U. S. Department of Commerce
National Bureau of Standards

Research Paper RP1950
Volume 42, January 1949

Part of the Journal of Research of the National Bureau of Standards

On the Precision of a Certain Procedure of Numerical Integration

By Harry D. Huskey
With an appendix by Douglas R. Hartree

An example of numerical integration is given that shows very systematic effects in the less significant digits. This lack of randomness gives rounding-off errors that exceed the predicted standard deviation by a factor of three.

The example considered in this paper shows that systematic rounding-off errors can occur in numerical integration, irrespective of the number of digits kept in the contributions to the integral. In the appendix this phenomenon is examined, and criteria are set up to detect the cases in which it may arise to a serious extent.

I. Introduction

The use of numerical methods has led to the study of the accumulation of errors in computations by various people.[1] In this paper we apply formulas developed by Rademacher to the errors involved in the integration of simultaneous linear differential equations. The system chosen for this application is

$$x'(t)=y(t),$$

$$y'(t)=-x(t).$$

The results of integrating these equations were easily checked by comparison with the sine and cosine tables published by the National Bureau of Standards.[2]

The errors involved in the numerical integration of these equations arise from two sources. One, called the truncation error, arises from replacing the differential equations by difference equations; the other, a round-off error, comes from the rounding-off procedure used in the computation. Formulas developed by Rademacher account for the truncation error. The rounding-off error can be estimated in a statistical manner, provided the dropped digits are randomly distributed. Rademacher suggests that this random property is satisfied provided the increments involved in the integration are not too small. We shall exhibit an integration where this assumption is satisfied, but the dropped digits vary from zero to four and back to zero over a range involving nearly three hundred steps in the integration. This causes the error to increase by a factor of twenty and to become almost three times the standard deviation as given by Rademacher's formulas.

In certain other runs the error exceeds the predicted standard deviation by a small factor. In two of these cases results were tabulated every five or ten steps in the integration and a frequency count of the digits taken. Standard statistical tests indicate that these numbers did not consist of randomly distributed digits.

These results show that one must be very careful in applying error estimates based on an assumption of randomness. To be safe it is best to use the estimates for the maximum rounding-off error.

1. Rademacher Theory

(a) Heun Method

We now indicate the method of solution studied by Rademacher and give the formulas developed by him. He starts with the system.

[1] F. Schlesinger, Astron. J. **30**, 183 (1917); D. Brouwer, Astron. J. **46**, 149 (1937); H. Rademacher, On the accumulation of errors in processes of integration on high-speed calculating machines, Proceedings of a Symposium on Large-Scale Digital Calculating Machinery (Harvard University Press, Cambridge, Mass., 1948).

[2] Tables of sines and cosines for radian arguments (National Bureau of Standards, 1940) MT4; Tables of Circular and Hyperbolic sines and cosines for radian arguments (National Bureau of Standards, 1939) MT3.

$$\left.\begin{aligned} x'(t)&=f(x,y)\\ y'(t)&=g(x,y)\end{aligned}\right\} \quad (1)$$

and the solution is to be found for an interval $t_0 \leq t \leq T$ by application of the Heun method. That is, having found x_{j-1} and y_{j-1} as approximations to the solutions at $t_{j-1}=t_0+(j-1)(\Delta t)$, the following formulas give x_j and y_j.

$$\left.\begin{aligned} x_j^*&=x_{j-1}+\Delta t\cdot f(x_{j-1},y_{j-1})\\ y_j^*&=y_{j-1}+\Delta t\cdot g(x_{j-1},y_{j-1})\end{aligned}\right\} \quad (2)$$

$$\left.\begin{aligned} x_j&=x_{j-1}+\frac{\Delta t}{2}[f(x_{j-1},y_{j-1}+f(x_j^*,y_j^*)]\\ y_j&=y_{j-1}+\frac{\Delta t}{2}[g(x_{j-1},y_{j-1})+g(x_j^*,y_j^*)]\end{aligned}\right\} \quad (3)$$

(b) Definitions

Let us make the following definitions:

(a) Let $x(t)$, $y(t)$ be solutions of eq 1 satisfying the condition that $x(t_o)=x_o$ and $y(t_o)=y_o$.

(b) Let x_j, y_j, $j=1,2,\ldots,n$, be the numbers obtained by successive application of eq 2 and 3.

(c) Let $\lambda(t)$, $\mu(t)$ be generic notation for solutions of the system

$$\left.\begin{aligned} d\lambda/dt&=-(\partial f/\partial x)\lambda-(\partial g/\partial x)\mu,\\ d\mu/dt&=-(\partial f/\partial y)\lambda-(\partial g/\partial y)\mu.\end{aligned}\right\} \quad (4)$$

(d) Let $u(t_j)=x(t_j)-x_j$ and $v(t_j)=y(t_j)-y_j$, $j=1,\ldots,n$. The numbers $u(t_j)$ and $v(t_j)$ are a measure of the truncation error in each step of the integration.

(c) Truncation Error

Rademacher derived the following formulas for the truncation error:

$$\left.\begin{aligned} &\lambda(T)u(T)+\mu(T)v(T)\sim\\ &-\frac{1}{12}(\Delta t)^2[\lambda(t)x''(t)+\mu(t)y''(t)]\Big|_{t_0}^{T}\\ &-\frac{1}{6}(\Delta t)^2\int_{t_0}^{T}[\lambda'(t)x''(t)+\mu'(t)y''(t)]dt\end{aligned}\right\} \quad (5)$$

The truncation errors $u(T)$ and $v(T)$ can be separately obtained from eq 5 by applying the proper terminal conditions to the solutions $\lambda(t)$ and $\mu(t)$ of eq 4. For example, $u(T)$ can be found by letting $\lambda(T)=1$ and $\mu(T)=0$.

(d) Rounding-Off Error

Thus far, it has been assumed that all computations are done exactly. In actual computing, this is not the case. The accumulators or registers of the computing machine accommodate only a limited number of digits. Thus eq 2 and 3 should be written as

$$\left.\begin{aligned} x_j^*&=x_{j-1}+\Delta t\cdot f(x_{j-1},y_{j-1})+\sum_m \epsilon_{jm}r_{jm}^{(1)}\\ y_j^*&=y_{j-1}+\Delta t\cdot g(x_{j-1},y_{j-1})+\sum_m \epsilon_{jm}r_{jm}^{(2)}\end{aligned}\right\} \quad (6)$$

$$\left.\begin{aligned} x_j&=x_{j-1}+\frac{\Delta t}{2}[f(x_j^*,y_j^*)+f(x_{j-1},y_{j-1})]+\sum_m \epsilon_{jm}r_{jm}^{(3)}\\ y_j&=y_{j-1}+\frac{\Delta t}{2}[g(x_j^*,y_j^*)+g(x_{j-1},y_{j-1})]+\sum_m \epsilon_{jm}r_{jm}^{(4)}\end{aligned}\right\} \quad (7)$$

The ϵ_{jm} satisfy $|\epsilon_{jm}|\leq 0.5$. The coefficients $r_{jm}^{(i)}$ depend not only upon the equations to be solved but upon the explicit procedure or order of operations in the process of solution. In the following discussion quantities with bars above them represent the actual numbers stored in the registers or accumulators of the computing machine. Although the analysis can be carried through using eq 6 and 7, Rademacher makes the simplifying assumption that

$$\overline{\Delta t\, f(\overline{x}_j,\overline{y}_j)}=\overline{\Delta t\,\overline{f(\overline{x}_j,\overline{y}_j)}}.$$

This means that he assumes that $f(x_j,y_j)$ can be computed sufficiently accurately so that when multiplied by Δt any inaccuracies it may have are lost in the digits that are dropped. Thus, eq 7 can be replaced by

$$\left.\begin{aligned} \overline{x}_j&=\overline{x}_{j-1}+\frac{\Delta t}{2}[f(\overline{x}_j^*,\overline{y}_j^*)+f(\overline{x}_{j-1},\overline{y}_{j-1})]+\epsilon_{j1}10^{-k}\\ \overline{y}_j&=\overline{y}_{j-1}+\frac{\Delta t}{2}[g(\overline{x}_j^*,\overline{y}_j^*)+g(\overline{x}_{j-1},\overline{y}_{j-1})]+\epsilon_{j2}10^{-k}.\end{aligned}\right\} \quad (8)$$

Note that if the parentheses are removed in eq 8 so that there are four multiplications, then there are four rounding-off terms, say ϵ_{jm}^{s} ($s=1,2$; $m=1,2$).

As in the case of the truncation error let us make the following definition: $\overline{u}_j=x_j-\overline{x}_j$ and $\overline{v}_j=y_j-\overline{y}_j$.

TABLE 1. *Sine-cosine rounding errors*

Δt		Angle, in radians							
		0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
	S. D.	4.082	5.773	7.071	8.164	9.129	10.00	10.80	11.55
	Sin (A)	1	1	1	3	5	8	13	19
2×10^{-3}	Cos (A)	2	3	4	9	9	15	3	9
	Sin (B)	0	−1	5	7	5	7	10	11
	Cos (B)	1	2	1	0	1	−3	−3	−5
	S. D.	5.774	8.166	10.00	11.55	12.91	14.14	15.28	16.33
	Sin (A)	1	−1	−5	2	−1	0	−5	−4
1×10^{-3}	Cos (A)	−2	1	9	6	4	2	9	9
	Sin (B)	4	0	−6	−5	−7	−10	−7	−8
	Cos (B)	−1	−2	0	−2	−2	−4	−3	−2
	S. D.	8.161	11.54	14.14	16.32	18.25	19.99	21.60	23.08
	Sin (A)	1	10	11	12	14	25	20	14
5×10^{-4}	Cos (A)	2	5	3	3	6	5	−2	−2
	Sin (B)	1	3	0	−3	−12	−15	−12	−17
	Cos (B)	2	1	3	−2	−8	−7	−3	−5
	S. D.	12.91	18.25	22.36	25.82	28.86	31.62	34.15	36.51
	Sin (A)	5	−17	−11	24	50	77	104	134
2×10^{-4}	Cos (A)	26	55	85	84	90	99	92	105
	Sin (B)	−2	6	3	−13	−19	−14	−11	−6
	Cos (B)	−3	−2	−8	−4	−9	4	1	−2
	S. D.	18.26	25.82	31.62	36.51	40.83	44.72	48.31	51.64
	Sin (A)	1	−2	−48	−53	−95	−97	−116	−148
1×10^{-4}	Cos (A)	−17	−36	−21	−41	−32	−32	−45	−39
	Sin (B)	−7	−10	9	8	−5	−15	−20	−4
	Cos (B)	−7	−15	−9	−13	−7	−5	−7	−1
	S. D.	25.81	36.51	44.71	51.63	57.73	63.24	68.30	73.02
	Sin (A)	32	21	60	34	45	4	9	10
5×10^{-5}	Cos (A)	8	60	42	21	25	−1	1	−7
	Sin (B)	18	−2	−16	−4	−5	10	16	32
	Cos (B)	−1	−21	−10	−14	−24	−26	−20	−20
	S. D.	40.82	57.73	70.71	81.64	91.29	100.0	108.0	115.5
	Sin (A)	−17	23	−3	8	−190	−222	−254	−317
2×10^{-5}	Cos (A)	3	5	16	−18	21	57	49	86
	Sin (B)	14	34	32	30	50	61	68	7
	Cos (B)	2	6	−9	−16	−44	−40	−79	−94
	S. D.	57.74	81.66	100.0	115.5	129.1	141.4	152.8	163.3
	Sin (A)	−5	−26	−35	−62	−110	−91	−87	−71
1×10^{-5}	Cos (A)	4	12	40	63	84	65	84	122
	Sin (B)	−2	−17	−4	−4	14	16	6	17
	Cos (B)	−10	0	−11	−16	−28	−24	−49	−53
	S. D.	81.61	115.4	141.4	163.2	182.5	199.9	216.0	230.8
	Sin (A)	19	11	11	−27	−52	−68	−46	−39
5×10^{-6}	Cos (A)	4	10	−6	−8	−10	17	4	−4
	Sin (B)	−5	−4	20	28	39	51	37	50
	Cos (B)	−2	1	−7	4	32	53	56	45
	S. D.	129.1	182.5	223.6	258.2	288.6	316.2	341.5	365.1
2×10^{-6}	Sin (A)	37	30	52	34	27	32	40	24
	Cos (A)	20	5	13	16	0	8	8	−7

Letting $\bar{u}_n=\bar{u}(T)$ and $\bar{v}_n=\bar{v}(T)$, the expression for the rounding-off error is

$$\lambda(T)u(T)+\mu(T)v(T)=-10^{-k}\sum_{j=1}^{n}(\epsilon_{j1}\lambda_j+\epsilon_{j2}\mu_j). \quad (9)$$

From the inequalities $|\epsilon_{jm}|\leq 0.5$, $m=1,2$, the maximum possible value of the rounding-off error is

$$|\lambda(T)u(T)+\mu(T)v(T)|\leq 10^{-k}\sum_{j=1}^{n}(|\lambda_j|+|\mu_j|)$$
$$\sim\frac{10^{-k}}{\Delta t}\int_{t_0}^{T}[|\lambda(t)|+|\mu(t)|]dt. \quad (10)$$

However, if the ϵ_{jm}, $m=1,2$, are random variables then the standard deviation of the rounding-off error is

$$\sigma[\lambda(T)u(T)+\mu(T)v(T)]\sim$$
$$\frac{10^{-k}}{\sqrt{3}}(\Delta t)^{-1/2}\left[\int_{t_0}^{T}[\lambda^2(t)+\mu^2(t)]dt\right]^{1/2}. \quad (11)$$

II. Example

1. Sine-cosine Integrations

To check the theory developed by Rademacher the system

$$\left.\begin{aligned} x'(t)&=y(t)\\ y'(t)&=-x(t)\end{aligned}\right\} \quad (12)$$

was integrated on the Electronic Numerical Integrator and Computer.[3] The range $0.1\leq t\leq 0.9$ radians was chosen as the integration interval, since neither function was zero in that interval. (While the function is near zero the increment $\Delta t f(\bar{x},\bar{y})$ is small and might lead to a systematic effect in the rounding-off.) All computations were done to 10 decimal digits. About 10 values of Δt were used ranging from 2×10^{-3} to 2×10^{-6}. A run "A" was made with the parentheses appearing in eq 8 removed; this gives four round-offs per integration step. A run "B" was made with the parentheses in; this gives two round-offs per integration step. The results of these runs are tabulated in table 1.

The first entry in each rectangle in table 1 is the run A standard deviation for the respective angle and increment as given by eq 11. For run B this standard deviation should be divided by two. Underneath are the residual errors (after the truncation error is removed) for the various runs and functions. A typical entry (such as the residual error of 15 for the cosine in run A, angle equal to 0.7 radians, and increment of 2×10^{-3}) is found as follows:

Integration result=	0.76484 19311
Truncation error as given by eq 5=	+0.00000 02577
	0.76484 21888
True value=	0.76484 21873
Residual error=	0.00000 00015

The most interesting feature in the table occurs in run A for the sine with an increment of 2×10^{-5}. For the angle changing from 0.5 to 0.6 radian the residual error jumps from +0.00000 00008 to −0.00000 000190. This integration was rerun, and results were printed more frequently. It was found that most of the disturbance occurred between 0.5211 and 0.5264 radian. Table 3 gives the results over this range with printings at every five integration steps, and table 2 exhibits a typical five steps between the values of table 3.

TABLE 2. *Sample step in the sine-cosine integration*

$\Delta t=0.00002$ $\qquad 0.52250\leq t\leq 0.52260$

$x_i^*=x_{i-1}+\Delta t y_{i-1}$ $\qquad x_i=x_{i-1}+\frac{\Delta t}{2}[y_i^*+y_{i-1}]$

$y_i^*=y_{i-1}-\Delta t x_{i-1}$ $\qquad y_i=y_{i-1}-\frac{\Delta t}{2}[x_i^*+x_{i-1}]$

t	x sin t	x^*	y cos t	y^*
0.52250	0. 49904 81273	0. 49904 81273	0.86657 42703	0. 86657 42703
	86657 4	1 73314	−49904 8	−99810
	86656 4	. 49906 54587	−49906 5	. 86656 42893
.52252	. 49906 54586	. 49906 54586	. 86656 42891	. 86656 42891
	86656 4	1 73312	−49906 5	−99814
	86655 4	. 49908 27898	−49908 2	. 86655 43077
.52254	. 49908 27897	. 49908 27897	. 86655 43076	. 86655 43076
	86655 4	1 73310	−49908 2	−99816
	86654 4	. 49910 01207	−49910 0	. 86654 43260
.52256	. 49910 01206	. 49910 01206	. 86654 43258	. 86654 43258
	86654 4	1 73308	−49910 0	−99820
	86653 4	. 49911 74514	−49911 7	. 86653 43438
.52258	. 49911 74513	. 49911 74513	. 86653 43436	. 86653 43436
	86653 4	1 73306	−49911 7	−99824
	86652 4	. 49913 47819	−49913 4	. 86652 43612
.52260	. 49913 47818	. 49913 47818	. 86652 43611	. 86652 43611

[3] The "ENIAC" was built by the Moore School of Electrical Engineering of the University of Pennsylvania and is now located at the Ballistic Research Laboratories of Aberdeen Proving Ground.

TABLE 3. *Sine-cosine integration*

$\Delta t = 0.00002$

t	Sine	E	Cosine	t	Sine	E	Cosine
0.5100	0.48817 72474	−4	0.87274 45090	0.5240	0.50034 74198	102	0.86582 47235
.5110	.48904 97478	−8	.87225 58955	.5241	.50043 39993	107	.86577 46845
.5120	.48992 17591	−11	.87176 64098	.5242	.50052 05738	112	.86572 46368
.5130	.49079 32802	−2	.87127 60521	.5243	.50060 71433	117	.86567 45803
.5140	.49166 43102	−2	.87078 48234	.5244	.50069 37078	122	.86562 45153
.5150	.49253 48499	−9	.87029 27237	.5245	.50078 02673	127	.86557 44416
.5160	.49340 48968	−18	.86979 97538	.5246	.50086 68218	132	.86552 43593
.5170	.49427 44483	−3	.86930 59143	.5247	.50095 33713	137	.86547 42683
.5180	.49514 35083	−13	.86881 12054	.5248	.50103 99158	142	.86542 41686
.5190	.49601 20698	+2	.86831 56275	.5249	.50112 64553	147	.86537 40603
.5200	.49688 01398	−18	.86781 91812	.5250	.50121 29898	152	.86532 39433
.5201	.49696 69193	−23	.86776 94889	.5251	.50129 95193	157	.86527 38178
.5202	.49705 36938	−18	.86771 97879	.5252	.50138 60438	152	.86522 36835
.5203	.49704 04633	−23	.86767 00782	.5253	.50147 25633	157	.86517 35405
.5204	.49722 72278	−18	.86762 03597	.5254	.50155 90778	162	.86512 33890
.5205	.49731 39873	−23	.86757 06327	.5255	.50164 55873	167	.86507 32287
.5206	.49740 07418	−18	.86752 08970	.5256	.50173 20918	172	.86502 30599
.5207	.49748 74913	−23	.86747 11525	.5257	.50181 85913	177	.86497 28824
.5208	.49757 42358	−18	.86742 13995	.5258	.50190 50858	172	.86492 26961
.5209	.49766 09753	−13	.86737 16378	.5259	.50199 15753	177	.86487 25013
.5210	.49774 77098	−18	.86732 18673	.5260	.50207 80598	182	.86482 22978
.5211	.49783 44393	−13	.86727 20881	.5261	.50216 45393	187	.86477 20856
.5212	.49792 11638	−8	.86722 23004	.5262	.50225 10138	182	.86472 18649
.5213	.49800 78833	−3	.86717 25039	.5263	.50233 74833	187	.86467 16354
.5214	.49809 45978	−8	.86712 26987	.5264	.50242 39478	192	.86462 13974
.5215	.49818 13073	−3	.86707 28850	.5265	.50251 04073	187	.86457 11507
.5216	.49826 80118	+2	.86702 30625	.5266	.50259 68618	192	.86452 08954
.5217	.49835 47113	7	.86697 32313	.5267	.50268 33113	187	.86447 06314
.5218	.49844 14058	12	.86692 33916	.5268	.50276 97558	192	.86442 03587
.5219	.49852 80953	7	.86687 35431	.5269	.50285 61953	187	.86437 00774
.5220	.49861 47798	12	.86682 36859	.5270	.50294 26298	192	.86431 97874
.5221	.49870 14593	17	.86677 38202	.5271	.50302 90593	187	.86426 94889
.5222	.49878 81338	22	.86672 39457	.5272	.50311 54838	192	.86421 91816
.5223	.49887 48033	27	.86667 40625	.5273	.50320 19033	187	.86416 88658
.5224	.49896 14678	32	.86662 41708	.5274	.50328 83178	192	.86411 85413
.5225	.49904 81273	37	.86657 42703	.5275	.50337 47273	187	.86406 82081
.5226	.49913 47818	42	.86652 43611	.5276	.50346 11318	182	.86401 78664
.5227	.49922 14313	47	.86647 44434	.5277	.50354 75313	187	.86396 75159
.5228	.49930 80758	52	.86642 45169	.5278	.50363 39258	182	.86391 71569
.5229	.49939 47153	57	.86637 45817	.5279	.50372 03153	177	.86386 67892
.5230	.49948 13498	62	.86632 46380	.5280	.50380 66998	172	.86381 64129
.5231	.49956 79793	57	.86627 46855	.5281	.50389 30793	167	.86376 60279
.5232	.49965 46038	62	.86622 47243	.5282	.50397 94538	162	.86371 56344
.5233	.49974 12233	67	.86617 47546	.5283	.50406 58233	157	.86366 52321
.5234	.49982 78378	72	.86612 47761	.5284	.50415 21873	157	.86361 48213
.5235	.49991 44473	77	.86607 47889	.5285	.50423 85458	162	.86356 44018
.5236	.50000 10518	82	.86602 47932	.5286	.50432 48993	167	.86351 39736
.5237	.50008 76513	87	.86597 47887	.5287	.50441 12478	172	.86346 35369
.5238	.50017 42458	92	.86592 47757	.5288	.50449 75913	177	.86341 30914
.5239	.50026 08353	97	.86587 47540	.5289	.50458 39298	182	.86336 26374

Thus, most of the change in error occurs in an interval of about 0.0053 radian. This represents about 260 integration steps and over a thousand round-offs. In one half of the multiplications, the digit being dropped in the rounding-off process (see the sixth digit in the cosine values of table 3) changes gradually from zero up to four and back to zero again.

The errors in the sine values (see column headed E in table 3) may be wrong by up to plus or minus five units, since they were obtained by subtracting the integration results (listed in table 3) from nine digit values of the true results taken from Tables of Circular and Hyperbolic Sines and Cosines.[4]

It will be noted that the last digits of the sine values in table 3, generally speaking, are alternately 3's and 8's. This can be traced to the

[4] See footnote 2.

alternate 2's and 7's or 1's and 6's in the fifth digit of the cosine values. This is another warning against unconsidered assumption of randomness in the less significant digits of numbers involved in computations.

Rademacher asserts there will be statistical independence of the dropped digits in the rounding-off process provided $(\Delta t/2)\ f(\overline{x}_{j-1},\overline{y}_{j-1})$ and $(\Delta t/2)\,f(\overline{x}_j,\overline{y}_j)$ differ in the place 10^{-k}. The example given here shows that Rademacher's condition is not sufficient. In fact, inspection of table 2 shows that the increments $\Delta t/2\ f(x,y)$ may differ in the place 10^{-k} and yet be alike for a large number of integration steps in the place 10^{-k-1}.

There are other examples listed in table 1 leading to large residual errors. For example, consider in run *A* the sine and cosine values for $\Delta t=2\text{x}10^{-4}$ and the sine values for $\Delta t=1\times 10^{-4}$. More frequent tabulation of results shows a steady increase of residual error with no such jumps as described above.

The author thanks various members of the staff of the Ballistics Research Laboratories and of the Moore School who assisted in running the problem, and extends his appreciation to L. S. Dederick of the Ballistics Research Laboratories, whose permission and cooperation made this paper possible.

III. Appendix

Note on Systematic Rounding-off Errors in Numerical Integration

By D. R. Hartree [5]

In this paper, which summarizes the results of a numerical study of truncation and rounding-off errors in the numerical solution of a differential equation by a step-by-step process, Huskey has exhibited a case in which rounding-off errors in a sequence of successive contributions to the solution are systematically of one sign and approximately equal in magnitude, although the leading digit rounded off is the sixth significant figure in each contribution. The result is that the rounding-off errors build up to a total substantially greater than would be estimated in the basis of a random distribution of rounding-off errors in the individual contributions. The purpose of this note is to examine this situation further and to establish criteria for identifying the conditions in which it is likely to occur, so that steps can be taken to deal with it, as for example by carrying an extra significant figure temporarily in the course of the solution.

Consider the numerical evaluation of $\int y dt$, *k* decimals being kept in the calculation.

[5] On the staff of the Institute for Numerical Analysis.

Systematic rounding-off errors occur when the leading digit rounded off remains the same in a number of successive contributions to the integral; that is, when for successive contributions, last integer digit of $10^{k+1}y\delta t$ is the same. When this occurs, [last integer digit of $\delta(10^{k+1}y\delta t)$]$=0$, or [last integer digit of $10^{k+1}\dot{y}(\delta t)^2$]$=0$; that is,

$$10n-0.5<10^{k+1}\dot{y}(\delta t)^2<10n+0.5, \qquad (13)$$

for some integer *n*. This will usually occur for some value or other of *t* if

$$10^k \max|\dot{y}|(\delta t)^2>1; \qquad (14)$$

it will also occur if

$$10^k \max|\dot{y}|(\delta t)^2<1/10. \qquad (15)$$

The range $\Delta\dot{y}$ of $\dot{y}$ over which the inequalities (eq. 13) are satisfied is

$$\Delta\dot{y}=1/[10^{k+1}(\delta t)^2],$$

and the number *N* of intervals required to cover this range is given approximately by

$$N(\delta t)\,|\ddot{y}|=\Delta\dot{y},$$

so that

$$N=1/[10^{k+1}(\delta t)^3|\ddot{y}|]. \qquad (16)$$

The accumulation of systematic errors is only serious if *N* is greater than 3 or 4, that is, if

$$4\cdot 10^{k+1}(\delta t)^3|\ddot{y}|<1; \qquad (17)$$

for this not to occur

$$(\delta t)^3>1/[4\cdot 10^{k+1}|\ddot{y}|]. \qquad (18)$$

The inequalities (eq. 13 and 17) together provide a criterion for identifying the situations in which accumulation of systematic rounding-off errors may be dangerous. Such a situation may arise in any numerical integration, not only in the solution of a differential equation, the context in which it was first found by Huskey. The inequality (eq. 17) shows how much more likely it is to arise with small values of the integration interval (δt) than with large values.

In the case considered particularly by Huskey, $y=\cos t$, $k=10$, $\delta t=2\cdot 10^{-5}$, $\dot{y}=\sin t$, so that eq 13 becomes

$$10n-0.5<40\sin t<10n+0.5;$$

this is satisfied for a range of *t* in the neighborhood of $\sin t=½$, which is just the region in which the phenomenon does occur; and it happens to be particularly marked in this case, since the digit which is rounded off systematically happens to be a 4 over a considerable range. Also

$$10^{k+1}(\delta t)^3|\ddot{y}|\approx 7\cdot 10^{-4},$$

so that, from eq. 16, *N* is about 1,400, and the inequality (eq. 17) is very far from being satisfied; hence it is not surprising that the phenomenon arose in a marked form. The condition (eq. 18) suggests that with the value of $k=10$, the interval length should certainly be greater than $\delta t=10^{-4}$.

WASHINGTON,, July 17, 1948.