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Theory of Wagner Ground Balance for Alternating-Current Bridges

By Richard K. Cook

A method is described for getting the Wagner ground balance of an alternating-current bridge when used with any three-terminal source. The basic idea is to insert impedances between the two ungrounded terminals of the source, and the corresponding terminals of the Wagner ground arm and the bridge, so as to "balance" approximately the currents from the source. The method is applicable to high-voltage type Schering bridges. Such a bridge, incorporating the principles of the new method for securing the Wagner ground balance, has been designed and built principally for measurement of the small capacitances, about 50 $\mu\mu$f, of some types of condenser microphones.

I. Introduction

A Wagner ground is used sometimes with alternating current bridges in order to eliminate the effects of capacity to ground. The same end can be achieved by a system of electrostatic shielding, but the required shielding can become very complicated. With a Wagner ground, it is possible to secure high precision with simple shielding.

The object of this paper is to show how almost any available three-terminal source can be used with a Wagner-grounded bridge. The simple theory that is usually given assumes that the source has only two accessible output terminals, which are connected to the two ungrounded input terminals of the bridge [1] [1]. The Wagner ground balance is readily found theoretically, but sometimes it is difficult or impossible to find experimentally, particularly in the case of high-voltage type Schering bridges used to measure small capacitances.

Ogawa [2] has pointed out that the experimental difficulty is due to the fact that available sources really have three output terminals, and the simple theory of a two-terminal source does not apply. He has given a comprehensive treatment of the general theory of Wagner-grounded alternating-current bridges and has solved the difficulty of securing balance by means of a specially constructed transformer satisfying an electrical condition which simplifies the Wagner ground balance equation. The theory has been applied by Astin [3] to a Schering bridge used for precision measurement of the power factors of air capacitors.

The theory given below shows how almost any available three-terminal source can be used, thus obviating the need for a special transformer. The theory is applied to the design of a high-voltage type Schering bridge for measurement of the small capacitances of condenser microphones at frequencies between 500 and 10,000 c/s. Such capacitance measurements are required when making absolute calibrations of condenser microphones by the reciprocity technique.

It should be pointed out that there are other ways of securing Wagner ground balance. For example, in a high-voltage type Schering bridge for capacitance measurements at power-supply frequencies, which has recently become commercially available [4], the Wagner ground balance is obtained by adjustment of the amplitude and phase (relative to ground) of the voltage applied to one of the input terminals of the bridge.

II. Theory of a Wagner-grounded Bridge

The basic idea of the new way for getting Wagner ground balance is as follows: In the circuit shown in figure 1, suppose the bridge to be balanced, so that points B_1 and B_2 are at the

[1] Figures in brackets indicate the literature references at the end of this paper.

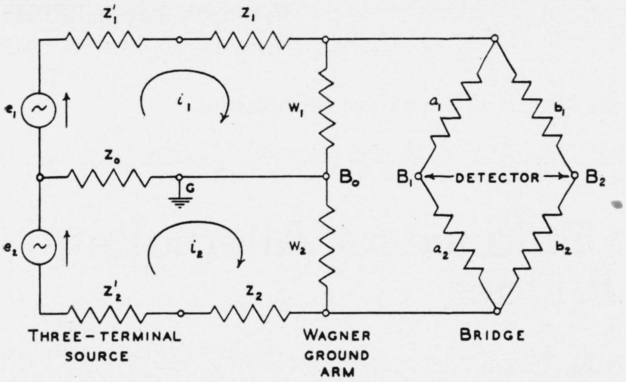

FIGURE 1.—*Circuit diagram of Wagner-grounded alternating-current bridge.*

same potential. Hence $a_1b_2=a_2b_1$. Suppose also that $i_1=i_2$; the wire B_0G will then carry no current. This will be referred to as the balanced ground-current condition. It is clear that this assumption eliminates the effect of z_0, so that the simple bridge theory will apply. The third assumption is that $a_1w_2=a_2w_1$. Then it follows from these three assumptions that the Wagner ground is balanced, or that points B_0 and B_1 are at the same potential. If $e_1=e_2$, the balanced ground-current condition can be realized by adjusting z_1 and z_2 so that z_1+z_1' plus the parallel combination of a_1, b_1, and w_1, is equal to z_2+z_2' plus the parallel combination of a_2, b_2, and w_2.

The mathematics needed for proof of the foregoing statements can be obtained from Ogawa's article, but a simplified derivation might be of interest. We assume a more general condition in which the ground-current might not be balanced, so that i_1 is not necessarily equal to i_2, but in which the potentials at B_0, B_1, and B_2 are the same (Wagner ground and bridge balanced). Also, we assume e_1 is not necessarily equal to e_2.

Before proceeding, a few words about three-terminal sources are in order. Ogawa has shown by a generalization of Thévenin's theorem that any three-terminal source can be represented by the network shown in figure 1. Subsequently a more general proof was given by Starr [5]. An equivalent network of this type applies whenever the potentials of the terminals are linear functions of the three currents through the terminals. Whether or not the network can be applied to a source that is available in the laboratory is a question that can, of course, be answered only by experiment. The simple bridge theory which is usually given tacitly assumes that $z_0=\infty$.

By virtue of the bridge balance, the circuit of figure 1 can be replaced by the simpler circuit of figure 2, in which the impedance s_1 is the parallel combination of a_1 and b_1, and s_2 is the parallel combination of a_2 and b_2. Also, $Z_1=z_1+z_1'$, $Z_2=z_2+z_2'$. As in figure 1, the detector impedance is not indicated, as it has no current at balance. Application of Maxwell's loop equations to the network yields the following equations:

$$\left.\begin{aligned}(w_1+Z_1+z_0)i_1-z_0i_2-w_1i_3-e_1=0\\ -z_0i_1+(w_2+Z_2+z_0)i_2-w_2i_3-e_2=0\\ w_1i_1+0-(w_1+s_1)i_3+0=0\\ 0+w_2i_2-(w_2+s_2)i_5+0=0.\end{aligned}\right\} \qquad (1)$$

As these equations are linear and homogeneous in $(i_1, i_2, i_3, 1)$, the necessary and sufficient condition for a nonzero solution for $(i_1, i_2, i_3, 1)$ is the vanishing of the determinant of the coefficients:

$$\begin{vmatrix} w_1+Z_1+z_0 & -z_0 & -w_1 & -e_1 \\ -z_0 & w_2+Z_2+z_0 & -w_2 & -e_2 \\ w_1 & 0 & -(w_1+s_1) & 0 \\ 0 & w_2 & -(w_2+s_2) & 0 \end{vmatrix}=0. \qquad (2)$$

Expansion of the determinant yields the fundamental balance equation,

$$e_1\left[s_2+Z_2+\frac{s_2Z_2}{w_2}+\left(\frac{s_2}{w_2}-\frac{s_1}{w_1}\right)z_0\right]=$$
$$e_2\left[s_1+Z_1+\frac{s_1Z_1}{w_1}-\left(\frac{s_2}{w_2}-\frac{s_1}{w_1}\right)z_0\right], \qquad (3)$$

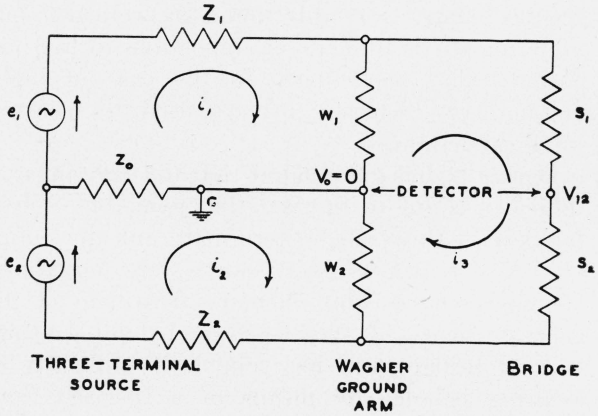

FIGURE 2.—*Simplified circuit diagram of Wagner-grounded bridge.*

which is basically the same as Ogawa's eq 62.

If $i_1=i_2$, then eq 1 show it is necessary that

$$\frac{s_2}{w_2}=\frac{s_1}{w_1}. \tag{4}$$

If in addition, $e_1=e_2$, then eq 3 reduces to

$$Z_2+\frac{s_2w_2}{s_2+w_2}=Z_1+\frac{s_1w_1}{s_1+w_1}. \tag{5}$$

This is the algebraic form of the statement made in the first paragraph of this section.

Suppose that $z_0\rightarrow\infty$. Then it is found that eq 3 reduces to eq 4, provided $e_1+e_2\neq 0$. This is the usual Wagner ground balance condition obtained by application of the simple theory.

We define the unbalance voltage to be the amount by which V_{12} exceeds ground potential when the Wagner ground is not balanced. This quantity is needed for studying the convergence of the sequence of balancing operations. It is justifiable to ignore the impedance of the amplifier-detector system which might be employed to observe V_{12}. The reason for this is that the impedance will affect only the magnitude and phase of V_{12}, and will not affect the convergence of the sequence of balancing operations when the unbalance current is small in comparison with the main bridge currents. We find

$$V_{12}=\frac{1}{\Delta}\left\{e_1\left[s_2+Z_2+\frac{s_2Z_2}{w_2}+\left(\frac{s_2}{w_2}-\frac{s_1}{w_1}\right)z_0\right]-\right.$$
$$\left.e_2\left[s_1+Z_1+\frac{s_1Z_1}{w_1}+\left(\frac{s_1}{w_1}-\frac{s_2}{w_2}\right)z_0\right]\right\}, \tag{6}$$

where

$$\Delta=\left(Z_1+s_1+\frac{Z_1s_1}{w_1}\right)\left(1+\frac{Z_2}{w_2}+\frac{z_0}{w_2}\right)+$$
$$\left(Z_2+s_2+\frac{Z_2s_2}{w_2}\right)\left(1+\frac{Z_1}{w_1}+\frac{z_0}{w_1}\right)+$$
$$z_0\left(\frac{Z_1}{w_1}+\frac{s_2}{w_2}+\frac{Z_1s_2}{w_1w_2}+\frac{Z_2}{w_2}+\frac{s_1}{w_1}+\frac{Z_2s_1}{w_2w_1}\right).$$

The way in which eq 3 was derived shows it to be a necessary condition for Wagner ground balance. Equation 6 shows that eq 3 is also sufficient, if $\Delta\neq 0$.

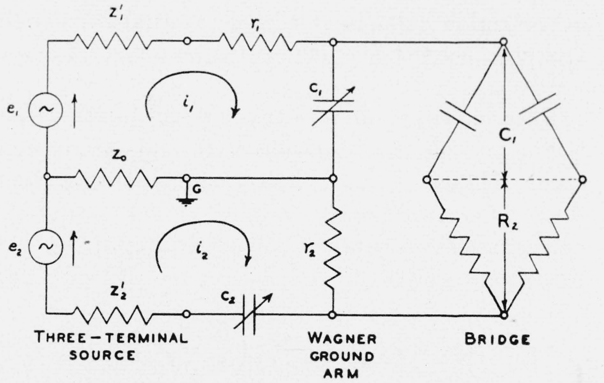

FIGURE 3.—*High-voltage type Schering bridge with Wagner ground.*

III. Application to a Schering Bridge

The application of the Wagner-grounding technique described above to a high-voltage type Schering bridge, which is useful for measuring the capacitance of condenser microphones, is easily accomplished. A circuit diagram for such a bridge is shown in figure 3. Here C_1 is the parallel combination of the two capacitances in the high-voltage arms, and R_2 is the parallel combination of the two resistances in the low-voltage arms.

It is important to study how the Wagner ground balance is obtained when the circuit parameters are varied. A practical advantage of the Schering bridge is that balance can be obtained by variation of capacitance only. The Wagner ground balance of the Schering bridge circuit shown in figure 3 can also be obtained by variation of capacitance only, as the following analysis shows.

As before, we shall suppose that both the bridge and the Wagner ground are balanced. Also, we make the following simplifying assumptions, which can be readily fulfilled experimentally. (1) The three-terminal source is the secondary of a center-tapped transformer. (2) The transformer is symmetrical enough so that $e_2=(1+\alpha+j\beta)e_1$, where $\alpha<<1$ and $\beta<<1$. (3) The terminals of the transformer secondary are externally connected to the center tap by equal low resistances, so that $z_1'\approx z_2'$, and $z_0\approx\frac{-z_1'}{2}$. z_1' and z_2' are positive resistances, but z_0' is a negative resistance. In what follows there will be no loss of generality in supposing that z_1' is included in r_1, and z_2' is included in the residual of c_2. (4) c_1, c_2, and C_1

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are variable capacitors whose residuals are small, and r_1, r_2, and R_2 are fixed resistors whose residuals are small.

First we shall suppose that all the small quantities α, β, and the residuals of the impedances are zero. Hence $e_1=e_2$. The object of this is to find the approximate values of the adjustable capacitances c_1 and c_2. Application of eq 3 to the circuit yields

$$\left.\begin{aligned} R_2=r_1+\left(\frac{c_1}{C_1}\right)r_1-2z_0\left(\frac{R_2}{r_2}-\frac{c_1}{C_1}\right); \\ \frac{1}{\omega c_2}\left(1+\frac{R_2}{r_2}\right)=\frac{1}{\omega C_1}. \end{aligned}\right\} \tag{7}$$

The first of these corresponds to the real part of eq 3, and the second corresponds to the imaginary part. Solutions of eq 7 for c_1 and c_2 are as follows:

$$\left.\begin{aligned} c_1=\left(\frac{R_2-r_1+\frac{2z_0R_2}{r_2}}{r_1+2z_0}\right)C_1; \\ c_2=\left(\frac{R_2}{r_2}+1\right)C_1. \end{aligned}\right\} \tag{8}$$

These equations indicate that if

$$r_1+2z_0>0,\text{ and }R_2-r_1+\frac{2z_0R_2}{r_2}>0, \tag{9}$$

the Wagner ground balance can be obtained by variation of capacitance only. The balance is independent of frequency.

It is to be noted that c_1 can be adjusted to satisfy the real part of eq 3, and c_2 can be separately adjusted to satisfy the imaginary part. This, together with eq 6, shows that the Wagner ground unbalance voltage, which is a vector in the complex plane, can be reduced to zero by a series of steps, successive steps being mutually perpendicular. This is a highly desirable property of a sequence of balancing operations for an alternating-current bridge.

The unbalance voltage can be reduced to zero by adjustment of other pairs of parameters besides c_1 and c_2. Equations 7 show that variation of r_1 and c_2 will also yield successive voltage steps that are mutually perpendicular. However, all other pairs will yield successive steps, which in general are not mutually perpendicular, an example of which is the combination r_2 and c_1. The pair r_1, c_1 cannot be used at all to get balance, as the second eq 7 contains neither of these parameters.

A more detailed analysis in which α, β, and the residuals are not assumed to be zero shows that the Wagner ground balance can still be secured by variation of the capacitances c_1 and c_2 only. However, the successive steps in the unbalance voltage, obtained by varying c_1 and c_2 successively, are vectors which are not quite perpendicular to one another.

It might be inconvenient to select the circuit parameters so that the inequalities (9) hold if a center-tapped transformer is used as a source, as for such a transformer, z_0 is a negative resistance. An examination of the fundamental balance eq 3 shows that the effect of z_0 disappears if the condition given by eq 4 holds. When this condition is applied to the Schering bridge circuit discussed above, we find

$$c_1=\frac{R_2}{r_2}C_1,\ r_1=\frac{r_2R_2}{r_2+R_2},\ c_2=c_1+C_1. \tag{10}$$

It is understood that r_1 includes the resistive impedance z_1'. The ground current is balanced in this case.

Another possibility is to introduce a positive external resistor, whose magnitude is $|z_0|$, at the center tap of the transformer. This would, in effect, reduce z_0 to zero, and the terms in eq 3 containing z_0 would vanish.

Still another possibility is to use two similar transformers driven by the same oscillator. The transformers would have to be arranged so that there is no mutual induction between them. The use of buffer amplifiers might also be required between the oscillator and transformers. The whole arrangement can be converted into a three-terminal source having $z_0=0$ by connecting a terminal on the secondary of one transformer to a terminal on the other.

IV. Description of a Bridge

A Wagner-grounded Schering bridge, based on the principles outlined above, has been constructed in our Sound Laboratory. The bridge is the high-voltage type and has been used mainly to measure condenser-microphone capacitances, which are about 50 $\mu\mu$f, at frequencies ranging from 500 to 10,000 c/s. Accuracy of the order of 0.1 percent can be obtained, but the microphones must of course be converted into three-terminal impedances by means of a grounded shield in order to realize this accuracy. The source is the center-tapped output transformer of a commercially available beat-frequency oscillator. The circuit of the high-voltage type bridge easily allows the introduction of the direct-current polarizing voltage needed for condenser microphones.

The resistance arms of the bridge are fixed resistors, each of about 50,000 ohms. The capacitance arms have the unknown condenser, usually about 50 $\mu\mu$f, and a standard condenser, variable over the range 40 to 60 $\mu\mu$f. Hence C_1 (see fig. 3) is about 100 $\mu\mu$f, and R_2 is about 25,000 ohms. The condensers c_1 and c_2 are variable up to about 1,000 $\mu\mu$f, and r_1 and r_2 are each approximately 3,000 ohms.

The three-terminal source has z_1' and z_2', which are mainly resistive, both approximately 800 ohms. The impedance z_0 is also mainly resistive, and is approximately -400 ohms. The voltages e_1 and e_2 are closely equal, both in magnitude and phase, and are about 8 volts for frequencies between 500 and 10,000 c/s.

V. References

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