COIL ARRANGEMENTS FOR PRODUCING A UNIFORM MAGNETIC FIELD

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ABSTRACT

A simple method is described for constructing coils which will give a magnetic field that is nearly uniform in magnitude and direction throughout a long cylindrical volume. A study is made of three types of coils and in each case formulas permitting the computation of the field are developed. Data are given showing the departure of the field from uniformity in specific cases and a procedure is outlined for the design of such coils.

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I. INTRODUCTION

The application of electron beams to physical measurements often makes it desirable that the beam shall not be deflected by the earth’s magnetic field. In cases where the beam direction cannot be made coincident with the direction of the local magnetic field, the neutralization of the component of the local field perpendicular to the direction of the beam is required. This is especially the case with low-voltage beams, which suffer relatively large deflection even in weak magnetic fields. In order that this neutralization may be accomplished, it is necessary to set up a magnetic field which will be quite uniform in magnitude and direction throughout the long, narrow cylinder which comprises the path of the beam and in such a direction as to compensate the local field tending to deflect the beam.

Such a field may be set up between a pair of plane rectangular coils whose long dimension is parallel to the path of the beam and whose length is somewhat greater than the distance over which compensation is desired. A modification of such a coil system which resulted in a considerable improvement in the uniformity of the
compensating field has been described by Beyerle\textsuperscript{1} and has been briefly discussed by the author,\textsuperscript{2} who has used it to compensate the horizontal component of the local magnetic field along the electron-beam path of a cathode-ray oscillograph.

In the present paper it is proposed to give a more complete analysis of the field near the axis of such coils and to show that certain further modifications in the shape of the compensating coils will result in a considerable gain in uniformity of the field.

II. MAGNETIC FIELD OF COMPENSATING COILS

1. GENERAL CONSIDERATIONS

The magnetic field produced at any point in space by an element of a linear conductor carrying current is given by Ampere's law which is conveniently expressed in vector notation as

\[ dH/I = [dl, \rho]/\rho^3 \]

where \( dl \) is the vector circuit element whose sense is that of the current \( I \), and \( \rho \) is the vector distance from the circuit element to the point at which the field is to be determined.

For the straight section of wire of length \((a, b)\) shown in figure 1 the field at a point \( P \), distant \( \rho_o \) from the wire, is given by

\[ H/I = i(o) + j(o) + k \int_{-b}^{a} \frac{\rho_0 dl}{(\rho_0^2 + P^2)^{3/2}} \]

\[ = k \frac{1}{\rho_0} \left[ \frac{a}{(\rho_0^2 + a^2)^{3/2}} + \frac{b}{(\rho_0^2 + b^2)^{3/2}} \right] \] (1)

where \( i, j, k \) are unit vectors along the coordinate axes. The field \( H \) at the point \( P \) is perpendicular to the plane of the paper and toward the reader.

In the discussion which follows, a number of coil shapes, differing from the coils shown in figure 2 solely in the shape of the short end portions, will be considered. For convenience, these several shapes will each be designated by the shape of its short end portion, or head. For example, the plane rectangular coils of figure 2 will be referred to as a straight-head coil pair. Although the coils may, and usually will, consist of a number of turns forming a compact bundle, their fields in the region considered will be represented with all necessary accuracy by the formulas here developed for single turns formed of linear elements.

2. STRAIGHT-HEAD COILS

Since the straight-head coils of figure 2 are made up entirely of straight sections an expression for the field at any point can be immediately written from equation 1, as the resultant of the field due to each of the 8 straight sections which form the coil pair.

We will use as a reference line the vertical axis equidistant from each of the 4 vertical coil sides. The field at a point distant \((x_1, y_1)\)

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\textsuperscript{1} Beyerle, Arch. f. Elektrot., 25, 298 (1931).
\textsuperscript{2} F. K. Harris, B8 J. Research 12, 87 (1934).
Figure 1.

Figure 2.—Straight-head coils.
from this axis and distant \((a, b)\) from the upper and lower coil heads is
given by the expressions

\[
\frac{RH_x}{I} = -\frac{K}{A} \left[ \frac{c}{\sqrt{(c^2 + A)}} + \frac{d}{\sqrt{(d^2 + A)}} \right] + \frac{L}{C} \left[ \frac{c}{\sqrt{(c^2 + C)}} + \frac{d}{\sqrt{(d^2 + C)}} \right]
\]

\[
\frac{RH_y}{I} = \frac{M}{A} \left[ \frac{c}{\sqrt{(c^2 + A)}} + \frac{d}{\sqrt{(d^2 + A)}} \right] + \frac{N}{B} \left[ \frac{c}{\sqrt{(c^2 + B)}} + \frac{d}{\sqrt{(d^2 + B)}} \right]
\]

\[
\frac{RH_z}{I} = -\frac{K}{e^2 + K^2} \left[ \frac{M}{\sqrt{(c^2 + A)}} + \frac{N}{\sqrt{(c^2 + B)}} \right] + \frac{K}{d^2 + K^2} \left[ \frac{M}{\sqrt{(d^2 + A)}} \right]
\]

\[
\frac{RH_y}{I} = M \left[ \frac{c}{\sqrt{(c^2 + A)}} + \frac{d}{\sqrt{(d^2 + A)}} \right] + N \left[ \frac{c}{\sqrt{(c^2 + B)}} + \frac{d}{\sqrt{(d^2 + B)}} \right]
\]

\[
\frac{RH_z}{I} = -\frac{K}{e^2 + K^2} \left[ \frac{M}{\sqrt{(c^2 + A)}} + \frac{N}{\sqrt{(c^2 + B)}} \right] + \frac{K}{d^2 + K^2} \left[ \frac{M}{\sqrt{(d^2 + A)}} \right]
\]

\[
\frac{RH_y}{I} = M \left[ \frac{c}{\sqrt{(c^2 + A)}} + \frac{d}{\sqrt{(d^2 + A)}} \right] + N \left[ \frac{c}{\sqrt{(c^2 + B)}} + \frac{d}{\sqrt{(d^2 + B)}} \right]
\]

Here \(c = a/R; \quad d = b/R; \quad A = h - s - t; \quad B = h + s - t; \quad C = h - s + t; \quad D = h + s + t; \quad K = \cos \phi - g; \quad L = \cos \phi + g; \quad M = \sin \phi - f; \quad N = \sin \phi + f,
\]

where \(f = x_1/R; \quad g = y_1/R; \quad h = 1 + f^2 + g^2; \quad s = 2f \sin \phi; \quad t = 2g \cos \phi.
\]

\(R\) is the distance from the axis to the vertical coil sides and \(\phi\) is half of the angular opening of the coil sides as seen from the axis (see fig. 2).

For points on the axis the \(x\)-component and \(z\)-component of the field are zero and the \(y\)-component reduces to the expression

\[
\frac{RH_y}{I} = 4 \sin \phi \left[ \frac{c}{\sqrt{(1 + c^2)}} \left( 1 + \frac{1}{c^2 + \cos^2 \phi} \right) + \frac{d}{\sqrt{(1 + d^2)}} \left( 1 + \frac{1}{d^2 + \cos^2 \phi} \right) \right]
\]

3. CIRCULAR-HEAD COILS (FIELD AT AXIS)

If, as suggested by Beyerle, the coil ends be bent into arcs of a circle as shown in figure 3 we must solve Ampere’s equation for a circuit section which forms the arc of a circle.

For points on the axis we have for the field resulting from the upper right-hand circuit section

\[
\frac{H}{I} = -i \int_{\phi_1}^{\phi} aR \sin \phi d\phi + j \int_{\phi_1}^{\phi} aR \cos \phi d\phi - k \int_{\phi_1}^{\phi} R^2 d\phi
\]

\[
= i(\phi) + j2aR \sin \phi_1 - k2R^2 \phi_1
\]

(4)
Figure 3.—Circular-head coils.
By using equations 1 and 4 we can write an expression for the field at the axis of the coil pair

$$\frac{R H_y}{I} = 4 \sin \phi \left[ \frac{c}{\sqrt{1 + c^2}} \left( \frac{2 + c^2}{1 + c^2} \right) + \frac{d}{\sqrt{1 + d^2}} \left( \frac{2 + d^2}{1 + d^2} \right) \right]$$

(5)

The $x$-component and $z$-component of the field are again zero.

For points not on the axis of the coils the field can be expressed in terms of infinite series.\(^3\) That expression is not, however, convenient for computing, and an approximation has been developed for evaluating the field in this case. This approximation requires the use of

\(^3\) See appendix.
material in the section immediately below, and will therefore be discussed later.

4. BENT-HEAD COILS

A coil pair of the type shown in figure 4 can be wound as rectangular coils and the ends then bent back to form the rectangular heads shown in the figure. For our purposes it will be necessary to consider only coils shaped so that \( E \gg R \), since for smaller values of \( E \) the field set up by the coils is increasingly less uniform.

The expression for the field set up by the coils is

\[
\frac{RH_x}{I} = -K \frac{c}{A} \left[ \frac{d}{\sqrt{(e^2 + A)}} + \frac{d}{\sqrt{(e^2 + D)}} \right] + \frac{K}{B} \left[ \frac{c}{\sqrt{(e^2 + B)}} + \frac{d}{\sqrt{(e^2 + D)}} \right] + \frac{L}{C} \left[ \frac{c}{\sqrt{(e^2 + C)}} + \frac{d}{\sqrt{(e^2 + D)}} \right] + \frac{L}{D} \left[ \frac{c}{\sqrt{(e^2 + D)}} + \frac{d}{\sqrt{(e^2 + D)}} \right] + \frac{e^2 + M^2}{\sqrt{(e^2 + P)}} - \frac{e^2 + N^2}{\sqrt{(e^2 + Q)}} + \frac{d}{\sqrt{(e^2 + D)}} - \frac{d^2 + M^2}{\sqrt{(d^2 + P)}} - \frac{d^2 + N^2}{\sqrt{(d^2 + Q)}} + \frac{c}{\sqrt{(e^2 + C)}} + \frac{d}{\sqrt{(d^2 + D)}} \right] \]

\[
\frac{RH_y}{I} = M \left[ \frac{c}{\sqrt{(e^2 + A)}} + \frac{d}{\sqrt{(d^2 + A)}} \right] + N \left[ \frac{c}{\sqrt{(e^2 + B)}} + \frac{d}{\sqrt{(d^2 + B)}} \right] + \frac{e^2 + M^2}{\sqrt{(e^2 + C)}} + \frac{d^2 + M^2}{\sqrt{(d^2 + C)}} + \frac{e^2 + N^2}{\sqrt{(e^2 + D)}} + \frac{d^2 + N^2}{\sqrt{(d^2 + D)}} \right] \]

\[
\frac{RH_z}{I} = -M \left[ \frac{c}{\sqrt{(e^2 + P)}} + \frac{d}{\sqrt{(d^2 + P)}} \right] - N \left[ \frac{c}{\sqrt{(e^2 + Q)}} + \frac{d}{\sqrt{(d^2 + Q)}} \right] - M \left[ \frac{c}{\sqrt{(e^2 + R)}} + \frac{d}{\sqrt{(d^2 + R)}} \right] - N \left[ \frac{c}{\sqrt{(e^2 + S)}} + \frac{d}{\sqrt{(d^2 + S)}} \right] \]

\[
\frac{M}{c^2 + M^2} \left[ \frac{F}{\sqrt{(e^2 + P)}} - \frac{G}{\sqrt{(e^2 + Q)}} + \frac{L}{\sqrt{(e^2 + D)}} - \frac{K}{\sqrt{(e^2 + A)}} \right] - \frac{N}{c^2 + N^2} \left[ \frac{F}{\sqrt{(e^2 + R)}} - \frac{G}{\sqrt{(e^2 + D)}} + \frac{L}{\sqrt{(d^2 + A)}} \right] \]
Here, in addition to the symbols already defined, we have

\[\begin{align*}
1 - g + \epsilon &= F; \\
1 + g + \epsilon &= G; \\
\sin^2 \phi + h - 2g - s + 2\epsilon(1 - g + \epsilon/2) &= P; \\
\sin^2 \phi + h + 2g - s + 2\epsilon(1 + g + \epsilon/2) &= Q; \\
\sin^2 \phi + h - 2g + s + 2\epsilon(1 - g + \epsilon/2) &= R; \\
\sin^2 \phi + h + 2g + s + 2\epsilon(1 + g + \epsilon/2) &= S;
\end{align*}\]

where (referring to fig. 4) \(\epsilon = \frac{E}{R} - 1\)

At the axis equations 6 reduce to

\[\begin{align*}
H_x &= 0 \\
\frac{RH_y}{I} &= 4 \sin \phi \left[ \frac{c}{\sqrt{(1 + c^2)}} \left( 1 + \frac{1}{\sqrt{(1 + c^2)} \sqrt{(1 + \epsilon)^2 + c^2 + \sin^2 \phi} \left( 1 + \frac{2\epsilon + \epsilon^2}{1 + c^2} \right)} \right) \right. \\
&\left. + \frac{d}{\sqrt{(1 + d^2)}} \left( 1 + \frac{1}{\sqrt{(1 + d^2)} \sqrt{(1 + \epsilon)^2 + d^2 + \sin^2 \phi} \left( 1 + \frac{2\epsilon + \epsilon^2}{1 + d^2} \right)} \right) \right] \\
H_z &= 0
\end{align*}\]

5. CIRCULAR-HEAD COILS (APPROXIMATION)

If, for the coils of figure 4, we set \(E = R\ (\epsilon = 0)\), and if at the same time, we consider the coil shape shown in figure 2 we will have two coil shapes (figs. 2 and 4) between which the circular-head coil pair of figure 3 is intermediate. The long sides of the coils are coincident in the three cases and the end effect for the circular-head coils is intermediate between the end effects in the boundary cases of figures 2 and 4.

The field at the axis of the bent-head coils for the boundary case \((\epsilon = 0)\) may be written immediately from equation 7.

\[\begin{align*}
\frac{RH_y}{I} &= 4 \sin \phi \left[ \frac{c}{\sqrt{(1 + c^2)}} \left( 1 + \frac{1}{\sqrt{(1 + c^2)} \sqrt{(1 + \epsilon)^2 + c^2 + \sin^2 \phi} \left( 1 + \frac{2\epsilon + \epsilon^2}{1 + c^2} \right)} \right) \right. \\
&\left. + \frac{d}{\sqrt{(1 + d^2)}} \left( 1 + \frac{1}{\sqrt{(1 + d^2)} \sqrt{(1 + \epsilon)^2 + d^2 + \sin^2 \phi} \left( 1 + \frac{2\epsilon + \epsilon^2}{1 + d^2} \right)} \right) \right]
\]

Let \(H_1(x,y,z)\) denote any particular component of field at the point \(P(x,y,z)\) for the straight-head coils, let \(H_2(x,y,z)\) denote the same component of field at \(P\) for the circular-head coils, and \(H_3(x,y,z)\) that for the bent-head coils with \(\epsilon = 0\). Now \(H_2(x,y,z)\) cannot conveniently be computed (except for points \((o,o,z)\) on the axis) but is intermediate between \(H_1(x,y,z)\) and \(H_3(x,y,z)\), which can be computed from equations 2 and 6, respectively.

The following method of approximating to any component of \(H_2(x,y,z)\) suggests itself and has been adopted. If we define a function \(K(x,y,z)\) by

\[K(x,y,z) = \frac{H_2(x,y,z) - H_3(x,y,z)}{H_1(x,y,z) - H_3(x,y,z)}\]
then

\[ H_2(x, y, z) = H_3(x, y, z) + K(x, y, z)[H_1(x, y, z) - H_3(x, y, z)] \]  

(9a)

For points on the axis equation 9a becomes

\[ H_2(o, o, z) = H_3(o, o, z) + K(o, o, z)[H_1(o, o, z) - H_3(o, o, z)] \]  

(9b)

It will be noted that \( H_1(o, o, z) \), \( H_2(o, o, z) \), and \( H_3(o, o, z) \) can be computed from equations 3, 5, and 8, respectively. Hence \( K(o, o, z) \) can be evaluated at the axis for the \( y \)-component of field. The \( K(o, o, z) \) thus computed is less than unity and varies rather slowly with \( z \).

The \( y \)-component of field at points near the axis (\( x \) and \( y \) small) can be closely approximated by using \( K(o, o, z) \) in place of \( K(x, y, z) \) in equation 9a thus,

\[ H_2(x, y, z) \approx H_3(x, y, z) + K(o, o, z)[H_1(x, y, z) - H_3(x, y, z)] \]  

(10)

\( H_1(x, y, z) \) being computed from equation 2 and \( H_3(x, y, z) \) from equation 6 with \( \varepsilon = 0 \). That this approximation is good for small values of \( x \) and \( y \) is evident from the following considerations. The difference \( H_1 - H_3 \) is small compared to \( H_3 \). Near the axis \( K(x, y, z) \) differs by only a small quantity from \( K(o, o, z) \) since it is a continuous function of \( (x, y, z) \). Hence only terms of higher order than \( H_1 - H_3 \) are neglected.

The \( x \)-component and \( z \)-component of field at the point \( (x, y, z) \) may also be approximated from equation 10 if \( K(x, y, z) \) be assumed to have for them the value \( K(o, o, z) \) computed at the axis for the \( y \)-component. This assumption may be justified as yielding a sufficiently good approximation for practical purposes by the check obtained between computed and experimental values for the case discussed in the following section.

### III. FIELD UNIFORMITY

Computations have been carried out for the particular case of a pair of circular-head coils such as that shown in figure 3, for which the length \( ^5 \) is \( 9R \) and \( \phi = \pi/4 \).

As a check on the computations and on the approximation just outlined, a pair of coils was set up, and by means of a search coil, the field over a considerable distance around the axis was mapped.

The set-up used for this purpose is shown schematically in figure 5. Current at a frequency of 1,000 cycles per second was sent through the coils and the emf induced in the search coil was balanced by means of the mutual inductor. In order to obtain silence in the phones at the balance point, it was necessary to insert a small quadrature voltage from the drop across a resistance. The field strength for any position of the search coil was computed from the reading of the mutual inductor. The reproducibility of the results was limited to about 1 percent by the precision with which the coordinates and plane of the search coil could be located.

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4 In the case discussed below, for which computations have been carried out, the value of \( K(o, o, z) \) was found to vary from about 0.2 near the coil heads to 0.3 at the center.

4 This ratio was used in the compensating coils set up for a cathode-ray oscillograph (see footnote 2) and will show in general the field distribution within any coil pair of this type. The ratio of length to radius for any specific case would be expected to depend on the particular problem in compensation involved.
Figure 5.—Diagram of connections for experimental determination of field.

C, search coil; M, variable mutual inductor; S, variable shunt; P, telephone receiver.
That the approximation outlined in section II-5 is adequate, appears from figures 6, 7, and 8 in which are shown the results of the comparison between computed and experimental field values within the circular-head coils of figure 3.

Figure 6 is a contour map of the y-component of the field for a horizontal plane midway between the coil heads. The contour lines are drawn from computed values, with the value at the center arbitrarily fixed at 100. The solid circles represent experimental values of the field. Each plotted point represents the average of 4 deter-

![Contour Map](image_url)

**Figure 6.** — *y*-component of field in transverse median plane of circular-head coils. Numbers attached to contour lines and to points represent computed and experimental field values, respectively.

minations, 1 in each quadrant. The spread of these individual determinations amounts to a percent or less.

The x-component of the field in the same plane is shown in figure 7. Here the field is zero at the center and reverses in direction on passing from one quadrant to the next. The z-component of the field is zero throughout this plane.

Figure 8 shows the three components of the field plotted against vertical distance below the upper coil head for the entire length of the coils. The y-component is plotted along the axis. The x-component and z-component are both zero along the axis and so are plotted for values along vertical lines at the positions indicated by the coordinates at the top of each plot.
The left-hand side of figure 9 shows the uniformity of the resultant field. The solid lines are the traces of figures of revolution within which the resultant field does not depart from the value at the axis and midway between the coil heads by more than the percentage stated on each contour line. For example, within the inner spindle-shaped figure extending over about 0.4 of the coil length, the $y$-component of the field does not vary by more than 1 percent from the value which it has at the point $(0, 0.5L)$ and the $x$-component and $z$-component of the field are each less than 1 percent of the $y$-component. If a field variation of 3 percent is permitted over the cylinder within which compensation is desired 0.85 of the coil length may be used.

The constriction in the upper portion of the figures is a result of the saddle-back shape of the $y$-component of the field along the axis as shown in figure 8. An equal constriction exists for the lower end of the coil, which is not shown in figure 9.

The right-hand side of figure 9 shows the uniformity of the field for the coil pair with rectangular heads bent back so as to be just tangent to the arc used in the circular-head coils. This is the coil shape shown in figure 4 with $\epsilon=0$ and is the outer boundary case which was used in the approximation of section II-5. It will be seen that

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**Figure 7.**—$x$-component of field in transverse median plane of circular-head coils. Numbers attached to contour lines and to points represent computed and experimental field values, respectively.

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<table>
<thead>
<tr>
<th>$x$-Component of Field</th>
<th>$y$-Component of Field</th>
<th>$z$-Component of Field</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\pm 3$</td>
<td>$\pm 1$</td>
<td>$\pm 0.5$</td>
</tr>
</tbody>
</table>

---

[Diagram of field distribution]
the coil length over which the field variation is less than 2 percent is increased, by the change in shape, from 0.6 of the coil length to 0.8 of the coil length. The constriction in the upper part of the figure has been considerably reduced from the circular-head case but at the cost of shortening the figure somewhat.

Uniformity of the field within the coils can be improved by a further modification of the coil head. If the return lead in the head
Figure 9.—Total field variation near axis of circular-head coils (left) and bent-head coils (right).
Numbers attached to contour lines represent field values.
is moved back farther from the axis, i.e., if $\epsilon$ is increased, the constriction shown in figure 9 is decreased. However as this constriction decreases it moves in toward the center of the coil to decrease the total length over which the field is effectively uniform. The exact point at which a gain in field uniformity, resulting from an increase of $\epsilon$, ceases must be determined from the conditions imposed by any particular problem in compensation. This is shown in figure 10. In this figure

![Diagram](https://example.com/diagram.png)

**Figure 10.—Variation of $y$-component of field along axis.**

the $y$-component of the field at the axis is plotted against vertical distance along the coil for a variety of coil heads.

It will be seen that if uniformity to within 1 percent is desired the coil pair corresponding to $\epsilon=0.25$ has nearly the optimum shape. In this case, compensation of a uniform local magnetic field to within 1 percent could be secured over a circular cylinder whose length is approximately 0.7 of the coil length and whose diameter is approximately 0.05 of the distance between diagonally-opposite corners of the coil pair.
IV. CHOICE OF COIL SHAPE

It is apparent from the results given above that the coil shape described by Beyerle is not the shape best adapted for compensating the local magnetic field over a long cylindrical volume. A coil of the type designated above as bent-head is simpler to construct, since it can be wound on a rectangular form and the ends then be bent back a proper distance. Furthermore, for a coil which occupies the same space, i.e., with $\epsilon=0$, there is a distinct gain in the uniformity of the field which is set up close to the axis as compared with the field resulting from the circular-head coils. If a more uniform field is desired heads can be formed with $\epsilon>0$ by bending back the ends of the rectangular coils at an appropriate point.

![Graph - Variation of y-component of field along axis.](image-url)
There is also a gain in uniformity of field in bending back the coil ends as compared with rectangular coils having the same total length of wire. This is shown in figure 11. Here the field along the axis is shown for two cases: (1), for the straight-head coils shown in figure 2; and (2), for the same coils with the ends bent back as shown in figure 4 with \( \varepsilon = 0.25 \). It will be seen that the length over which the field is uniform to within 1 percent is 50 percent greater in the second case than in the first.

By the aid of equation 7 and of figures 9 and 10 it should be possible to design coils which, for a straight electron beam, will effectively neutralize the component of the local magnetic field perpendicular to the direction of the beam. When the diameter of the cylinder over which compensation is desired and the degree of uniformity needed for the compensating field are known, the diameter of the coils can be fixed from figure 9. The length of the coils can then be tentatively fixed from the length of the cylinder which must have a uniform field, and the shape of the coil heads from the degree of uniformity needed, by the aid of figure 10. Since, however, figures 9 and 10 have been constructed for coils whose ratio of length to radius is 9, it may be advisable when the coils are to have a ratio greatly different from that to check one's conclusions by computing by means of equation 7 the field distribution of a pair of bent-head coils having the desired ratio.

It must be remembered that the analysis given above was worked out for a specific problem in compensation and, except for the formulas given, does not claim generality. It is hoped that the data which are presented are sufficient to indicate the general characteristics of the field and to assist in the design of coils for other cases where magnetic compensation is needed over a long, narrow cylinder.

**APPENDIX**

The solution of the magnetic field in the case of circular-head coils is considerably more complicated for points not on the axis than the simple solution given in equation 5. Here we need to consider only the coil heads themselves. The field due to the straight sides of the coils is included in the cases already discussed and may be added directly to the field found for the coil heads.

We must first write an expression for the field at any point resulting from a current flowing through a circular arc. Using Ampere's law, as already stated, and referring to figure 12, we have, for the arc forming the upper right coil head,

\[
\frac{H}{I} = -i \int_{-\phi_1}^{\phi_1} \frac{aR \sin \phi d\phi}{[u^2 + a^2 + 2R(x_1 \sin \phi - y_1 \cos \phi)]^{3/2}} \\
+ j \int_{-\phi_1}^{\phi_1} \frac{aR \cos \phi d\phi}{[u^2 + a^2 + 2R(x_1 \sin \phi - y_1 \cos \phi)]^{3/2}} \\
+ k \int_{-\phi_1}^{\phi_1} \frac{(y_1 \cos \phi - x_1 \sin \phi - R)R d\phi}{[u^2 + a^2 + 2R(x_1 \sin \phi - y_1 \cos \phi)]^{3/2}}
\]

where \( u^2 = x_1^2 + y_1^2 + R^2 \).
Figure 12.—Curved end sections of circular-head coils.
On combining the fields resulting from the four arcs forming the coil heads and reverting to the coordinates and nomenclature previously defined we may express the total end effect as

\[
\frac{RH_x}{I} = -\int \frac{c \sin \phi d\phi}{[c^2 + h + 2(f \sin \phi - g \cos \phi)]^3} - \int \frac{d \sin \phi d\phi}{[d^2 + h + 2(f \sin \phi - g \cos \phi)]^3}
\]

\[
= \int \frac{c \cos \phi d\phi}{[c^2 + h + 2(f \sin \phi - g \cos \phi)]^3} + \int \frac{d \cos \phi d\phi}{[d^2 + h + 2(f \sin \phi - g \cos \phi)]^3}
\]

\[
\frac{RH_y}{I} = \int \frac{c \cos \phi d\phi}{[c^2 + h + 2(f \sin \phi - g \cos \phi)]^3} + \int \frac{d \cos \phi d\phi}{[d^2 + h + 2(f \sin \phi - g \cos \phi)]^3}
\]

\[
= \int \frac{(g \cos \phi - f \sin \phi - 1) d\phi}{[c^2 + h + 2(f \sin \phi - g \cos \phi)]^3} - \int \frac{(g \cos \phi - f \sin \phi - 1) d\phi}{[d^2 + h + 2(f \sin \phi - g \cos \phi)]^3}
\]

\[
+ \int \frac{(g \cos \phi - f \sin \phi + 1) d\phi}{[c^2 + h + 2(f \sin \phi - g \cos \phi)]^3} - \int \frac{(g \cos \phi - f \sin \phi + 1) d\phi}{[d^2 + h + 2(f \sin \phi - g \cos \phi)]^3}
\]

where all the integrals are taken between the limits \(-\phi_1\) and \(\phi_1\).

Since the approximation given above has been found satisfactory it will suffice here to work out only the y-component of the field. The x-component and z-component may be obtained by a procedure similar to that indicated below.

We will solve the integrals containing \(c\) and the solution for the integrals in \(d\) may be obtained by substitution. We have, then

\[
\int \frac{c \cos \phi d\phi}{[c^2 + h + 2(f \sin \phi - g \cos \phi)]^3} + \int \frac{c \cos \phi d\phi}{[c^2 + h + 2(f \sin \phi - g \cos \phi)]^3}
\]

If we let

\[
\frac{f}{\sqrt{h-1}} = \cos \alpha; \quad \frac{g}{\sqrt{h-1}} = \sin \alpha; \quad \beta = \varphi - \alpha;
\]

we have

\[
\frac{cf}{\sqrt{h-1}} \left[ \int \frac{\cos \beta d\beta}{[c^2 + h + 2\sqrt{(h-1)} \sin \beta]^3} + \int \frac{\cos \beta d\beta}{[c^2 + h + 2\sqrt{(h-1)} \sin \beta]^3} \right]
\]

\[
- \frac{cg}{\sqrt{h-1}} \left[ \int \frac{\sin \beta d\beta}{[c^2 + h + 2\sqrt{(h-1)} \sin \beta]^3} + \int \frac{\sin \beta d\beta}{[c^2 + h + 2\sqrt{(h-1)} \sin \beta]^3} \right].
\]

The first expression integrates immediately to

\[
\frac{cf}{h-1} \left[ \frac{1}{\sqrt{(c^2 + h - 2(f \sin \phi - g \cos \phi))}} - \frac{1}{\sqrt{(c^2 + h + 2(f \sin \phi - g \cos \phi))}} \right]_{\gamma_1}
\]
which yields on application of the limits

\[
\begin{align*}
\frac{c}{h-1} & \left[ \frac{1}{\sqrt{(c^2 + A)}} - \frac{1}{\sqrt{(c^2 + B)}} + \frac{1}{\sqrt{(c^2 + C)}} - \frac{1}{\sqrt{(c^2 + D)}} \right].
\end{align*}
\]

We may integrate the second expression as an infinite series by expanding into a power series an expression of the type

\[
\frac{\sin \beta}{p + q \sin \beta} = \sin \beta - \frac{3}{2} \frac{q}{p} \sin^2 \beta + 3 \frac{5}{2} \frac{(q/p)^2}{2} \sin^3 \beta - \ldots.
\]

The resulting expressions may now be combined with that for the field resulting from the straight sections of the coils obtained from equation 2. For the special case considered experimentally \((\phi = \pi/4)\) the complete expression for the \(y\)-component of the field is the following

\[
\begin{align*}
R_{H_0} & = \frac{M}{A} \left[ \frac{c}{\sqrt{(c^2 + A)}} + \frac{d}{\sqrt{(d^2 + A)}} \right] + \frac{N}{B} \left[ \frac{c}{\sqrt{(c^2 + B)}} + \frac{d}{\sqrt{(d^2 + B)}} \right] \\
& + \frac{M}{C} \left[ \frac{c}{\sqrt{(c^2 + C)}} + \frac{d}{\sqrt{(d^2 + C)}} \right] + \frac{N}{D} \left[ \frac{c}{\sqrt{(c^2 + D)}} + \frac{d}{\sqrt{(d^2 + D)}} \right] \\
& + \frac{c}{h-1} \left[ \frac{1}{\sqrt{(c^2 + A)}} - \frac{1}{\sqrt{(c^2 + B)}} + \frac{1}{\sqrt{(c^2 + C)}} - \frac{1}{\sqrt{(c^2 + D)}} \right] \\
& + \frac{d}{h-1} \left[ \frac{1}{\sqrt{(d^2 + A)}} - \frac{1}{\sqrt{(d^2 + B)}} + \frac{1}{\sqrt{(d^2 + C)}} - \frac{1}{\sqrt{(d^2 + D)}} \right] \\
& + 2 \sqrt{2} (2) \frac{c g^2}{(h-1)(c^2 + h)^3} \left[ 1 + \frac{5}{2} \frac{(h-1)}{(c^2 + h)^2} \frac{1}{2(h-1) + 2} \right] \\
& + \frac{63}{8} \frac{(h-1)^2}{(c^2 + h)^4} \frac{1}{4(h-1)^2} \frac{1}{2(h-1) + 2} \left[ g^2 + 3f^2 \right] + \frac{2}{3(h-1) + 3} \left[ g^2 + f^2 \right] \\
& + \frac{429}{16} \frac{(h-1)^3}{(c^2 + h)^6} \left[ g^2 + 10f^2 g^2 + 5f^4 \right] \\
& + \frac{3}{10(h-1)^2} \left[ g^2 + 3f^2 \right] + \frac{4}{5(h-1)} \left[ g^2 + f^2 \right] + \frac{16}{5} \left[ g^2 + f^2 \right] + \ldots.
\end{align*}
\]

A number of field values were computed with this formula and found to agree to within about 0.1 percent with the corresponding values computed by means of the approximation described earlier in the paper. These values ranged as far as 0.4\(R\) from the axis. It can therefore be stated that the approximation given in section II-5 is sufficiently accurate for computing the field at the small distances from the axis that are of particular importance.

\textit{Washington, June 25, 1934.}