Theory for Axial Rigidity of Structural Members Having Ovaloid or Square Perforations

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Formulas are derived for computing the over-all lengthening (or shortening) of a tension (or compression) member having a uniform gross cross section and a series of similar perforations of approximately ovaloid or approximately square shape uniformly distributed along the length.

I. Introduction

In a previous paper [1] the author demonstrated a method for computing the axial rigidity of a tension or compression member having a uniform gross cross section and a series of similar perforations of circular or elliptical shape uniformly distributed along the length. The axial-rigidity factor $K$ is defined so that $KEA_x$ is the rigidity that should be used in place of $EA_t$ in the ordinary formula for computation of the extension of the member. Here $E$ is Young's modulus of elasticity, and $A_x$ is the gross cross-sectional area.

The present paper extends this method to other shapes of perforation, originally described in reference [2]. The boundary of the hole has the parametric equation

$$x = p \cos \beta + r \cos 3\beta, \quad y = q \sin \beta - r \sin 3\beta.$$  \hspace{1cm} (1)

Equation 1 represents a closed curve having symmetry about the $z$-axis and about the $y$-axis, and which, for appropriate values of $p$, $q$, and $r$, does not cross itself. By adjustment of the values of $p$, $q$, and $r$, a variety of curves is obtained, including a good approximation to an ovaloid (a square with a semicircle erected on each of two opposite sides) and a good approximation to a square with rounded corners, as well as exact ellipses ($r=0$) of any eccentricity. The approximate ovaloid obtained by taking

$$p = 2.063, \quad q = 1.108, \quad r = -0.079,$$  \hspace{1cm} (2)

is shown compared to the actual ovaloid in figure 1. The approximate square obtained by taking

$$p = q = 1, \quad r = -0.14$$  \hspace{1cm} (3)

is shown in figure 2. The sides of the square are parallel to the axes of coordinates. By taking

$$p = q = 1, \quad r = 0.14,$$  \hspace{1cm} (4)

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1 Numbers in brackets indicate literature references at the end of this paper.
the same square, but with the diagonals parallel to the axes of coordinates, is obtained. The radius of curvature at the midpoint of the fillet is about 0.086 times the length of the side of the square.

It is shown in [1, eq 2] that for a member having perforations spaced 2l apart along the x-axis (parallel to the load), the axial rigidity factor \( K \) is given by

\[
\frac{1}{K} = 1 + \frac{E}{P l} \int u_x dy dz,
\]

in which \( u_x \) is the \( x \)-component of the displacement of the boundary of the hole, and the surface integral is extended over half the boundary of the perforation. The values of the load \( P \) and the displacement \( u_x \) may be assumed, for sufficiently large values of \( l \) and of \( A_x \) compared to the dimensions of the perforation, to be the same as in an infinite plate under uniform stress \( S_x \) disturbed by a single perforation.

For a member having circular holes, and of gross cross-sectional area \( A_x \) and net cross-sectional area \( A_n \),

\[
P = S_x A_x C(n) = S_x A_x \left(1 - \frac{1}{2n^2} - \frac{1}{2n^4}\right),
\]

where \( S_x \) is the mean stress on the gross cross-sectional area and \( n = A_x/(A_x - A_n) \). Equation 6 is derived in [1]. \( C(n) \) may be considered to be a correction factor arising from the finite area of the member. It has been shown [3] that \( C(n) \) is nearly independent of eccentricity in case of an elliptical hole, and since it is nearly unity in any practical case, eq 6 may be used without serious error for a hole of any shape.

The displacement \( u_x \) may be calculated from the stress function given in [2] or by the method of Muschelisvili.

\[ \text{II. Displacements} \]

1. Displacement-function method

This method \(^4\) presupposes a knowledge of the Airy stress-function, \( \phi \). The form of \( \phi \) is exhibited in [2] in terms of curvilinear coordinates \( (\alpha, \beta) \) such that eq 1 of the boundary of the hole reduces to \( \alpha = \alpha_0 \). The parameters \( p, q, \) and \( r \) of [2], which, together with \( \alpha_0 \), define the shape of the hole, are related to the \( p, q, \) and \( r \) of eq 1 by

\[
e^{\alpha_0} = \frac{p + q}{2}, \quad a b e^{-\alpha_0} = \frac{p - q}{2}, \quad a c e^{-3\alpha_0} = r.
\]

\(^4\) See p. 130 of [3].
where $\mu$ is Poisson’s ratio. In the present case the only values of $u$ required are $u_p$, those on the boundary of the perforation. On this (free) boundary, $\partial \phi/\partial z$ is constant, and hence zero, since $\phi$ is even in $x$ and in $y$. Hence from eq 9, 

$$Eu_p = \frac{\partial \psi}{\partial y} (\alpha = \alpha_0).$$

(10)

According to [2, eq 18],

$$\phi = C_1 \phi_1 + C_2 \phi_2 + C_3 \phi_3 + C_4 \phi_4 + C_5 \phi_5 + C_6 \phi_6 + C_7 \phi_7 + C_8 \phi_8,$$

of which $\partial \phi/\partial z$ is the imaginary part, so that from eq 11

$$\frac{\partial \psi}{\partial y} = C_1 (8e^a \cos \beta + 8abe^{-a} \cos \beta - 4x)$$

$$+ 4C_2 e^{-a} \cos \beta - Dy + E'.$$

The constants $D$ and $E$ are obtained by substituting for $\partial \psi/\partial y$ in eq 9, which gives

$$Eu = C_1 (8e^a \cos \beta + 8abe^{-a} \cos \beta - 4x)$$

$$+ 4C_2 e^{-a} \cos \beta - Dy + E' - (1 + \nu) \frac{\partial \phi}{\partial z}.$$

For $x=0 (\beta = \pm \pi/2)$, $u=0$ for all $y$, and $\partial \phi/\partial z = 0$ by symmetry. Therefore, $D = E' = 0,$ and

$$\frac{\partial \psi}{\partial y} = C_1 (8e^a \cos \beta + 8abe^{-a} \cos \beta - 4x)$$

$$+ 4C_2 e^{-a} \cos \beta.$$

(12)

Substitution of eq 12, with $C_1$ and $C_2$ replaced by their values from [2, eq 23] into eq 10 gives

$$\frac{Eu_p}{S_x} = -x_0 + 2e^{ao} \cos \beta$$

$$+ 4C_2 e^{-a} \cos \beta + \frac{1}{1 - a e^{-\omega} (2 - a e^{-\omega} - a^2 b e^{-\omega})},$$

where $x_0$ is the value of $x$ on the boundary $\alpha = \alpha_0$, which, together with eq 7, gives

$$\frac{Eu_p}{S_x} = -x_0 + 2(p + q)^2 - 2pr$$

$$+ \frac{1}{p + q - 2r} \cos \beta.$$

(13)

2. Mushelisvili’s method

This method, which is treated in detail by Sokolnikoff [4], expresses the stresses, displacements, and the boundary conditions in terms of two analytic functions, $\phi$ and $\psi$ (not the same as those of the preceding section), of the complex variable $z=x+i y$. The functions $\phi$ and $\psi$, which satisfy the problem of [2], have been given by Morkovin [5], who uses the notation

$$s = (p + q)/2, \ t = (p - q)/2, \ r = r,'$$

(14)

and

$$\zeta = e^{a - \omega} + i \beta,$$

(15)

so that the function

$$z = \omega(\zeta) = s \zeta + t/\zeta + r/\zeta^3$$

(16)

maps the boundary $\alpha = \alpha_0$ onto the unit circle $|\zeta|=1$. The value of $\zeta$ on the unit circle is denoted by $\sigma$; thus $\overline{\sigma}$, the conjugate of $\sigma$, is $1/\sigma$. The notation

$$\phi(z) = \phi(\omega(\zeta)) = \phi(\zeta),$$

is used, and similarly for $\psi$. Rigidity of Perforated Members
The functions $\phi$ and $\psi$ as found by Morkovin are
\[ \phi(\xi) = sB\xi + \frac{a_1}{\xi} - \frac{Br}{\xi}, \]
\[ \psi(\xi) = -\left( \frac{s}{\xi} + t_2 + r\xi \right) \left( \frac{sB\xi^2 - a_1\xi^2 + 3rB}{s\xi^3 - r\xi^3 - 3r} \right) - \frac{sB}{\xi}, \]
\[ + s(B' + iT_x)\xi - a_1\xi + B\xi^3 + Bt \left( 1 + \frac{r}{s} \right) \xi, \]
where
\[ a_1 = Bt \left( \frac{r + s}{s - r} \right) + B' \left( \frac{s^2}{r - s} \right) + \frac{2sB}{r + s}, \]
and
\[ 4B = S_x + S_y, \quad 2B' = S_x - S_y. \]
In the present case, $S_x = T_{xy} = 0$, giving
\[ \phi(\xi) = \frac{S}{4} \left( \frac{s \xi + a - r \xi}{s \xi} \right), \quad (17) \]
\[ \psi(\xi) = \frac{S}{4} \left[ -\left( \frac{s}{\xi} + t_2 + r\xi \right) \left( \frac{s^2 \xi^4 - a_1 \xi^2 + 3r^2}{s^3 \xi^3 - r\xi^3 - 3r} \right) - \frac{2s\xi}{s - r} + ar + r^2 + t \left( 1 + \frac{r}{s} \right) \xi \right], \quad (18) \]
where
\[ a = \frac{4a_1}{S_x} - \frac{t(r + s) - 2s^2}{r - s}. \]

### III. Axial Rigidity

#### 1. General

Substitution of $P$ from eq 6 and $u_x$ from eq 13 into eq 5 gives
\[ \frac{1}{K} = 1 + \frac{t}{A_x C(n)} \left[ - \int xdy + 2 \left( \frac{p + q}{p + q - 2r} \right)^2 \left( \frac{p + q - 2r}{s^2 - 2s^2 - 2r^2} \right) \int \cos \beta dy \right], \quad (21) \]
where $t$, the thickness of the plate in which the perforation occurs, is $\int dx$. The two line integrals in eq 21, which are to be taken along half the boundary $\alpha = \alpha_0$, are obtained from eq 1, as
\[ \int xdy = \frac{1}{2} A_0 = \pi (pq - 3r^2), \quad \int \cos \beta dy = \frac{\pi q}{2}, \]
where $A_0$ is the area of the perforation.

Equation 21 becomes
\[ \frac{1}{K} = 1 + \frac{V_0}{V_x C(n)} \frac{q(p + q)(p + 2q)}{(p + q - 2r)(pq - 3r^2)}, \quad (22) \]
where $V_0 = A_0 t$ is the volume of the perforation and $V_x = 2A_0 t$ is the gross volume of one bay of the member.

For an ellipse, $p = a$, $q = b$, $r = 0$, where $a$ and $b$ are the semiaxes ($a$ parallel to the load) and eq 22 reduces to
\[ \frac{1}{K} = 1 + \frac{1 + \frac{2b}{a} V_0}{C(n) V_x}, \]
which is the same as [1, eq 48].

It remains to compute the coefficient of $V_0/V_x$ in eq 22 for the various cases of interest.

#### 2. Ovaloid; load parallel to long axis

Here $p = 2.063$, $q = 1.108$, $r = -0.079$ (eq 2). Equation 22 becomes
\[ \frac{1}{K} = 1 + \frac{2.048 V_0}{C(n) V_x}, \quad (23) \]
In practical numerical cases, eq 23 gives substantially the same results as [1, eq 13], derived.
by a more approximate method. However [1, eq. 13] may be used for an ovaloid of which the rectangular base is not square whereas eq 23 applies only to the shape shown in figure 1.

3. Ovaloid; load parallel to short axis

In this case, \( p = 1.108 \), \( q = 2.063 \), \( r = 0.079 \). Equal 23 becomes

\[
\frac{1}{K} = 1 + \frac{4.968 V_o}{C(n) V_g}
\]

(24)

This case and the following cases are not covered by previous work.

4. Square with Rounded Corners

For the approximate square of eq 3 and figure 2, \( p = 1 \), \( q = 1 \), \( r = -0.14 \). Eq. 22 becomes

\[
\frac{1}{K} = 1 + \frac{2.989 V_o}{C(n) V_g}
\]

(25)

for load parallel to the side of the square.

For load parallel to the diagonal of the square (eq 4) \( p = 1 \), \( r = 0.14 \), and eq 22 becomes

\[
\frac{1}{K} = 1 + \frac{3.596 V_o}{C(n) V_g}
\]

(26)

IV. Summary

Approximate formulas have been developed for the computation of the axial rigidity of a long tension or compression member containing a plate of constant thickness uniformly perforated with a series of similar holes of various shapes. The axial rigidity factor \( K \) is defined so that \( KE_A g \) is the rigidity that should be used in place of \( E_A g \) in the ordinary formula for computation of the extension of the extension of the member.

In many cases of interest the axial rigidity factor \( K \) is given by an equation of the form

\[
\frac{1}{K} = 1 + \frac{f}{C(n)} \frac{V_o}{V_g}
\]

(27)

where

\[ n = A_g/(A_g - A_n) \]

\( A_g \) being the gross and \( A_n \) the net cross-sectional area of the member.

V. References


WASHINGTON, May 21, 1946.