U. S. Department of Commerce
National Bureau of Standards

Research Paper RP1849
Volume 39, December 1947

Part of the Journal of Research of the National Bureau of Standards

Instability of Simply Supported Square Plate With Reinforced Circular Hole in Edge Compression

By Samuel Levy, Ruth M. Woolley, and Wilhelmina D. Kroll

A method is presented for computing the compressive buckling load of a simply supported elastic rectangular plate having a central circular hole reinforced by a circular doubler plate. Numerical results are presented for six square plates having hole diameters up to one-half of the plate length. Comparison of these results with those computed for plates without holes shows that an unreinforced circular hole causes a relatively small reduction in buckling load, and reinforcement of a circular hole by a doubler plate causes a substantial increase in buckling load.

I. Introduction

The stressed skin cover of airplane wings and fuselages has to be perforated by holes at a number of places to give access to the interior of the wing or fuselage. These holes must be reinforced to prevent weakening of the entire structure by the hole.

An ideal reinforcement would be one that confines the disturbance in the stress flow to the immediate neighborhood of the hole and which, at the same time, adds minimum weight to the structure.

Too little is known about reinforcements to approximate this ideal in practice. In the absence of a rational design procedure, it has been customary to reinforce holes by circular doubler plates riveted to one side of the sheet, thereby increasing the effective thickness and rigidity of the sheet at the edge of the hole.

The Bureau of Aeronautics, Navy Department, initiated a study of reinforcements around holes at the National Bureau of Standards to provide a better understanding of reinforcements and to indicate practical improvements in their design. The first phase of this investigation was a plane stress analysis of a doubler plate reinforcement around a circular hole in an infinite sheet under uniform compression in one direction or under shear.

The second phase of this investigation consisted of a check of the theoretical analysis by tests on plates having reinforced circular holes. It was found that the analysis gave an adequate description of the stresses and displacements provided that the doubler plate was fastened to the sheet by at least two concentric rows of rivets.

The third phase of this investigation is described in this paper. An analysis is given for the stabilizing effect of the doubler plate for square plates with central circular holes when the plates are subjected to edge compression in one direction.

II. Method of Analysis

An energy method for determining the buckling load of rectangular plates of constant thickness under compressive loads is presented by Timoshenko.[1] A review of this derivation shows that the method is also applicable to plates of variable thickness. In this case, Timoshenko's integrals I_1 and I_2 are

[1] S. Timoshenko, Theory of elastic stability pp. 307 to 326 (McGraw-Hill Book Co., New York, N. Y., 1936).

$$
\left.\begin{aligned}
I_1 &= \iint_{\text{surface}} D \left\{ \left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} \right)^2 - \right. \\
&\quad \left. 2(1-\mu)\left[\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2} - \left(\frac{\partial^2 w}{\partial x \partial y} \right)^2 \right] \right\} dx\, dy \\
I_2 &= \iint_{\text{surface}} h \left[\frac{\sigma_x}{S}\left(\frac{\partial w}{\partial x} \right)^2 + \frac{\sigma_y}{S}\left(\frac{\partial w}{\partial y} \right)^2 + \right. \\
&\quad \left. 2\,\frac{\tau_{xy}}{S}\frac{\partial w}{\partial x}\frac{\partial w}{\partial y} \right] dx\, dy
\end{aligned}\right\} \qquad (1)
$$

where

h=plate thickness (function of x and y).

x, y=rectangular coordinates with origin at center of plate and x-axis in direction of load.

w=lateral deflection of plate.

$D=Eh^3/12(1-\mu^2)$, flexural rigidity of plate (function of x and y).

μ=0.3, Poisson's ratio.

σ_x=tensile stress in x direction.

σ_y=tensile stress in y direction.

τ_{xy}=shear stress.

S=tensile stress in x direction far from hole.

The stress ratios σ_x/S, σ_y/S, and τ_{xy}/S just prior to buckling may be obtained as shown by Gurney.[2] The stresses σ_r, σ_θ, and $\tau_{r\theta}$ referred to polar coordinates r, θ are given, with some change in notation, as

$$
\left.\begin{aligned}
\sigma_r &= S\{F + KR^2/r^2 - (A + 3CR^4/r^4 + 2DR^2/r^2) \cos 2\theta\} \\
\sigma_\theta &= S\{F - KR^2/r^2 + (A + 6Br^2/R^2 + 3CR^4/r^4) \cos 2\theta\} \\
\tau_{r\theta} &= S\{(A + 3Br^2/R^2 - 3CR^4/r^4 - DR^2/r^2) \sin 2\theta\}
\end{aligned}\right\} \qquad (2)
$$

where

r, θ=polar coordinates with origin at center of hole,

θ=angle between radius r to point and direction of load,

R=radius of hole.

F, K, A, B, C, D=coefficients with different values in sheet and in doubler plate.

The values of F, K, A, B, C, and D may be determined from the dimensions for a particular doubler plate by using figure 1. The coefficients from "1" region, figure 1, are used in eq 2 when computing the stresses in the unreinforced sheet; the coefficients from "2" region are correspondingly used when computing stresses in the reinforced sheet. The abscissa in figure 1, $(T/t-1)(b^2/a^2-1)$, is the ratio of the volume of material in the reinforcement to the volume of material removed from the sheet to make the hole.

The stresses in rectangular coordinates may be computed from the stresses in polar coordinates as given in eq 2 by using the conversion formulas:

$$
\left.\begin{aligned}
\sigma_x &= \sigma_r \cos^2\theta + \sigma_\theta \sin^2\theta - \tau_{r\theta} \sin 2\theta \\
\sigma_y &= \sigma_r \sin^2\theta + \sigma_\theta \cos^2\theta + \tau_{r\theta} \sin 2\theta \\
\tau_{xy} &= \left(\frac{\sigma_r - \sigma_\theta}{2} \right) \sin 2\theta + \tau_{r\theta} \cos 2\theta.
\end{aligned}\right\} \qquad (3)
$$

The lateral deflection w of a simply supported rectangular plate of length a and width b will be approximated by the first terms in the trigonometric series

$$
\begin{aligned}
w &= a_{11} \cos\frac{\pi x}{a} \cos\frac{\pi y}{b} + a_{13} \cos\frac{\pi x}{a} \cos\frac{3\pi y}{b} + \\
&\quad a_{31} \cos\frac{3\pi x}{a} \cos\frac{\pi y}{b} + a_{33} \cos\frac{3\pi x}{a} \cos\frac{3\pi y}{b} + \ldots \qquad (4)
\end{aligned}
$$

Substitution of eq 4 in eq 1 and integration (see next section for method of numerical integration) reduces integrals I_1 and I_2 to quadratic expressions in the coefficients a_{11}, a_{13} . . . . .. The critical value of S at which buckling of the plate occurs is obtained from these expressions, according to Timoshenko, as that value of S which reduces to zero the determinant of the coefficients of a_{11}, a_{13}, etc. in the set of simultaneous equations,

$$
\left.\begin{aligned}
&\frac{\partial I_1}{\partial a_{11}} + S\frac{\partial I_2}{\partial a_{11}} = 0 \\
&\frac{\partial I_1}{\partial a_{13}} + S\frac{\partial I_2}{\partial a_{13}} = 0. \\
&\cdots\cdots\cdots \\
&\cdots\cdots\cdots
\end{aligned}\right\} \qquad (5)
$$

The sign before S in eq 5 is plus, and that before γ in the corresponding eq 212 given by Timoshenko [3] is minus. Where S is considered positive for tensile stress, as is done in this report, the signs in eq 5 are correct.

[2] G. Gurney, Brit. Rep. Memo. No. 1834 (Feb. 1938).

[3] S. Timoshenko, Theory of elastic stability (McGraw-Hill Book Co., New York, N. Y., 1936).

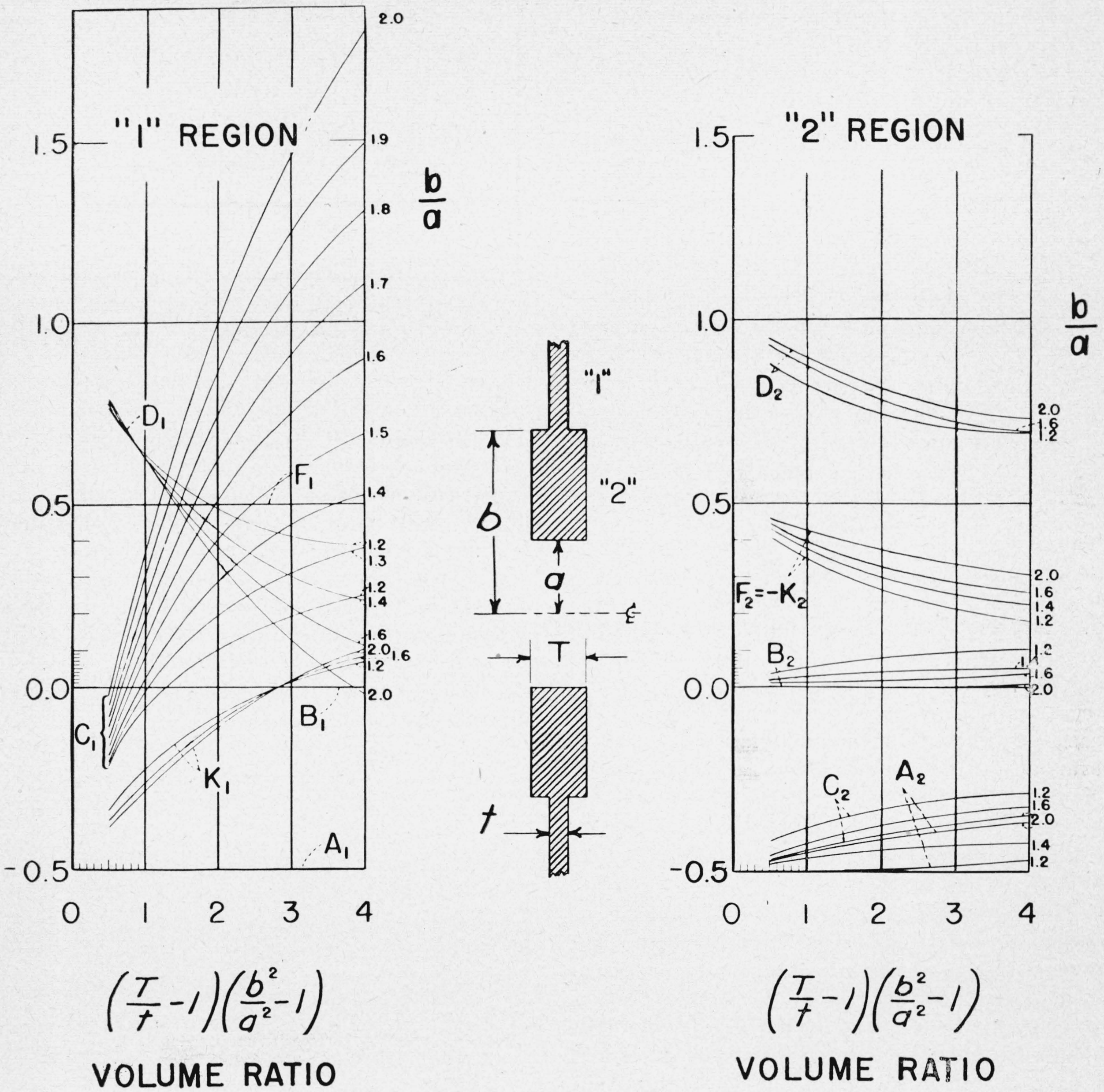

FIGURE 1.—*Values of coefficients F, K, A, B, C, and D to be used in eq 2 for computing the stress distribution in the sheet and reinforcement.*

765484—47——7

III. Numerical Integration

The evaluation of the integrals in eq 1 over the surface of the plate is made difficult by having a circular inner and a rectangular outer boundary and by involving a stress that is a complicated function when expressed in rectangular coordinates.

The integral I_1, eq 1, was evaluated as the resultant of three integrations:

$$I_1 = I_{1a} + I_{1b} - I_{1c} \tag{6}$$

where

I_{1a} = The integral I_1 for the surface enclosed by the outer rectangular boundary of the plate taking $D=D_1$, the value of D in the unreinforced portion of the plate.

I_{1b} = The integral I_1 for the surface enclosed by the outer circular boundary of the reinforcement taking $D=D_2-D_1$, where D_2 is the value of D in the reinforcement.

I_{1c} = The integral I_1 for the surface enclosed by the circular boundary of the hole taking $D=D_2$.

The double integration necessary to evaluate I_{1a} could in every case be done directly. However, this was not possible in evaluating I_{1b} and I_{1c}. The integrals I_{1b} and I_{1c} were obtained by integrating directly in respect to one variable and using Gauss' method of numerical integration,[4] for integration with respect to the other variable. In each case where numerical integration was used to evaluate I_1, the number of Gauss points was increased to the place where the addition of another point caused less than 1 percent change in the integral. In general, five Gauss points were enough for this purpose.

The integral I_2 was evaluated by considering the surface of the plate in three portions. The first portion, A, figure 2, was taken as the circular disk of the reinforced area; the second portion, B, figure 2, was taken as the circular disk between the outer edge of the reinforcement and the largest inscribed circle in the plate; and the third portion, C, figure 2, was taken as the remainder of the plate. The integral for each circular portion was determined, using Gauss' method of numerical integration, by first integrating numerically in a circumferential direction and then integrating numerically in a radial direction. The integral for the remainer of the plate was obtained also by using Gauss' method, first integrating numerically in the direction of the load and then at right angles to the direction of the load. Only one quadrant of the plate had to be considered because of symmetry. In each case, a sufficient number of Gauss points was used to reduce the estimated error to less than 1 percent. Twenty-three points in each quadrant were used for cases 1, 3, 4, and 5, table 1, thirty-three points for case 2, and twenty-eight points for case 6.

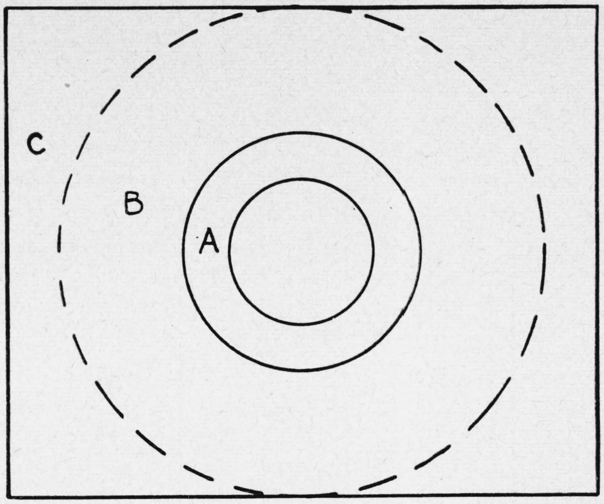

FIGURE 2.—*Subdivision of plate into three portions A, B, C, in evaluating I_2.*

TABLE 1.—*Dimensions for plates investigated*

Case	Width		Radius of hole, R	Outer radius of reinforcement	Thickness	
	Direction of load	Transverse to load			Reinforced region	Unreinforced region
1	a	a	0.125a	(1)	(1)	h
2	a	a	.0625a	(1)	(1)	h
3	a	a	.25a	(1)	(1)	h
4	a	a	.125a	0.25a	2.0h	h
5	a	a	.125a	.25a	1.5h	h
6	a	a	.125a	.1875a	3.4h	h

[1] Hole in plate not reinforced.

A typical distribution of Gauss points for computing I_2 is shown in figure 3 for one quadrant of a square plate with a central hole. For this distribution, the integral of a function F over the plate surface is

[4] E. T. Whittaker and G. Robinson, Calculus of observations, p. 159 (Blackie and Son, Ltd., London, 1944).

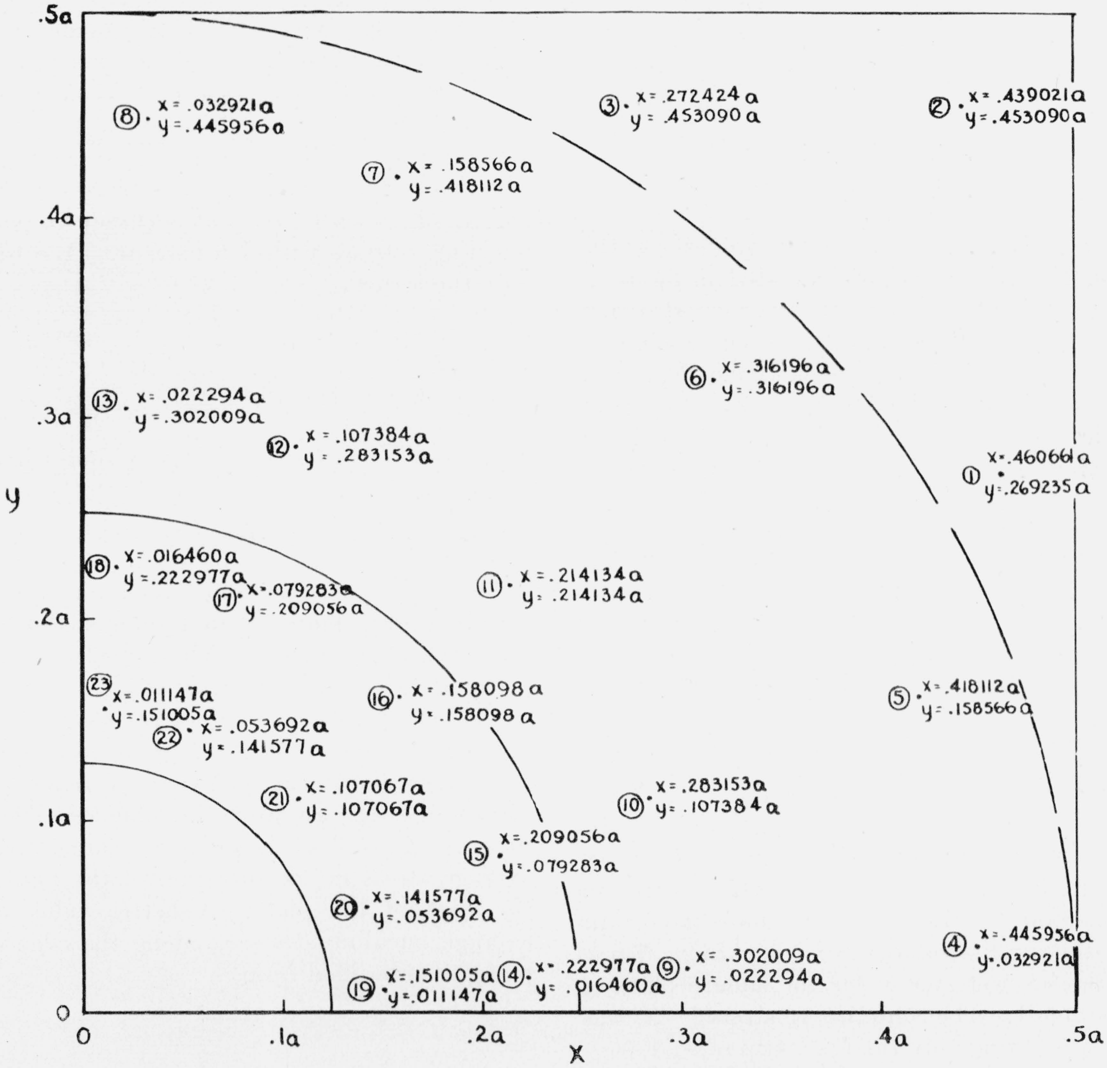

FIGURE 3.—*Distribution of Gauss points for computing I_2 in case 4.*

$$\iint_{\text{surface}} F dA = a^2[0.075316F_1+0.068366F_2+ 0.068366F_3+0.041605F_4+ 0.084048F_5+0.099898F_6+ 0.084048F_7+0.041605F_8+ 0.028176F_9+0.056919F_{10}+ 0.067653F_{11}+0.056919F_{12}+ 0.028176F_{13}+0.010401F_{14}+ 0.021012F_{15}+0.024975F_{16}+ 0.021012F_{17}+0.010401F_{18}+ 0.007044F_{19}+0.014230F_{20}+ 0.016913F_{21}+0.014230F_{22}+ 0.007044F_{23}] \tag{7}$$

where F_1, F_2 . . . F_{23} are the values of the function F at the positions indicated in figure 3.

An approximate check on the over-all adequacy of the numerical integration methods used was obtained by determining the buckling load of a square plate with no hole both by the exact method,[5] and by numerical integration with 23 points for I_2. The results differed by only 0.7 percent. In the case of a plate with a reinforced hole, it is likley that the more complicated stress distribution and the greater prominence of higher order terms in the series used for the deflection causes the error to be somewhat higher.

IV. Convergence of Deflection Function

The correct value of S for buckling of the plate would be that value of S which reduces the determinant of the coefficients of a_{11}, a_{13}, . . . in the infinite set of eq 5 to zero. In order to limit the work of computing S to a finite amount, preliminary computations were made to see which terms in the deflection function, eq 4, were most important in determining the value of S for buckling. These computations were made for the square plate of case 4, table 1. The compressive stress $(-S)$ was 10.33 Eh^2/a^2 using only the first term of eq 4,

$$w=a_{11}\cos\frac{\pi x}{a}\cos\frac{\pi y}{b}$$

Using only terms 1 and 2 of eq 4,

$$w=a_{11}\cos\frac{\pi x}{a}\cos\frac{\pi y}{b}+a_{13}\cos\frac{\pi x}{a}\cos\frac{3\pi y}{b}.$$

$(-S)$ was 9.43 Eh^2/a^2, a decrease of 8.6 percent. Using only terms 1 and 3,

$$w=a_{11}\cos\frac{\pi x}{a}\cos\frac{\pi y}{b}+a_{31}\cos\frac{3\pi x}{a}\cos\frac{\pi y}{b},$$

$(-S)$ was 8.80 Eh^2/a^2, a decrease of 14.7 percent. Using only terms 1 and 4,

$$w=a_{11}\cos\frac{\pi x}{a}\cos\frac{\pi y}{b}+a_{33}\cos\frac{3\pi x}{a}\cos\frac{3\pi y}{b},$$

$(-S)$ was 10.32 Eh^2/a^2, a decrease of 0.1 percent. Had additional terms of the trigonometric series been used with term 1, the decrease in the buckling stress would probably have been proportionately smaller. It is believed, on this basis, that the first three terms of eq 4 approximated the lateral deflection of the square plates with simply supported edges with sufficient accuracy to give the buckling stresses within 5 percent. Accordingly, only the terms

$$w=a_{11}\cos\frac{\pi x}{a}\cos\frac{\pi y}{b}+a_{13}\cos\frac{\pi x}{a}\cos\frac{3\pi y}{b}+ a_{31}\cos\frac{3\pi x}{a}\cos\frac{\pi y}{b}$$

were used in the remaining computations for square plates.

V. Results

The compressive buckling load was computed for six square plates with reinforced and unreinforced holes, figure 4. The dimensions for the plates are given in table 1.

The analysis gave the compressive buckling stress $(-S)$ far from the hole corresponding to the stress distribution derived in reference [2]. Values of $(-S)$ are given in table 2. The compressive stress $(-S)$ is somewhat larger than the average stress in the direction of the load along the edge of the plate. A better value of the average buckling stress σ_{cr} along the edge of the plate was obtained from

$$\sigma_{cr}=\frac{\sigma_1+\sigma_2}{2} \tag{8}$$

where σ_1 is the average compressive stress on the loaded edge of the plate and σ_2 the average compressive stress on the center cross section, obtained by dividing the load on that section by ah. The value of σ_1 was obtained by numerical integration and the value of σ_2 by direct integration of the stresses σ_x given by eq 3. Values of $(-S)$, σ_1, σ_2, and σ_{cr} are given in table 2. The average stresses

[5] S. Timoshenko, Theory of elastic stability, p. 331 (McGraw-Hill Book Co., New York, N. Y., 1936).

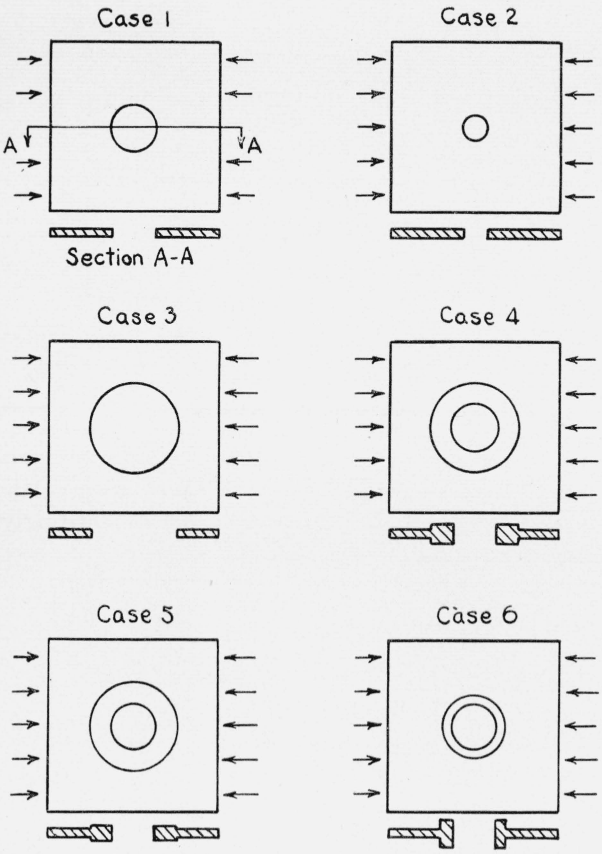

FIGURE 4.—*Loading of square plates with simply supported edges.*

Dimensions in table 1.

σ_1 and σ_2 differed from each other by less than 3 percent, indicating that the equilibrium condition is substantially satisfied.

TABLE 2.—*Critical compressive stress for plates investigated*

Case	Stress ($-S$) far from hole	Average stress σ_1 at $x=\frac{a}{2}, -\frac{a}{2}<y<\frac{a}{2}$	Average stress σ_2 corresponding to load at $x=0$ divided by ah	Buckling stress, σ_{cr}, average of σ_1 and σ_2
1	3.611 Eh^2/a^2	3.454 Eh^2/a^2	3.490 Eh^2/a^2	3.472 Eh^2/a^2
2	3.620 Eh^2/a^2	3.578 Eh^2/a^2	3.591 Eh^2/a^2	3.584 Eh^2/a^2
3	3.750 Eh^2/a^2	3.076 Eh^2/a^2	3.164 Eh^2/a^2	3.120 Eh^2/a^2
4	8.021 Eh^2/a^2	7.984 Eh^2/a^2	8.069 Eh^2/a^2	8.027 Eh^2/a^2
5	5.700 Eh^2/a^2	5.578 Eh^2/a^2	5.646 Eh^2/a^2	5.612 Eh^2/a^2
6	7.945 Eh^2/a^2	7.881 Eh^2/a^2	7.984 Eh^2/a^2	7.932 Eh^2/a^2

The buckling stress of a simply supported square plate of constant thickness without a hole is given on p. 331 of reference [3] (taking Poisson's ratio as 0.3) as

$$\sigma_{cr}=3.615\ Eh^2/a^2. \tag{9}$$

Comparison of this value with the values computed for cases 1 to 3 of table 2 shows that unreinforced central holes, having a diameter of ⅛, ¼ and ½ of the length of a side of a square plate, reduce its buckling load by only 1, 4, and 14 percent, respectively. Comparison of the value in eq 9 with the values for cases 4 to 6 of table 2 shows that the reinforcements increased the buckling stress over that for the plate without a hole by 55 to 122 percent.

The effect of the shape of the reinforcement on the buckling load is indicated by comparing cases 4 and 6, table 2. For these cases, the hole size and volume of reinforcing material are identical but the reinforcing material is concentrated nearer the edge of the hole in case 6 (see fig. 4). The computed buckling stresses are the same within 1.2 percent.

VI. Conclusions

A numerical procedure was developed for estimating the buckling stress of simply supported rectangular plates with circular holes and doubler plate reinforcement. The procedure provides a convenient method for solving the integrals for the energy stored in the plate.

The computations showed that the buckling stress of square plates is reduced only a small amount by the presence of unreinforced holes. For the cases considered, the greatest reduction, 14 percent, corresponded to case 3, a square plate with a hole diameter 0.5 times the length of the plate.

Reinforcements of the doubler-plate type are shown by the computations to cause marked increases in the buckling stress. For case 4 with reinforcement, the computed buckling stress was 2.3 times that for case 1 with no reinforcement.

The buckling stress seems to be insensitive to change in shape of reinforcements of given volume. The computed buckling stresses were the same within 1.2 percent for cases 4 and 6, which differed in having doubler plates of different thickness and radius, but of the same volume.

The authors thank the Bureau of Aeronautics, Navy Department, for their cooperation in carrying out this investigation and for permission to publish the results.

WASHINGTON, August 19, 1947.