} \sin nx \cos nu - \cos nx \sin nu \right] \xi'(u) du.
\]  

(23)

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Since \( \xi(u) \) is sectionally smooth, \( |\xi'(u)| \) is bounded. Hence the series in eq 23 remain uniformly convergent when multiplied by \( \xi'(u) \) and may be integrated termwise. We note that

\[
\begin{align*}
1 \int_{-\pi}^{\pi} \cos nu\xi'(u)du &= \frac{n}{\pi} \int_{-\pi}^{\pi} \sin nu\xi(u)du = nb_n, \\
1 \int_{-\pi}^{\pi} \sin nu\xi'(u)du &= -\frac{n}{\pi} \int_{-\pi}^{\pi} \cos nu\xi(u)du = -na_n,
\end{align*}
\]

(24)

where \( a_n \) and \( b_n \) are the Fourier coefficients of \( \xi \).

In view of eq 22 and 24, the termwise integration in eq 23 yields

\[
\frac{1}{\pi} \int_{-\pi}^{\pi} G(x-u, t)\xi'(u)du = \xi(x) + \frac{t}{4} \sum_{n=1}^{\infty} \beta_n \left( b_n \cos nx - a_n \sin nx \right) - \frac{\varphi}{32} \sum_{n=1}^{\infty} \alpha_n \left( a_n \cos nx + b_n \sin nx \right).
\]

(25)

But, by eq 13 and 14, the right-hand side of eq 25 is \( y(x, t) \), the solution corresponding to the initial values \( \xi(x) \). This completes the proof of eq 19.

The representations (eq 20 and 21) assume a more symmetric and unified form when the Lebesgue-Stieltjes integral is used. It can be shown that

\[
f(x) = \frac{1}{\pi} \int_{C} \sigma_0(x-u)df(u),
\]

and that

\[
v(x, t) = \frac{1}{\pi} \int_{C} G(x-u, t)df(u),
\]

(26)

(27)

where eq 26 and 27 include Lebesgue-Stieltjes integrals over the circle \( C \). Whenever \( f(x) \) has a discontinuity (say at \( x_l \)), the integral in eq 26 fails to converge as a Riemann-Stieltjes integral for \( x=x_l \), because the functions \( \sigma_0(x_l-u) \) and \( f(u) \) both have a discontinuity for \( u=0 \). The same holds for eq 27. The integrals (eq 26 and 27) are convergent for all \( x \) as Lebesgue-Stieltjes integrals. Moreover, the formula (eq 27) yields the solution of the problem when \( f(x) \) is an arbitrary function of bounded variation; the above proof of eq 19 can be modified to serve as a proof of eq 27.

V. Computation of the Green’s Function

For the purpose of using the representation (eq 21) and for its own meteorological interest, it was desired to compute the Green’s function \( G(x, t) \). A tabulation to three decimal places, accurate to approximately 0.001, appeared sufficiently accurate. An \( x \) interval of 5 degrees of longitude (\( \pi/36 \) radian) is convenient in meteorology. It was decided to compute \( G(x, t) \) for various times \( \tau \) up to 96 hours at latitudes \( \phi \) from \( 32^0 19' \) to \( 55^0 \). Since the length unit is here the radian of longitude, the expression \( d\phi/du \) in section I takes the value \( \cos \phi \). Then \( 4\beta=8\Omega \cos^2 \phi \), where \( \Omega=7.292 \times 10^{-5} \) radian/second. Now 24 hours corresponds to \( \tau=86,400 \) seconds, or to \( t=50.40 \) \( \cos^2 \phi \). At latitude \( 32^0 19' \), \( \cos^2 \phi=0.714 \), whence \( t=36 \) at 24 hours. The largest value of \( t \) for which \( G(x, t) \) was computed corresponds to 96 hours at latitude \( 32^0 19' \); it is \( t=144 \).

Let \( z=t/4 \), for convenience. By eq 14

\[
G(x, z) = \sum_{n=1}^{\infty} \frac{1}{n} \sin \left( nx + \frac{z}{n} \right).
\]

(28)

In summary, a method is required to compute \( G(x, z) \) to an accuracy of approximately \( \pm 0.001 \) for \( x= -\pi(\pi/36) \pi \) and for various positive values of \( z \) up to 36. The present section will present one such procedure, an application of a method for improving the convergence of certain Fourier series, given on pp. 84 to 88 of [10]. The procedure presented below is not an exact description of the methods actually used in making the table of section VIII. It is assumed in section V that computing machinery is available capable of dealing with numbers of 10 decimal digits, but no more than 10.

Of the tolerable error 0.001, the amount 0.0005 must be reserved for round-off in the final tabulation to three decimal places. Suppose that 0.0004 is allowed for truncation errors,\textsuperscript{12} and 0.0001 for computing errors resulting from round-offs during the calculation with 10-digit numbers. To have a truncation error as low as 0.0004 from use of a partial sum of eq 28 would require about 23,000

\textsuperscript{10} The limit \( \phi=32^0 19' \) arose unintentionally.

\textsuperscript{11} The notation \( x=a(b) \) means \( x=a, a+b, a+2b, a+3b, \ldots, b-a, b \).

\textsuperscript{12} Truncation errors are errors that result from use of approximate mathematical formulas, e. g., use of partial sums of infinite series.
terms for \( x=\pi/36 \) and \( x=35\pi/36 \). The convergence must obviously be improved.

1. Representation by Truncated Double Sum

We write

\[
G(x, z) = \sum_{n=1}^{\infty} \frac{1}{n} \sin nx \cos \frac{z}{n} + \sum_{n=1}^{\infty} \frac{1}{n} \cos nx \sin \frac{z}{n} = \Sigma_1 + \Sigma_2; \quad (29)
\]

for simplicity we consider only the first sum \( \Sigma_1 \) in eq 29. It can be shown that the term \( \Sigma_2 \) behaves similarly throughout the analysis. Expanding \( \cos (z/n) \) in its Maclaurin series, one has

\[
\Sigma_1 = \sum_{n=1}^{\infty} \frac{1}{n} \sin nx \sum_{r=0}^{\infty} \frac{(-1)^rz^{2r}}{(2r)!n^{2r+1}} = \sum_{n=1}^{\infty} \sum_{r=0}^{\infty} \frac{(-1)^rz^{2r} \sin nx}{(2r)!n^{2r+1}}. \quad (30)
\]

One type of truncation of eq 30 consists in omitting all terms for \( n \geq N+1 \), \( r \geq R \). Let the error caused by this truncation be called \( \epsilon_1 \). We shall estimate \( \epsilon_1 \) for \( 0 \leq z \leq Z \):

\[
|\epsilon_1| = \left| \sum_{n=N+1}^{\infty} \sum_{r=0}^{N} \frac{(-1)^rz^{2r} \sin nx}{(2r)!n^{2r+1}} \right|
\]

\[
\leq \sum_{r=0}^{N} \frac{Z^{2r}}{(2r)!} \sum_{n=N+1}^{\infty} \frac{1}{n^{2r+1}} < \sum_{r=0}^{N} \frac{Z^{2r}}{(2r)!} \int_{N}^{\infty} \frac{d\chi}{\chi^{2r+1}}
\]

\[
= \sum_{r=0}^{N} \frac{(Z/N)^{2r}}{(2r)!} \leq 4 \sqrt{\pi} \sum_{r=0}^{N} \frac{1}{r^{3/2}} \left( \frac{eZ}{2rN} \right)^{2r}.
\]

The last step above uses the Stirling expression for the factorial function, which, according to p. 74 of [8], is a one-sided estimate: \( \Gamma(s) > s^{s-e^2} \sqrt{2\pi}s \). Continuing, one finds for \( eZ/2RN < 1 \) that

\[
|\epsilon_1| < \frac{1}{4 \sqrt{\pi}} \left( \frac{eZ}{2RN} \right)^{2r} \left( \frac{1}{R^{3/2}} \sum_{r=0}^{\infty} \frac{1}{(2rN)^{2r}} \right)^{2r}
\]

\[
= \frac{1}{4 \sqrt{\pi}} \left( \frac{eZ}{2RN} \right)^{2r} \left[ 1 - \left( \frac{eZ}{2RN} \right)^{2r} \right]^{-1} = F(N, R). \quad (31)
\]

The first estimate in eq 31 seems crude, but it does not affect the values of \( N \) or \( R \) very much. Thus \( F(N, R) \) is an upper bound for the truncation error \( |\epsilon_1| \) introduced by omitting terms of type \( n \geq N+1 \), \( r \geq R \) in \( \Sigma_1 \).

One can reasonably tolerate a truncation error \( |\epsilon_1| \) of \( 5 \times 10^{-5} \). The corresponding admissible values of \( N \) and \( R \) come from setting \( F(N, R) = 5 \times 10^{-5} \) for \( Z = 36 \). The following pairs of values of \( N, R \) were obtained from eq 31 by a numerical calculation followed by a round-off of \( N \) to an integer value:

<table>
<thead>
<tr>
<th>( N )</th>
<th>( R )</th>
</tr>
</thead>
<tbody>
<tr>
<td>18</td>
<td>5</td>
</tr>
<tr>
<td>13</td>
<td>6</td>
</tr>
<tr>
<td>11</td>
<td>7</td>
</tr>
<tr>
<td>8</td>
<td>10</td>
</tr>
<tr>
<td>4</td>
<td>15</td>
</tr>
<tr>
<td>3</td>
<td>20</td>
</tr>
</tbody>
</table>

The selection of the most suitable pair of values \( N, R \) from eq 32 will be postponed until we have discussed the summation of the remaining terms of eq 30.

2. Computation of the Double Sum

The terms of \( \Sigma_1 \) for \( n=1, 2, \ldots, N \) and all \( r \) may be left in the form

\[
\Sigma_1 = \sum_{n=1}^{N} \frac{1}{n} \cos \frac{z}{n} \sin nx, \quad (33)
\]

and may be computed from this formula. The terms for \( n \geq N+1 \) and \( r = 0, 1, \ldots, R-1 \) may be written in the form

\[
\Sigma_2 = \frac{eZ}{2rN} \sum_{r=0}^{R-1} \frac{z^{2r}}{(2r)!} \sigma_{2r}^{(N)}(x), \quad (34)
\]

where

\[
\sigma_{2r}^{(N)}(x) = (-1)^r \sum_{n=N+1}^{\infty} \frac{\sin nx}{n^{2r+1}}. \quad (35)
\]

Once \( \sigma_{2r}^{(N)}(x) \) has been tabulated for the one value of \( N \) to be selected below, \( \Sigma_2 \) may be computed directly from eq 34. Two methods are needed to get \( \sigma_{2r}^{(N)}(x) \), as the calculating machinery is assumed to be limited to ten decimal digits.

The first method is to use the identity

\[
\sigma_{2r}^{(N)}(x) = \sigma_{2r}(x) - (-1)^r \sum_{n=N+1}^{\infty} \frac{\sin nx}{n^{2r+1}}, \quad (36)
\]

where

\[
\sigma_{2r}(x) = \sigma_{2r}^{(0)}(x) = (-1)^r \sum_{n=1}^{\infty} \frac{\sin nx}{n^{2r+1}}. \quad (37)
\]

In section VI it will be shown that for \( 0 \leq x \leq \pi \), \( \sigma_{2r}(x) \) is essentially a Bernoulli polynomial in the variable \( x/2\pi \). Hence \( \sigma_{2r}(x) \) can readily be com-
null
eq 29 involves the same considerations, we may therefore propose $N=18$, $R=5$ as being the optimal values to use in getting $G(x, z)$ by eq 33 and 34 for one value of $x$ and one value of $z$. The number of multiplications will be $2W_1(N, R)$.

Getting $G(x, z)$ for 72 values of $x$ and one value of $z$ involves no change of $N, R$. Since $\sin n x$ [cos $n x$] is an odd [even] function of $x$, the number of multiplications in getting $G(x, z)$ for all $x$ will be $72W_1(N, R)$. However, getting $G(x, z)$ for several values of $z$ changes the analysis, because the functions $\sigma_6^{(n)}(x)$, once computed, serve for each new $z$ without change. To get $\Sigma_1$ for 13 values of $z$ and one value of $x$, for example, will require, in addition to the multiplications in eq 41, only 12$N$ multiplications from eq 33 and $12R$ from eq 34. The total number of multiplications will then be


The minimum of $W_{13}(N, R)$ is 361, and occurs for $N=13, R=6$. The optimal choice of $N, R$ has changed, though not greatly. Since we expect to use 13 values of $z$, we adopt the values $N=13, R=6$.

4. Summary of the Computation Method

With the above choice of $N$ and $R$, the computation of $\Sigma_1$ may proceed as follows:

(a) Compute

$$\Sigma'_1 = \sum_{n=1}^{13} \left( \frac{1}{n} \cos \frac{z}{n} \right) \sin n x.$$  

(b) For $r=0, 1, 2$, compute

$$\sigma_r^{(13)}(x) = \sigma_r(x) - (-1)^r \sum_{n=1}^{13} \frac{\sin n x}{n^{2r+1}},$$  

where $\sigma_r(x)$ is computed from section VI.

(c) Compute

$$\sigma_0^{(13)}(x) = -\sum_{n=14}^{54} \frac{\sin n x}{n^2},$$

$$\sigma_2^{(13)}(x) = \sum_{n=14}^{92} \frac{\sin n x}{n^3},$$

$$\sigma_10^{(13)}(x) = -\sum_{n=14}^{52} \frac{\sin n x}{n^{11}}.$$  

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(d) Compute

$$\Sigma'_2 = \sum_{r=0}^{\infty} \frac{z^{2r}}{(2r)!} \sigma_r^{(13)}(x).$$

(e) Compute $\Sigma_1 = \Sigma'_1 + \Sigma'_2$.

The number of multiplications involved in getting $\Sigma_1$ for one $x$-value and for one $z$-value is: (a) 13; (b) 39; (c) 69; (d) 6; (e) 0. When getting $\Sigma_1$ for one $x$-value and 13 $z$-values, one adds 156 multiplications to (a) and 72 to (d). The total for 13 $z$-values is 355 multiplications per $x$-value. (The slight discrepancy with the number 361 in subsection 3 is due to the rough estimate previously made for step (e).) For all $x$-values (essentially 36), one gets a total of 12,780 multiplications to get $\Sigma_1$.

To get $\Sigma_2$ in eq 29, one follows analogous steps involving $\sigma_{2r+1}(x)$, $\sigma_{2r+1}^{(13)}(x)$, etc. There will be approximately 12,750 more multiplications, making a total of about 25,500 multiplications to get $G(x, z)$ for the 72 $x$-values and 11 $z$-values.

The total truncation error in getting $\Sigma_1$ is bounded by $2 \times 10^{-4}$. This is divided into four truncation errors of $5 \times 10^{-5}$, one for each of the three steps in (c), and one for the terms left out of (e). The truncation error for $\Sigma_2$ is also bounded by $2 \times 10^{-4}$, making a total truncation error of $4 \times 10^{-4}$. The final round-off of the final answer to three decimal places may introduce an error of $5 \times 10^{-4}$. The third source of error is the accumulation of round-offs from adding five-decimal-place terms. Each term is accurate to $5 \times 10^{-4}$; with an assumed rectangular distribution these terms have a dispersion near $3 \times 10^{-6}$. Each value of $G(x, z)$ is obtained from the addition of about 270 such terms. The dispersion $\sigma$ of the sum is therefore about $\sqrt{270 \times 5 \times 10^{-6}}$, or about $8 \times 10^{-5}$. One may expect the accumulated error to exceed $2.5 \sigma = 2 \times 10^{-4}$ in only 1.3 percent of the cases. The sum of the three errors is effectively bounded by $11 \times 10^{-4}$, or slightly more than 0.001.

VI. The Polynomials \{ $\sigma_k(x)$ \}

In section V we made use of certain functions $\sigma_k(x)$ defined as follows:

$$\sigma_{2r}(x) = (-1)^r \sum_{n=1}^{\infty} \frac{\sin n x}{n^{2r+1}}, \quad (r=0, 1, 2, \ldots);$$

$$\sigma_{2r+1}(x) = (-1)^{r+1} \sum_{n=1}^{\infty} \frac{\cos n x}{n^{2r+2}}, \quad (r=0, 1, 2, \ldots).$$  

\( \text{\quad (44)} \)
The function \( \sigma_0(x) \) was used in section IV; see eq. 18. For \( k > 0 \) the series \( \sigma_k(x) \) in eq. 44 are absolutely convergent; hence they represent continuous functions. Since \( \sigma_{2r}(x) \) is odd and \( \sigma_{2r+1}(x) \) is even, it is necessary to sum the series (eq. 44) only for \( 0 \leq x \leq \pi \).

As stated in eq. 18,

\[
\sigma_0(x) = \frac{\pi}{2} - \frac{1}{2} x, \quad (0 < x \leq \pi).
\]

(45)

Now

\[
\sigma_1(x) = -\sum_{n=1}^{\infty} \frac{\cos nx}{n^2} = \int_0^\pi \sigma_0(\xi) d\xi - \sum_{n=1}^{\infty} \frac{1}{n^2}.
\]

Hence

\[
\sigma_1(x) = -\frac{\pi^2}{6} + \frac{\pi}{2} x - \frac{1}{4} x^2, \quad (0 \leq x \leq \pi).
\]

Similarly, for \( 0 \leq x \leq \pi \), one finds

\[
\begin{align*}
\sigma_2(x) &= -\frac{\pi^2}{6} x^2 + \frac{\pi}{4} x^2 - \frac{1}{12} x^3; \\
\sigma_3(x) &= \frac{\pi^4}{90} - \frac{\pi^2}{24} x^2 + \frac{\pi}{12} x^3 - \frac{1}{48} x^4; \\
\sigma_4(x) &= \frac{\pi^4}{90} x^2 - \frac{\pi^2}{36} x^2 + \frac{\pi}{48} x^4 - \frac{1}{240} x^6.
\end{align*}
\]

(46)

Use has been made of the formulas \( \sum_{n=1}^{\infty} n^{-2} = \pi^2/6; \sum_{n=1}^{\infty} n^{-4} = \pi^4/90 \).

The functions \( \sigma_k(x) \) are therefore all polynomials. Their use in improving the convergence of Fourier series is pointed out on pp. 84 to 88 of [10]. Although they may be easily tabulated from eq 46, they may also be adapted from existing tables because they are essentially Bernoulli polynomials. Let \( \{ B_k(x) \} \) be the Bernoulli polynomials given on p. 181 of [4].

**Lemma.** For \( 0 \leq x \leq \pi \), and \( k = 0, 1, 2, \ldots \),

\[
\sigma_k(x) = -\frac{(2\pi)^{k+1}}{2(k+1)!} B_{k+1}\left(\frac{x}{2\pi}\right).
\]

(47)

Proof: Define the Bernoulli number \( B_n \) by the relation \( B_n = B_n(0) \). These are the Bernoulli numbers used on p. 21 of [8]; they are \( B_0 = 1, B_1 = -1/2, B_2 = 1/6, B_3 = 0, B_4 = -1/30, B_5 = 0, B_6 = 1/42, \ldots \), Davis uses other notations in [4]. Now fix \( x \) in the interval \( 0 \leq x \leq \pi \). For each \( k = 1, 2, 3, \ldots \),

\[
\sigma_k(x) = \int_0^x \sigma_{k-1}(\xi) d\xi - \frac{k\pi}{2} \sum_{n=1}^{\infty} \frac{1}{n^{k+1}} = \int_0^x \sigma_{k-1}(\xi) d\xi - \frac{(2\pi)^{k+1} B_{k+1}}{2(k+1)!}.
\]

(48)

The last step is by eq 9 on p. 21 of [8], which is correct except for sign. Hence, letting \( x = 2\pi t \),

\[
2(2\pi)^{-k-1}\sigma_k(2\pi t) = \int_0^t 2(2\pi)^{-k} \sigma_{k-1}(2\pi n) d\eta - \frac{B_{k+1}}{(k+1)!} \eta^{k+1} \quad (k = 1, 2, 3, \ldots).
\]

(49)

Now define \( 2(2\pi)^{-k} \sigma_{k-1}(2\pi t) = -\frac{B_0}{0!} t^{k-0} \). Use eq 47 formally to get \( 2(2\pi)^{-k} \sigma_0(2\pi t) = -\frac{B_0}{0!} t^{0} \).

(47)

(48)

Note that eq 48 agrees with eq 45 for \( \sigma_0(x) \). Hence eq 48 is a correct formula, although it was only derived formally.

We now apply formula (eq 47) repeatedly, getting always correct expressions:

\[
2(2\pi)^{-k} \sigma_1(2\pi t) = -\frac{B_0}{0!} t^1 - \frac{B_1}{1!} t - \frac{B_2}{2!} t^2,
\]

\[
2(2\pi)^{-k} \sigma_2(2\pi t) = -\frac{B_0}{0!} t^2 - \frac{B_1}{1!} t^2 - \frac{B_2}{2!} t^2 - \frac{B_3}{3!} t^2
\]

\[
2(2\pi)^{-k} \sigma_3(2\pi t) = -\sum_{j=0}^{k} \frac{B_j}{j!} \eta^{j+1} - \frac{B_{k+1}}{(k+1)!} \eta^{k+1}.
\]

(50)

Hence

\[
2(k+1)! (2\pi)^{-k} \sigma_k(2\pi t) = -\sum_{j=0}^{k} \frac{B_j}{j!} \eta^{j+1} - \frac{B_{k+1}}{(k+1)!} \eta^{k+1}.
\]

(51)

But it follows from the top of p. 188 of [4] that

\[
B_{k+1}(t) = \sum_{j=0}^{k} \left(\frac{B_{k+1-j}}{j!}\right) B_{j} t^j.
\]

(52)

Comparing eq 49 and 50, we see that

\[
B_{k+1}(t) = -2(k+1)! (2\pi)^{-k} \sigma_k(2\pi t).
\]

Let \( 2\pi t = x \), and the lemma is proved.

**VII. Solution by a Difference Equation**

Our first approximate solution of the problem stated in section II consisted of the approximate evaluation of the integral (eq 21) by means of numerical integration formulas, using the approxi-
imate values of $G(x, z)$ tabulated in section VIII below. A second approximate solution of the problem consists in solving with appropriate boundary conditions a difference equation that is closely related to the differential equation (eq 3). The latter method is considered in detail in [6], where proofs may be found; only a summary is given in the present section.

For any positive integer $2N$, let $h = \pi / 2N$; let $k > 0$ be arbitrary. A net is formed of all points $(x, t)$ of form $(\mu h, \nu k)$, where $\mu$ and $\nu$ are integers satisfying the conditions

$$\mu + \nu \equiv 0 \pmod{2}, |\mu| \leq 2N, \nu \geq 0. \quad (51)$$

Where necessary we extend the net and the functional values periodically in $x$ with period $2\pi$. The differential equation (eq 3) is approximated by the difference equation,

$$v(x+h, t+k) - v(x-h, t+k) = v(x+h, t-k) - v(x-h, t-k) - hkv(x, t). \quad (52)$$

The boundary conditions of the difference-equation problem are prescribed values of $v(x, t)$ on the two rows $t=0, t=k$. Assume that for $t=k$,

$$\sum_{x} v(x, t) = 0, \quad (53)$$

where the sum is extended over all points of the second row of the net. The boundary conditions and eq 52 then determine the value of $v(x, t)$ on the row $t=2k$ up to an additive constant. The additive constant and hence $v(x, 2k)$ are determined uniquely by requiring that eq 53 hold also for $t=2k$. Continuing row after row, one thus determines $v(x, t)$ over the whole net. Let the function so determined be denoted by $v^{(N)}(x, t)$; it depends on $N$, on $k$, and on the initial values prescribed for the first two rows. The problem of [6] is to see whether $v^{(N)}(x, t) \to v(x, t)$ as $N \to \infty$.

Let the initial values $v(x, 0)$ on the first row be defined by the relation $v(x, 0) = f(x)$, where $f(x)$ is the function of eq 1. Let $k$ be fixed. Then it is possible to choose the initial values $v(x, k)$ on the second row of the net in such a manner that, as $N \to \infty$, $v^{(N)}(x, t) \to v(x, t)$ for each $t$ of the net and for each $x$ that is an abscissa of continuity of $f(x)$. If $k$ is allowed to vary with $N$ in such a manner that $k \to 0$ as $N \to \infty$, then $v^{(N)}(x, t) \to v(x, t)$ for each $t \geq 0$ and for each $x$ that is an abscissa of continuity of $f(x)$. In neither case may one, in general, expect the convergence to be uniform in $x$ or $t$.

The method referred to for choosing the values $v(x, k)$ on the second row is not an economical one, and in a practical computation one would prefer a cheaper though approximate method. Two things are shown in [6] about the effects of an approximation of the values of $v(x, k)$: First, they may introduce ultimate instability into the solution. Even though the solution $v(x, t)$ of eq 3 be identically zero, it is possible that for fixed $N$ and $x$,

$$\lim_{t \to \infty} v^{(N)}(x, t) = \pm \infty.$$

Second, the approximation does not prevent convergence of $v^{(N)}(x, t)$ to $v(x, t)$, provided that the error of the approximation of $v(x, k)$ vanishes as $N \to \infty$. One reasonable way of causing the error to vanish is to let $k \to 0$.

These results show that the difference-equation method is a feasible method of solving the problem of this paper.

**VIII. Table of the Green’s Function**

In this section is tabulated the Green’s function $G(x, z)$, as computed in the Computation Unit of the Institute for Numerical Analysis. The value of the time parameter $z$ corresponding to $k$ hours at latitude $\phi$ is

$$z = 0.52502 h \cos^{2} \phi. \quad (51)$$

(Except for the last digit of the constant, formula (eq 51) can be verified from the introduction to section V.) Meteorological considerations suggested that $h$ should be chosen in convenient multiples of 12 hours, and that $\phi$ should be $35^\circ$, $45^\circ$, or $55^\circ$. The latitude $32^\circ 19'$ resulted from a numerical error by the author. A limited number of pairs of values of $h$ and $\phi$ were selected for the computation; these pairs are shown in table 1, together with the corresponding values of $z$ determined from eq 51.

For each of the 13 values of $z$ given in table 1 (and for $z=0$) and for $x = \pi (\pi / 36) \pi$, the Green’s function $G(x, z)$ is presented in table 2 to 5 decimal places. Since $G(x, z)$ has a discontinuity of the first kind at $x = 0$, the values $G(-0, z), G(0, z)$, and $G(+0, z)$ are all given. In every case, $G(0, z) = \frac{1}{2} [G(-0, z) + G(0, z)]$ and $G(+0, z) = G(-0, z) = \pi$. The computational procedure followed that of section V in general outline, with certain deviations. It was decided to use $N=18, R=6$. The auxiliary functions $\sigma_{a}^{(N)}(x)$ and

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The values of $G^{(3)}(x,0)$ were obtained it was possible to generate, very easily, the function $G(x,z)$ for any values of $z$ in the range $0 < z \leq 36$. To summarize,

(a) $G_1(x,z)$ was computed from eq (52).

(b) The functions $G^{(k)}(x,0)$ were computed for $k = 0, 1, 2, \ldots, 10$.

(c) $G_2(x,z)$ was obtained from eq (53).

(d) $G_1(x,z)$ and $G_2(x,z)$ were added, to yield $G(x,z)$.

The subsidiary computations in (a) and (b) were carried to nine decimal places, those in (c) to at least seven decimal places in the partial products. Table 2 is believed to be accurate to $\pm 0.00002$ for all $x$ and all $z$. The cosine component of $G(x,z)$ was given a final check by use of the following formula:

$$\sum_{k=-36}^{35} \frac{G(k\pi/36,z)}{2} = \sum_{\nu=1}^{\infty} \frac{1}{\nu^2} \sin \left( \frac{z}{2\nu} \right) = p(z).$$

The check showed a deviation between the sum and $p(z)$, which was never greater than 0.00005. The sine component of $G(x,z)$ was given a partial check by the formula

$$2 \sum_{\nu=1}^{\infty} \left[ \frac{1}{4\nu+1} \cos \left( \frac{z}{4\nu+1} \right) - \frac{1}{4\nu-1} \cos \left( \frac{z}{4\nu-1} \right) \right].$$

Comparable agreement was found. Finally, the table differences with respect to $x$ are very satisfactory.

### Table 1. Values of the time parameter, $z$, corresponding to hours, $h$, at latitude, $\phi$

<table>
<thead>
<tr>
<th>$\phi$</th>
<th>32°19'</th>
<th>35°</th>
<th>45°</th>
<th>55°</th>
</tr>
</thead>
<tbody>
<tr>
<td>$h$</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>12</td>
<td>3.150</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>24</td>
<td>8.455</td>
<td>6.300</td>
<td>4.145</td>
<td></td>
</tr>
<tr>
<td>48</td>
<td>16.900</td>
<td>12.600</td>
<td>8.291</td>
<td></td>
</tr>
<tr>
<td>72</td>
<td>27.060</td>
<td>18.900</td>
<td></td>
<td></td>
</tr>
<tr>
<td>96</td>
<td>36.600</td>
<td>25.200</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Table 2. Table of Green's Function G(z, x).

<table>
<thead>
<tr>
<th>x</th>
<th>0.0000</th>
<th>3.1502</th>
<th>14.1543</th>
<th>6.30024</th>
<th>8.20898</th>
<th>8.47505</th>
<th>9.00000</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.0000</td>
<td>0.0000</td>
<td>0.1000</td>
<td>0.4000</td>
<td>0.9000</td>
<td>1.0000</td>
<td>4.5000</td>
<td>1.5000</td>
</tr>
<tr>
<td>0.0100</td>
<td>0.0100</td>
<td>0.1000</td>
<td>0.4000</td>
<td>0.9000</td>
<td>1.0000</td>
<td>4.5000</td>
<td>1.5000</td>
</tr>
<tr>
<td>0.0200</td>
<td>0.0200</td>
<td>0.1000</td>
<td>0.4000</td>
<td>0.9000</td>
<td>1.0000</td>
<td>4.5000</td>
<td>1.5000</td>
</tr>
<tr>
<td>0.0300</td>
<td>0.0300</td>
<td>0.1000</td>
<td>0.4000</td>
<td>0.9000</td>
<td>1.0000</td>
<td>4.5000</td>
<td>1.5000</td>
</tr>
</tbody>
</table>

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<td>0.4000</td>
<td>0.9000</td>
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</tr>
<tr>
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<td>0.4000</td>
<td>0.9000</td>
<td>1.0000</td>
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<tr>
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<td>0.1000</td>
<td>0.4000</td>
<td>0.9000</td>
<td>1.0000</td>
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<td>0.4000</td>
<td>0.9000</td>
<td>1.0000</td>
<td>4.5000</td>
<td>1.5000</td>
</tr>
</tbody>
</table>

Solution of the Telegrapher's Equation
### Table 2. Table of Green's Function $G(x, z)$

<table>
<thead>
<tr>
<th>$x$</th>
<th>$G(x, z)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>12.6048</td>
<td>16.01015</td>
</tr>
<tr>
<td>18.0000</td>
<td>25.00006</td>
</tr>
<tr>
<td>27.0000</td>
<td>36.00000</td>
</tr>
</tbody>
</table>

### Degrees

| -30 | -71.22 | -27.239 | -63.977 | -94.777 | -76.527 | 1.68936 | -1.49433 |
| -25 | -61.086 | -33.909 | -64.843 | -181.155 | -60.566 | 1.29972 | -1.56697 |
| -20 | -48.373 | -41.091 | -66.252 | -161.133 | -42.926 | 1.30167 | -1.60479 |
| -10 | -20.989 | -55.874 | -68.530 | -112.866 | -8.0102 | 1.42266 | -1.58980 |
| -5 | -11.239 | -62.925 | -68.889 | -88.436 | 0.98924 | 1.44862 | -1.52570 |
| 0 | 0.15858 | -70.554 | -68.784 | -62.091 | 2.52034 | 1.45984 | -1.44108 |
| 5 | 1.58665 | -80.926 | 0.2806 | 1.51879 | 1.82377 | 3.02161 | 0.10929 |
| 10 | 3.15744 | 2.40065 | 2.4573 | 3.08808 | 3.39452 | 4.59243 | 1.70092 |
| 15 | 0.67142 | -0.81525 | -0.84342 | -0.37249 | -0.37249 | 0.51092 | -2.58009 |
| 20 | -5.67290 | -2.04179 | -1.8761 | -1.21962 | -0.23460 | -0.24746 | -1.70699 |
| 25 | -8.30767 | -2.07925 | -1.75697 | -0.97055 | -0.36803 | -0.46072 | -0.49519 |
| 30 | -6.88260 | -1.63134 | -1.60767 | -0.26009 | 1.28508 | 1.28508 | -0.00356 |
| 35 | -24.0151 | -1.06998 | -0.48829 | -0.14113 | 1.76763 | 2.06837 | -2.68223 |
| 40 | -3.09379 | -0.00220 | 0.05023 | 0.03794 | 1.80306 | 1.81177 | -0.65249 |
| 45 | -0.02109 | -1.7771 | -0.28871 | -1.22401 | -1.69102 | 1.32871 | -1.08153 |
| 50 | 1.30192 | -0.39489 | -0.52728 | 1.28534 | 1.32178 | 0.71992 | -1.17421 |
| 55 | 1.67948 | -0.16770 | -0.48756 | 1.17420 | 0.92058 | 0.15777 | -0.96052 |
| 60 | 1.91096 | -0.32146 | -0.23364 | 0.05067 | -0.60655 | -0.25779 | -0.56209 |
| 65 | 2.03312 | -0.55228 | -0.11399 | -0.07164 | -0.41141 | -0.49264 | -1.18201 |
| 70 | 2.04891 | -0.81290 | -0.11033 | -0.44820 | -0.35772 | -0.56289 | -0.31487 |
| 75 | 1.97550 | -0.60250 | -0.35755 | -0.20988 | -0.44447 | -0.18226 | -0.42631 |
| 80 | 1.83275 | -0.17946 | -0.42520 | -0.12229 | -0.61561 | -0.37836 | 0.76541 |
| 85 | 1.64002 | -0.43832 | -0.48723 | -0.07903 | -0.83961 | -0.22221 | -0.74192 |
| 90 | 1.41914 | -0.80708 | -0.46049 | -0.11749 | -1.06304 | -0.08486 | -0.58453 |

### IX. References

2. J. Charney, manuscript in preparation for Journal of Meteorology on the work reported in [1].
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6. G. E. Forsythe, Manuscript in preparation on the approximate solution by difference equations of the telegrapher's equation with boundary values on only one characteristic.
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Los Angeles, March 8, 1949.