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AN ABSOLUTE DETERMINATION OF THE AMPERE

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ABSTRACT

The current balance originally used by Rosa, Dorsey, and Miller in 1911 has been somewhat modified and used to determine the absolute value of the ampere. The value in absolute amperes of the current in the coils of the balance was determined from the measured constants of the coils and the electromagnetic force between them. The most important constant which could be directly measured was the ratio of the radii of the coils, which was measured by an electrical method. The values obtained on four sets of coils indicated that there was no error in the measurement of the ratio of the radii greater than three parts in a million.

The current through the balance was measured not only in absolute units but also in terms of the international units of electromotive force and resistance as maintained at the Bureau of Standards. The final result of measurements with four different coil combinations was:

1 B.S. International Ampere=0.999928 Absolute Ampere

The authors estimate that this result differs from the true value by less than 20 parts in a million.

This result agrees within experimental error with that obtained in 1911, indicating that the unit of current as realized at the Bureau of Standards by standard cells and standard resistances has not changed appreciably since then.

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I. INTRODUCTION

A determination of the absolute value of the international ampere as maintained at the Bureau of Standards has been completed. The method was that used by Rosa, Dorsey, and Miller in 1911, and a large part of their apparatus was employed. This investigation was undertaken to ascertain whether there had been an important change in the unit of current since 1911, and to determine the amount by which the present international unit of current differs from the absolute unit. The determination has been carried through with the idea of obtaining a reliable result at as early a date as possible.

The absolute determination of current had its inception in the work of Gauss.[1] The tangent galvanometer [2] provided the first

[1] Gauss, Pogg. Ann., vol. 28, p. 241, 1833.
[2] Pouillet, Pogg. Ann., vol. 42, p. 281, 1837

method [3] for an absolute current determination and this suggested to Weber [4] a unified system of absolute electrical units. The current balance was first used by Cazin [5] and later made an instrument of precision by Rayleigh.[6] In only two previous determinations [7, 8] did the results have an accuracy [9] greater than one part in 10,000. This report gives the results of a third precise determination.

1. IMPORTANCE OF A NEW ABSOLUTE DETERMINATION OF THE AMPERE

The International Committee of Weights and Measures decided in 1928 to adopt the absolute electrical units and to base the value of working standards on the results of absolute determinations. The national laboratories were requested to complete as soon as possible the absolute determinations which they had planned. The results herein reported are a part of the contribution of the Bureau of Standards.

This determination also gives information concerning the constancy of our own standards. In the absolute determination of the ampere which was made at the Bureau of Standards in 1911, the results were expressed in terms of the units of resistance and electromotive force of this Bureau. These units have been maintained by means of groups of standard resistances and standard cells. Individual members of each group have changed relative to the mean by appreciable amounts so that some doubt exists as to the accuracy of maintenance of these units of resistance and electromotive force.[10] Hence an absolute determination gives information concerning the changes which may have taken place in the electrical units.

2. RELATION OF THE PRESENT WORK TO THAT OF 1911

The current balance used in 1911 was chosen for this investigation partly because a large part of the apparatus was already available. Another reason was the desirability of having the Bureau of Standards method sufficiently different from that in use at the National Physical Laboratory so that the same systematic errors will not enter in the determinations at the two laboratories. Only these two types of current balances have been so perfected that precise results can be obtained by their use. Extensive development work will be necessary before determinations by any other method [11] will have the same accuracy as those by these current balances.

In the preliminary determination of the present series, made in 1927, all of the available apparatus which was used in 1911 was again used although it was set up in a different laboratory. Later the apparatus was modified in many details. The results of all the measurements that have been made are herein reported.

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[3] Weber, Deutscher Naturforscher Verein, Berichte, p. 154, 1841.
[4] Weber, Ann. der Phy. u. Chem., vol. 82, p. 337, 1851. Translated in Phil. Mag., series 4, vol. 22, p. 226 ff. and p. 261 ff. 1861.
[5] Cazin, Ann de Chimie et de Physique, series 4, vol. 1, p. 257, 1864.
[6] Rayleigh, B. A. Report p. 445, 1882. Proc. Camb. Phil. Soc., vol. 5, p. 50. Phil. Trans. 175, p. 411, 1884.
[7] Ayrton, Mather and Smith, Roy. Soc., Phil. Trans. vol. 207A, p. 463, 1908.
[8] Rosa, Dorsey, and Miller, B.S. Bull., vol. 8, p. 269, 1911.
[9] The most recent determination was that of Shaw, but the accuracy of the results obtained did not exceed one part in 10,000. Roy. Soc., Phil. Trans. vol. 214A, p. 147, 1914.
[10] H. L. Curtis, The Establishment and Maintenance of the Electrical Units, Bull. National Research Council no. 93, p. 80, 1933.
[11] Some progress has been made by the authors towards the construction of an electrodynamometer of the Pellat type. Pellat, Bull. de la Soc. Internat. des Electriciens, series 2, vol. 8, p. 573, 1908. The mathematical theory has been developed by Snow, B.S. Jour. Research, vol. 1 (RP24), p. 685, 1928.

The current balance was of the Rayleigh type having a moving coil which was suspended with its plane horizontal from 1 of the pans of a balance, and 2 fixed coils which were placed, one above and the other below, the moving coil. Each of the coils had a square cross section, the linear dimensions of which were small as compared to the radius of the coil. The coils were horizontal and coaxial, and the fixed coils were at such a distance from the moving coil that the electromagnetic force on the latter was a maximum.

In order to determine in absolute measure the current in the coils, it was necessary to measure not only the electro-magnetic force on the moving coil but also the ratio of the effective radii of the coils, and to know the number of turns in each of the coils. The experimental work, therefore, consisted of two distinct parts; one, the determination of the ratio of the radii of the coils used; and second, the determination of the force between the coils produced by the current which was to be measured.

II. DESCRIPTION OF THE CURRENT BALANCE

The current balance and accessories were installed in basement rooms in the electrical building, the balance room having only two small outside windows with light-tight shutters. The temperature change of the balance room was often as small as 0.2° C during an afternoon, a condition favorable for the operation of a precision balance. Adjacent to the balance room was a small observation room from which the balance could be operated without entering the balance room. The rooms were constructed with a minimum of magnetic material so placed that all magnetic masses of appreciable size were at a considerable distance from the balance. Inside the balance room were nonmagnetic piers each having a heavy top of statuary marble.

1. THE BALANCE

A 2-kilogram precision balance with a 30-cm beam was used, the same one as was employed in the work of 1911, when all the magnetic parts except the knife edges were removed. In the course of the present investigation, the knife edges were reground and heavy agate planes substituted for those originally furnished. As shown in figures 1 and 2, the balance was mounted on three marble blocks attached to the marble top of the coil case. On top of the blocks was a hole-slot-plane system arranged to hold the three supporting feet of the balance case. Slides were attached to the pieces holding the hole and the slot so that the balance with the attached moving coil could be moved in two horizontal directions, thus providing the horizontal adjustments of the moving coil with respect to the fixed coils. The moving coil was suspended from the right-hand pan of the balance by means of a tube through which the leads to the moving coil were carried.

A device was installed by which a cylindrical weight could be added to, or removed from, the pan without arresting the balance. The device had, projecting over the pan, an arm with 3 ivory fingers which could be moved between 4 ivory fingers attached to the pan. Grooves in both sets of fingers prevented a cylindrical weight from rolling out of place. A cam operated from a distance raised and lowered the arm carrying the three ivory fingers, thereby either removing the weight from the balance pan or replacing it. As the weights were placed on the pan to which the moving coil was attached, neither the ratio of the balance arms nor the bending of the beam affected the results.

(a) OPTICAL SYSTEM

The optical system for determining the position of the balance beam consisted of a lamp and scale with a mirror on the balance beam. There were three distinctive features of this system: (1) The light beam was symmetrical with respect to the 2 arms of the balance beam so that any heating produced by the light affected the 2 arms equally; (2) the effective length of the optical lever was doubled by an arrangement of mirrors which gave 2 reflections of the light from the mirror on the balance beam, thus doubling the motion of the spot of light on the scale so that this motion was then about 72 times the motion of the moving coil; (3) right-angled prisms with optical surfaces were used for most of the reflectors, thus giving a very distinct image on the reading scale. By employing these features the optical system using a lamp and scale was sufficiently accurate and easily operated.

(b) SENSITIVITY OF THE BALANCE

The sensitivity of the balance, with a half period of about 19 seconds, was usually about 3 milliradians per milligram. With the optical system described above, 1 mg placed on the pan produced a deflection of 16 mm on the scale. It was possible to make a weighing to about 0.01 mg since the position of the spot of light on the scale could be read to 0.2 mm. As the total load on the pan was more than 1 kg, a weighing to 0.01 mg required a sensitivity of 1 part in 100 million of the total load on the pan. This sensitivity was necessary to obtain a precision of one part in a million in the electromagnetic force between the coils, since this force seldom exceeded 10 grams.

In addition to the precautions normally taken in precision weighings, the following requirements were essential for the extreme precision needed in this investigation: (1) The balance must not be arrested during a weighing; (2) the temperature of the balance beam must be kept very constant; and (3) the density of the weights on the two pans must be approximately the same. An arrestment of the balance usually changed the rest point, the amount of change being different for each arrestment. A method of weighing will be described later which avoided this difficulty. The balance beam was made of a material which had an expansivity of nearly 20 parts in a million per degree centigrade. A change in temperature of 0.0005° C of one arm relative to the other affected a weighing by a part in 100 million. The method of weighing employed in this investigation eliminated the effect of any slow, uniform drift of temperature. The density of the counterweights was such that the volume occupied by them was about the same as the volume of the coil, so that changes in barometric pressure did not appreciably affect the rest point of the balance during a weighing.

In addition to the difficulties that are inherent in any weighing of extreme precision, the current balance has the added difficulty that the current heats the moving coil, thus causing, in the surrounding air, convection currents which produce a force on the coil. In a closed space which has reached temperature equilibrium, these convection currents become very stable so that the force is very constant. Hence weighings were made only after the balance and its surroundings had been under constant temperature conditions for several hours. The convection currents, when a moving coil was dissipating 2 watts, produced a force on it of as much as 0.2 mg, but, so long as this force remained constant, it caused no error in the determination of the electromagnetic force which was always obtained by reversing the current in the fixed coils.

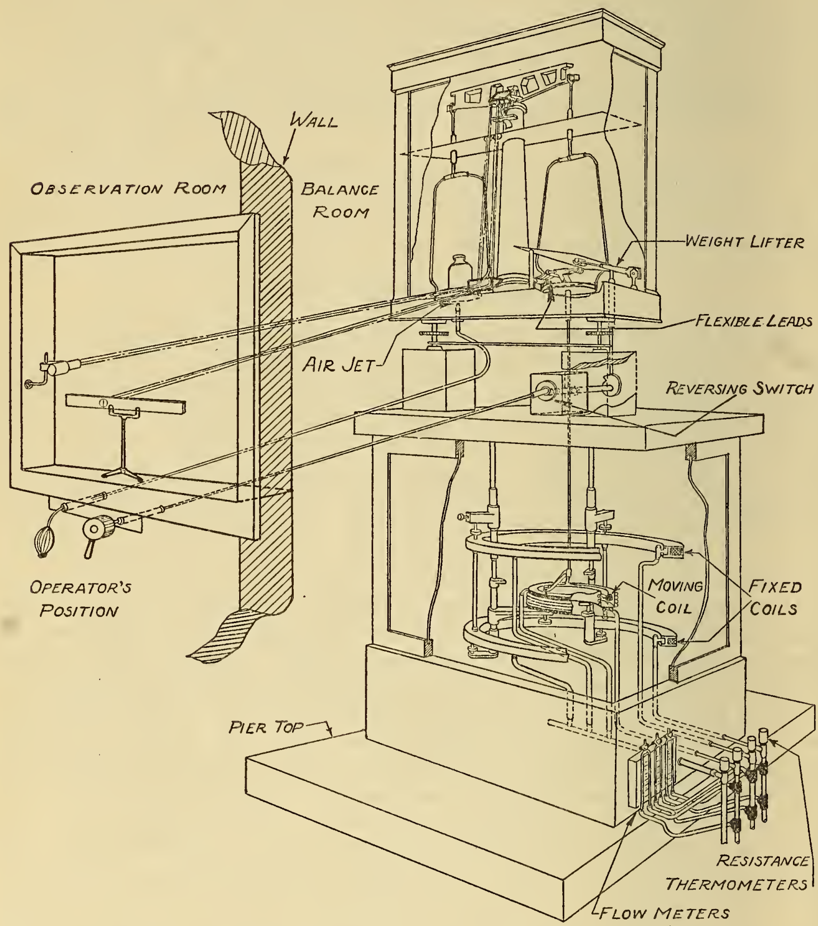

FIGURE 1.—*Assembled current balance and the operator's position.*

(c) THE WEIGHTS

For each combination of coils, 3 special weights were prepared, corresponding to 3 different currents. All these weights were in the form of cylinders with rounded ends. The lengths varied from 2.5 to 3.5 cm. Some were of a gold-platinum alloy, others of a platinum-iridium alloy. They have been frequently calibrated by the mass section of this Bureau and no significant changes in mass have

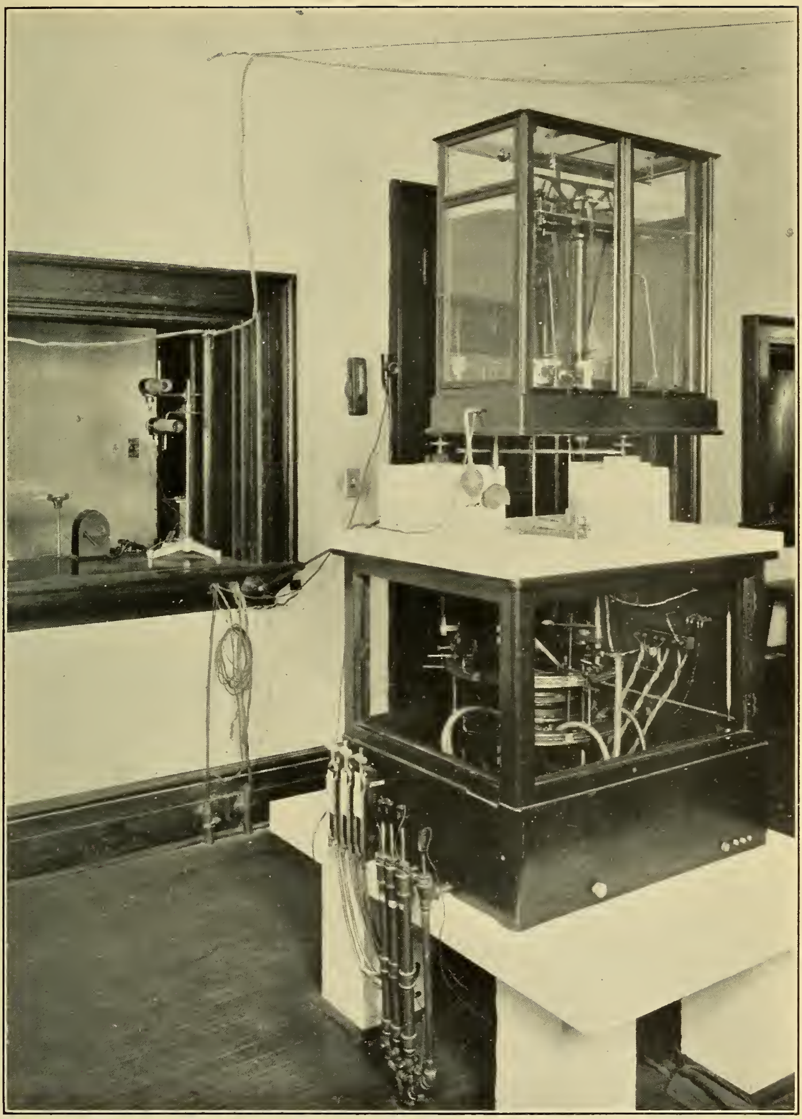

FIGURE 2.—*Photograph of the current balance as used in the final observations, showing the operator's position in an adjoining room.*

The window between the two rooms was closed when observations were being made. Every part of the entire assembly, including the rubber hose for making the water connections, the brass water pipes, and the resistance thermometers in the pipes, was carefully tested and found to be nonmagnetic. The only exceptions were the three steel knife edges of the balance, which were more than a meter from the coils.

been observed. The nominal values of the weights are given in table 2, p. 677.

(d) PROCEDURE IN MAKING A WEIGHING

For several hours before the actual weighings started, a current of approximately the same value as that to be used later was sent through the coils. When conditions had become stable the balance beam was released. With the direction of the current such that the electromagnetic force was downward and with the weight off the pan, the counterweight on the opposite pan was adjusted to give a suitable value of the rest point. These adjustments were necessarily made with the operator in the balance room. The operator then went to the observation room and after a short interval (about 15 minutes) began a set of observations. By means of air jets that could be directed under the pan, the amplitude of swing was adjusted to a convenient value, usually about 2 centimeters on the scale. The position of the rest point was then determined by observing 3 turning points at the right and 2 at the left. The direction of the electromagnetic force was then reversed by reversing the current through the fixed coils, and at the same time the weight was placed on the coil pan of the balance so that the total force acting on the balance beam was only slightly changed. A second rest point was then observed in order to obtain the difference in the rest points with the weight on the pan and with it off. As a check, the current in the fixed coils was again reversed and the weight removed. If conditions had remained exactly the same, this third rest point would exactly agree with the first. This was seldom the case, because conditions changed almost continuously. Hence, six or more rest points were usually necessary in order to determine, by a graphical method, the difference in the rest points with the weight on and with it off. From this difference and the sensitivity of the balance, the difference between the gravitational force on the weight and twice the electromagnetic force between the coils could be obtained.

2. COILS AND THEIR MOUNTINGS

The coils were mounted in a case with sides of wood and glass, the bottom of the case being the marble top of a pier, and the top being a marble slab upon which the balance was supported and to the underside of which the fixed coils were attached. The tube by which the moving coil was suspended from the pan of the balance passed through a hole in the marble slab.

(a) THE COILS

Rosa, Dorsey, and Miller constructed 4 moving coils designated as M1, M2, M3, and M4, and 3 pairs of fixed coils designated as S1, S2; L1, L2; and L3, L4. They stated that moving coils M2 and M3 were somewhat superior to M1 and M4 and that L1, L2 were somewhat inferior to either L3, L4 or S1, S2. Hence, for this investigation the moving coils M2 and M3 and the 2 pairs of fixed coils S1, S2, and L3, L4 have been used. The important constants of these coils as determined at the time of winding are given in table 1.

TABLE 1.—*Constants of coils*

[All measured constants were determined at time of winding]

Kind of coil	Designation	Number of windings	Number of turns per layer in each winding	Number of layers	Total number of turns in each winding	Mean radius of coil	Diameter of wire over insulation	Radial depth of winding	Axial width of winding channel	Computed width of winding [a]	Measured width less computed width [b]
						cm	*cm*	*cm*	*cm*	*cm*	*cm*
Moving	M2	2	6	12	72	12. 499	0. 076$_8$	0. 954$_8$	0. 956$_4$	0. 921$_6$	0. 034$_8$
Do	M3	2	7	14	98	10. 030	. 070$_1$	1. 029$_4$	. 996$_7$	. 981$_4$	. 015$_3$
Fixed	S1	2	14	28	392	19. 97	. 054$_6$	1. 528	1. 580	1. 529	. 051
Do	S2	2	14	28	392	19. 96	. 054$_8$	1. 522	1. 579	1. 529	. 050
Do	L3	2	17. 972	36	647	25. 00	. 051$_0$	1. 943	1. 969	1. 833	. 136
Do	L4	2	17. 972	36	647	25. 00	. 053$_6$	1. 925	1. 965	1. 927	. 038

[a] The computed width of the winding is the product of the total number of turns per layer and the diameter of the wire over the insulation.

[b] In all computations of the force the axial width of the coil was assumed to be equal to the width of the winding channel.

All coils were of enamel-insulated wire wound bifilarly on brass forms. A channel having approximately the dimensions desired for the finished coil was turned in each coil form. The bottom and sides of the channel were insulated with a thin layer of paper. Two wires entered the bottom of the channel from axial holes which were at opposite ends of a diameter. The correct number of turns for the bottom layer was wound, then each wire was brought up to the next layer at a predetermined point. In most coils this point was on the radius at which the winding started. However, in coils L3 and L4 this was not the case, but in each layer the wire was brought up for the next layer at a point 10° (one thirty-sixth of a turn) before reaching the radius at which the layer started. In every coil there was the same number of turns in each of the two windings. The end of each winding was brought out through an axial hole in the form on the same radius as the beginning of the winding. Terminal blocks were so mounted that the external leads could be held in a definite position and could be soldered to the leads from the coil. The form of a terminal block is shown in figure 3.

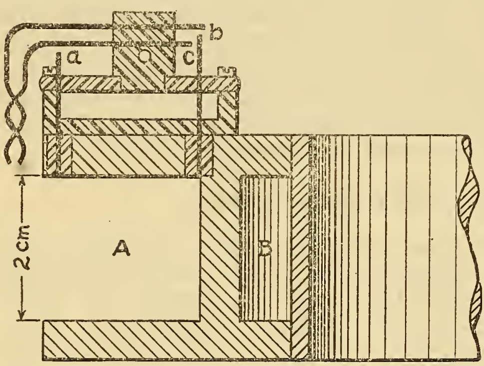

FIGURE 3.—*Crosssection of the form of a large fixed coil at a terminal block, showing the leads.*

The channel in which the wires were wound is A; the water channel is B. The electrical connections between the leads from the winding and the external leads were made by drops of solder: at a and b for normal operation; at b and c for measuring the effect of the leads.

In winding the coils, one or more strips of "onionskin" paper about 0.05 mm thick were placed between layers. In order to prevent moisture from penetrating the coil, the winding was carefully sealed by the following method: ". . . the paper covering the outer layer of wire was saturated with paraffin melted in with a clean hot soldering copper, the paraffin being well melted to the sides of the form. Then a strip of muslin, well soaked in a hot mixture of beeswax and Venice turpentine, was wrapped around the coil and melted to the underlying paraffin; over the whole was wrapped a strip of binder's cloth soaked in hot paraffin and melted to the muslin and form."[12] The sealing of the coils has not been disturbed since 1910. In this investigation there has been no indication of any imperfection in this sealing.

(b) COIL MOUNTING

As shown in figures 1 and 2, the fixed coils were supported from the marble top of the coil case, and the moving coil was suspended from a pan of the balance. The two fixed coils were maintained parallel and approximately coaxial by three *spacing* rods, and were attached to the marble top by *supporting* rods. A separate set of spacing rods of uniform length was required for each coil combination, the length being such that the electromagnetic force of each fixed coil on the moving coil is a maximum when the moving coil is midway between the fixed coils. The supporting rods permitted the leveling of these coils. The suspension of the moving coil was a tube, the upper end of which was attached to a pan of the balance and the lower end of which carried a tripod. The moving coil was attached to the feet of the tripod by adjusting screws which permitted the coil to be leveled.

(c) COOLING SYSTEM

A cooling system in which water was circulated in a closed circuit was arranged to carry off the heat that was produced by the current in the coils. The water, returning warm from the current balance, flowed through a cooling coil to lower its temperature, then into a tank in which the water was thoroughly stirred and carefully thermostated, the regulation being to about 0.02° C. The water was then pumped by a centrifugal pump to a distributing point inside the coil case. At the distributing point there was a connection to each of the fixed coils and to the water jacket of the moving coil, each of which had a separate return to a junction point from which the water was returned to the cooling coil. In the return circuit of each coil was a valve for regulating the flow, a flowmeter for measuring the flow, and the coil of a resistance thermometer for measuring the temperature of the water. The temperature of the water when it reached the distributing point was measured by a fourth resistance thermometer.

The cooling water for each fixed coil was circulated through a channel in the form on which the coil was wound, as shown in figure 3. The water entered the channel at one end of a diameter and left at the opposite end so that the circulation was in opposite directions in the two halves of the circumference. As a result, the temperature distribution in the form was not exactly uniform. As an example, with 1 ampere in the large fixed coils, the temperature of the outflowing water was 0.5° C. higher than the inflowing. The nonuniformity in temperature caused a slight distortion of the coil which, if appreciable, would cause the results with different currents to be different. In no case was any distortion indicated.

[12] See p. 286 in reference of footnote 8.

Two different types of water jackets have been used for cooling the moving coil, and in addition weighings have been made without any water jacket. The first water jacket consisted of a double-walled copper cylinder having a metal top and bottom with the water circulating in the space between the two walls. In this water jacket, cooling was effected in part by radiation and in part by convection currents which were set up in the air inside the jacket. These convection currents were not always sufficiently steady to permit the most accurate weighing. The second type of water jacket, shown in figure 4, consisted of a channel in a solid piece of brass to which was soldered a spiral copper tube, in which water was circulated. With the coil in this channel, the balance was very steady although the temperature of the coil was somewhat higher than when used in the open air or in the cylindrical water jacket. The type of water jacket

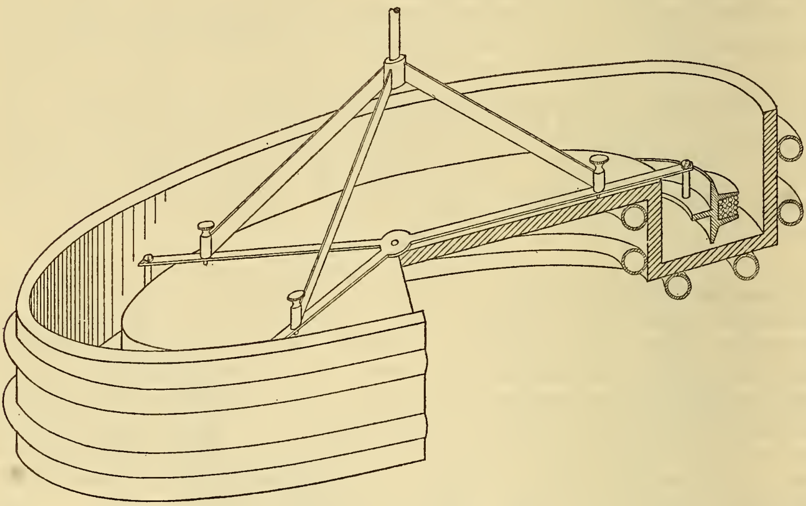

FIGURE 4.—*The second type of water jacket, consisting of a channel in a piece of brass.*

A section is removed to show the moving coil and its mounting. The coil was placed near the bottom of the channel, so that air currents were minimized. The water jacket was cooled by water flowing in the copper tube that was attached to the outside of the channel. The transfer of heat from the brass to the copper tube was facilitated by soldering the tube to the brass.

produced no effect on the result, but the balance was somewhat steadier when the second type was used.

III. ELECTRICAL CIRCUIT OF THE CURRENT BALANCE

The electrical circuit of the current balance was arranged to meet the following conditions: (1) The operator should be able to maintain a constant current; (2) the operator should be able to measure the current in B.S. international amperes by direct comparison with a standard of resistance and a standard cell; (3) the operator should be able to remove a weight from the pan of the balance at a time so nearly simultaneous with the reversal of the current in the fixed coils that the swinging of the balance would not be greatly disturbed; (4) the leads connecting the moving coil to the rest of the circuit should be of such a character as not to affect materially the sensitivity or reliability of the balance; (5) those parts of the circuit in which thermoelectromotive forces would affect the result should be so designed that these forces would be a minimum; (6) the insulation resistance between the balance coils and ground should be so high that the effect of leakage currents would be negligible; (7) the frame of the moving coil should be at the same potential as the surrounding water jacket so that electrostatic forces in the balance would be eliminated; (8) the leads should be arranged to have as small a magnetic field as possible. These conditions were met with as simple a circuit as could be devised. A diagram of the circuit is shown in figure 5.

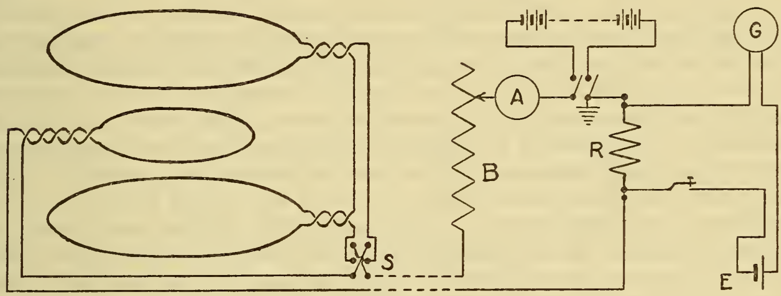

FIGURE 5.—*Electrical circuit of the current balance.*

The current in the current balance coils and the resistance standard, *R*, was adjusted by the rheostat, *B*, until the fall in potential across *R* was exactly equal to the electromotive force of the standard cell, *E*, as shown by the zero deflection of the galvanometer, *G*. In this manner the current was measured directly in B.S. international amperes by the ratio of *E* to *R*. The ammeter, *A*, was used for obtaining a rough adjustment of the current. The circular reversing switch, *S*, reversed the current through the fixed coils only, thus reversing the electromagnetic force on the moving coil.

1. CONSTANCY OF CURRENT

The constancy of the current depended largely on the storage battery which was used as a source, but was influenced to some extent by the temperature conditions in the copper portions of the circuit. The battery which gave most satisfactory constancy was a 100-volt, 400-ampere-hour lead storage battery. The current changed so slowly that the operator could maintain it constant to a part in a million by adjusting a resistor once or twice a minute.

2. MEASUREMENT OF THE CURRENT IN B.S. INTERNATIONAL AMPERES

The current was measured in B.S. international amperes by comparing the potential drop produced by it in a standard resistance with the electromotive force of a standard cell. As shown in figure 5 a standard cell, E, in series with a key and sensitive galvanometer, was connected directly to the potential terminals of a four-terminal resistance standard, R. The current was then adjusted by the rheostat, B, until the deflection of the galvanometer was zero. For this condition the current, I, in B.S. amperes was equal to E/R, where E is the B.S. value for the electromotive force of that particular standard cell, and R is the B.S. value of the resistance of that particular standard resistor.

The standard resistors used in this set-up were specially designed to carry a large current without change in resistance. They were made of manganin strip 5 millimeters wide, and thick enough to give the desired resistance in a length of about 5 meters. This strip was wound between a series of insulated posts and the whole immersed in oil maintained at a constant temperature. The temperature coefficient of resistance was usually about 5×10^{-6} per degree centigrade and the load coefficient about 1×10^{-6} per watt. In order that the current could be determined to a part in a million, the temperature had to be measured to 0.2° C and the power to 1 watt. These resistors were frequently measured by the resistance section of this Bureau. Their value at any particular time was known to one or two parts in a million.

The standard cell employed at any particular time was compared on the day of use with the reference standards of this Bureau. In many of the measurements, there was used a standard cell located in the same bath as the primary group of standard cells, the connection to the current-balance circuit being made through a specially insulated line which connected the observation room with the standard-cell laboratory. In the rest of the measurements, there was used one of a group of three standard cells (neutral saturated cadmium cells) which were kept at the bottom of a well extending 3 meters below the floor of the laboratory. The daily drift of temperature of the cells as measured by a resistance thermometer was never more than 0.02° C. In no case did the change in temperature between time of use and time of comparison with the standard group amount to as much as 0.01° C, which would change the electromotive force of the cell less than ½ microvolt.

The cell employed in any set of observations was connected almost continuously for several hours. Small currents, seldom more than 0.01 microampere, passed through the cell, sometimes in the positive, sometimes in the negative direction. No change in the electromotive force of the cell due to this cause was ever detected.

The sensitivity of the galvanometer was 0.8 radians per microampere and its external critical damping resistance was 500 ohms. At a scale distance of 6.2 meters a current of 2×10^{-9} ampere produced 1 centimeter scale deflection. Since the resistance of the galvanometer was about 42 ohms and that of the standard cell about 500 ohms, an unbalanced electromotive force in the galvanometer circuit of 1 microvolt, corresponding to a change in the current through the current balance of a part in a million, produced a deflection of nearly 1 centimeter. The galvanometer had ample sensitivity and, as normally used, was critically damped.

A disadvantage of the method just described for measuring the current in B.S. amperes was its lack of flexibility. For a given resistance and standard cell, a definite current was required. With this current and a given set of coils in the current balance, the weight necessary to counterbalance the electromagnetic force was fixed. Hence a weight had to be constructed for each resistance that was used in conjunction with each set of coils. Twelve weights were made, the nominal values of which are given in table 2. Extremely precise adjustment of the weights is not necessary since differences in force as large as 1 milligram could be obtained from changes in the rest point of the balance.

TABLE 2.—*Nominal masses of weights used with different resistances and different coil combinations*

Coil combination	Composition	Density	R=1 ohm I=1.018 amp	R=1.5 ohm I=0.679 amp	R=2.0 ohm I=0.509 amp
	Percent	*g/cc*	*grams*	*grams*	*grams*
S1 S2 M3	90 Au 10 Pt	19.5	8.81	3.91	2.20
L3 L4 M2	90 Au 10 Pt	19.5	10.55	4.68	2.64
S1 S2 M2	90 Pt 10 Ir	21.5	11.92	5.29	2.96
L3 L4 M3	90 Au 10 Pt	19.5	8.43	3.75	2.09

3. REVERSAL OF CURRENT IN THE FIXED COILS

The reversal of the current in the fixed coils presented both an electrical and a mechanical problem. The reversal should not produce such high induced electromotive forces as to endanger the insulation of the coils, and the effective time of reversal should so nearly coincide with the time at which the weight is added or removed that no large change would be produced in the amplitude of swing of the balance. Both of these problems were solved by the use of a step-by-step reversing switch which was mounted on the same shaft as a cam which controlled the addition or removal of the weight. The reversing switch [13] was of the rotary type so designed that a rotation of about 160° first decreased the current in five steps to about one tenth its original value, reversed this small current, then increased it to its original value in five steps. The complete operation of the switch could be accomplished in a fraction of a second. The cam of the weight lifter was so oriented on the shaft of the reversing switch that the weight would be added or removed at the proper time. An operator learned from experience the position in the swing of the balance at which a certain speed of rotation of the shaft produced a minimum disturbance of the balance. The reversal did not ordinarily change the amplitude of swing by more than a few centimeters on the scale. The amplitude of swing could be adjusted from the observation room by means of air jets under the pans. Any lateral swinging of the pans could be observed by an auxiliary optical system and could be damped by lightly pressing against the pan a camel's-hair brush which could be operated from the observation room.

4. LEADS TO THE MOVING COIL

The moving coil was connected to the electrical circuit at terminals in the balance case from which flexible leads extended to terminals on the pan of the balance. Rigid leads, passing through the tube that supported the moving coil, connected the terminals on the pan to wires that were connected to the windings of the coil. The sensitivity of the balance was not affected by the rigid leads, but was affected by the flexible leads. The flexible leads were made as pliable as possible in order that they would produce a minimum effect on the sensitivity of the balance, and were so arranged that air currents produced by the heating of them did not appreciably disturb the balance. They consisted of two sets of fine copper wires, each set extending from a terminal in a terminal block mounted on the balance

[13] See p. 295 in reference of footnote 8.

pan to a terminal in an identical block on the balance case. Each terminal block consisted of two strips which were cut from a threaded brass tube and which were insulated by being attached to opposite sides of two short amber cylinders. About 36 copper wires each 0.025 mm in diameter extended from each strip of one block to the corresponding strip of the other block. The strips on the pan were connected to the rigid leads of the moving coil, while the strips in the balance case were connected to the main circuit. A sketch of the leads showing their relation to the weight lifter and balance pan is given in figure 6.

In order to construct a set of leads, the terminal blocks were removed from the balance and mounted in a frame which held them 8 centimeters apart. The fine copper wire was wound around the two cylinders, the spacing between wires being 1 or 2 millimeters. The wires were soldered to the strips, then the portions of wire between the two strips of each block were removed. The two sets were then mounted in the balance with the strips horizontal and the wires were annealed by heating to a dull red by passing a current through them. During the annealing process, a small glass rod was placed on each of the two sets of wires, the upper and the lower, so as to obtain uniformity in the shape of the system of wires. The two sets when connected in series had a resistance of about 0.13 ohm. The heat generated in these wires by the current was sufficient to cause, in the air, appreciable convection currents which were deflected away from the balance beam by a shield placed directly above the wires. However, there was a buoyant effect on the leads themselves which, with a current of 1 ampere, produced a force on the pan of the balance of 0.15 mg. This force was very constant, thus introducing no difficulty in making a weighing and producing no error in the result since it was independent of the direction of the current.

In addition to the current leads just described, there were two potential leads connected to the terminals of the winding of the moving coil, and one grounding lead connected to its frame. The portion of each of these leads which extended from the balance case to the pan consisted of a single fine wire. The potential leads were used in measuring the resistance of the moving coil. The grounding lead was used to connect the frame of the coil to earth.

The flexible leads appreciably affected the sensitivity of the balance. In one case a sensitivity of 2.2 cm/mg was reduced to 1.2 cm/mg by installing a set of leads. However, sufficient sensitivity could always be obtained by adjusting the center of gravity of the moving system of the balance.

5. THERMAL ELECTROMOTIVE FORCES

The only part of the circuit in which a thermal electromotive force would influence the result was the closed loop which included the standard cell, galvanometer, and standard of resistance. All parts of this loop were of copper except the standard resistor and standard cell, both of which were kept at a constant temperature. The conducting parts of the galvanometer were entirely of copper, and all keys, switches, and binding posts were made of copper. Two different tests were used to determine the thermal electromotive force in this loop. In the first test, a series of observations was made in which

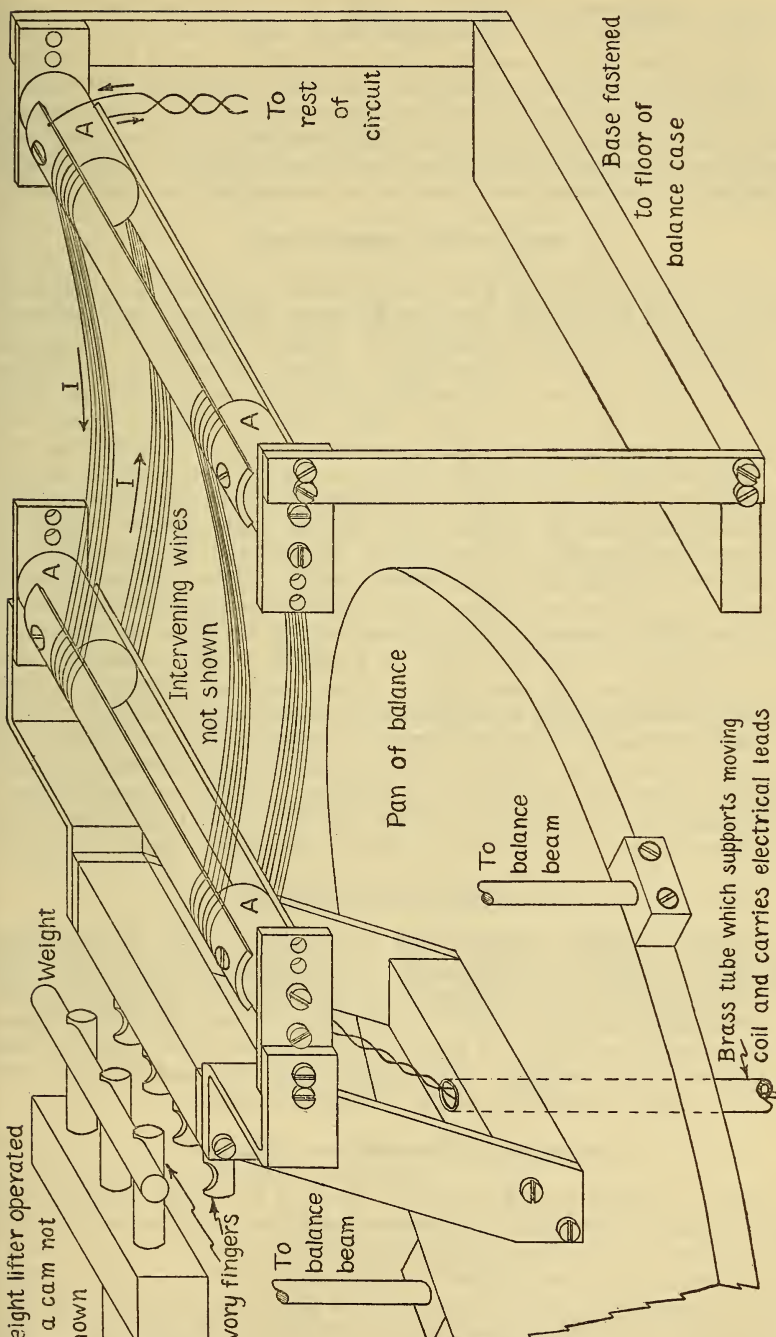

FIGURE 6.—*The flexible leads and the weight lifter, showing their relation to the pan of the balance.*

The amber cylinders, marked A, served to separate and insulate the brass strips to which the fine, flexible wires were attached. The potential and grounding leads are not shown.

the storage battery and standard cell were reversed at the end of each observation. Several such series were made, and in no case was the indicated thermal electromotive force more than 2 or 3 microvolts, which was within experimental error. In the second test, the standard cell was replaced by a 500-ohm copper coil. With the current balance circuit disconnected from the terminals of the standard resistor, the deflection of the galvanometer was observed. In no case was the thermal electromotive force in the loop as large as 1 microvolt.

6. INSULATION RESISTANCE

The insulation resistance was measured not only between the two windings of each of the coils, but also between the circuit and ground. The resistance between windings was always sufficiently high to insure that there was no appreciable leakage between turns. The resistance to ground was frequently measured to make sure that leakage currents were so small that they would not affect either the force produced in the current balance or the reading of the galvanometer. To produce a measurable effect on the force measured in the current balance and hence on the absolute value of the current, the leakage current must be of the order of one microampere which is many times the largest ever observed. However, a thousandth of a microampere through the galvanometer would make a deflection of 5 millimeters on the galvanometer scale, which would affect the measurement in B.S. international amperes, of the current through the current balance, by nearly a part in a million. By grounding the main circuit between the resistance standard and the battery, no part of the galvanometer loop was different from earth potential by more than 1 volt, so that if the insulation resistance between this loop and ground was more than 1,000 megohms, the leakage current did not appreciably affect the galvanometer reading. As the measured insulation resistance to ground was never less than 6,000 megohms, the leakage current never appreciably affected the galvanometer reading.

7. ELECTROSTATIC FORCES

Electrostatic forces between the moving coil and surrounding bodies were avoided by grounding all the metal parts which were not a part of the circuit. The grounding of the frame of the moving coil was particularly desirable since its windings were completely insulated from the frame, and the frame was insulated from earth by the agate plane of the balance. This grounding was accomplished by means of a separate flexible lead as already described.

8. MAGNETIC EFFECT OF LEADS

In order to minimize the magnetic effect of the leads carrying the current to each of the coils, the conductors in the neighborhood of the current balance were all made from twisted pairs of wires. An exception was the flexible leads which carried the current to the moving coil, but these were so far distant from the coils that the electromagnetic force resulting from them was small. The effect of the leads on the measured force was experimentally determined for every set-up by a method described later.

IV. COMPUTATION OF THE MAXIMUM FORCE FOR UNIT CURRENT

The computation of the maximum force between the coils of a current balance when unit current flows through each of them may be considered under three headings: (1) The maximum force between two circular coaxial filaments; (2) the maximum force between two coaxial coils of finite cross section; (3) the maximum force between three maladjusted coils; i.e., two fixed coils and one moving coil, the fixed coils being not exactly coaxial and not quite at the correct spacing, so that at no position of the moving coil can both exert the maximum force on it. The first two of these were computed from observed data taken on the coils before mounting them in the current balance. The effect of lack of coaxiality and of variation from the position for maximum force was determined from experimental data taken in the course of weighings.

1. MAXIMUM FORCE BETWEEN CIRCULAR FILAMENTS

The maximum force between two filaments located at the circumference of coaxial circles is a function of the ratio of the radii of the circles and of the currents in the filaments. This force for unit current in each filament can be expressed in mathematical notation by use of the following symbols: [14]

Let a_1 = radius of the filament at the circumference of the larger circle

a_2 = radius of the filament at the circumference of the smaller circle.

$\alpha = a_2/a_1$ = ratio of radii of the filaments (always less than unity)

z_m = the axial distance between the circular filaments when the force is a maximum

$y_m = \frac{z_m}{a_1}$ = ratio of axial distance for maximum force to the radius of the larger filament

F_m = force in dynes between the filaments carrying unit current in the cgs electromagnetic system when the distance between their planes is such as to give the maximum force.

The units used in measuring the radii are immaterial so long as the same unit is used for both since only the ratio enters the formula.

Then Maxwell's elliptic-integral formula can be written as

$$F_m = \frac{\pi y_m k}{\sqrt{\alpha}} \left\{ \frac{2-k^2}{1-k^2} E - 2K \right\} \tag{1}$$

where K and E are the complete elliptic integrals of the first and second kind, respectively, to modulus k and

$$k^2 = \frac{4\alpha}{(1+\alpha)^2 + y_m^2} \tag{2}$$

The formula for F_m is expressed as a function of α and y_m. However, it can be shown that y_m is also a function of α so that F_m is a

[14] The nomenclature used in this paper corresponds with that used by Snow (footnote 25). This is somewhat different from that used by Rosa, Dorsey, and Miller (footnote 8), and by Grover (footnote 15).

function of α only. No formula for computing F_m has been developed which does not first require the computing of y_m. An exact formula for computing y_m as a function of α has been given by Grover [15] who also gives a table of y_m as a function of α. The following empirical formula derived from Grover's table gives y_m with an accuracy of at least a part in a thousand in the indicated range of α, which accuracy is sufficient for computing F_m to a part in a million. The formula is

$$y_m = 0.5 - \frac{9}{20}\alpha^2 - \frac{1}{16}\alpha^4 \quad 0 < \alpha < 0.75 \tag{3}$$

It has been shown [16] that a variation in y_m produces a variation in F_m which is given by the equation

$$\frac{\Delta F_m}{F_m} = -c\left(\frac{\Delta y_m}{y_m}\right)^2 \tag{4}$$

where c varies between 0.6 and 0.7 for the values of α used in this investigation. This equation substantiates the statement of the required accuracy in y_m that was made in the previous paragraph. While y_m does not need to be accurately known, the value of α is required with great precision. This can be seen from the formula [17]

$$\frac{\Delta F_m}{F_m} = \epsilon \frac{\Delta \alpha}{\alpha} \tag{5}$$

where ϵ varies between 2.0 and 4.2 in the range of α considered. Actual values of ϵ for the coils used are given in table 11.

The values of the elliptic integrals necessary for computing F_m are given in various tables, but only in Legendre's [18] with sufficient accuracy, since the difference of two quantities having about the same value is required. The interpolation in Legendre's tables is exceedingly laborious. Rosa, Dorsey, and Miller [19] used these tables for computing their table XXX from which F_m can be determined, but the range of the table is not sufficient to cover one combination of coils used in this investigation. Grover [20] has given, for the entire range of α, tables which are easy to use and by which all values of F_m in this paper have been computed. However, all values have been checked by a method based on the use of arithmetico-geometric means [21] for computing the elliptic integrals. This method so simplifies the computation of F_m that the use of auxiliary tables is unnecessary.

The first step of the computation is to form a table of arithmetico-geometric means defined as follows: (Note that the a's in this table are not the radii of the coils).

---

[15] B.S. Bull., vol. 12, p. 317, 1915.
[16] Equation 64 on p. 344 of Grover's paper. Reference, footnote 15.
[17] The ϵ in the formula is the same as the ϵ given on p. 330 of the paper describing the work in 1911. Reference, footnote 8.
[18] A photographic reproduction of the original tables was published by K. Wittwer, Stuttgart, in 1931.
[19] Reference footnote 8.
[20] Reference footnote 15.
[21] For a complete discussion of the method see King, On the Direct Numerical Calculation of Elliptic Functions and Integrals. Cambridge Univ. Press, 1924. See also Grover, Phil. Mag., series 7, vol. 15, p. 1115, 1933.

$$
\begin{array}{lll}
a_0=1 & b_0=\sqrt{1-k^2} & c_0=k \\
a_1=\tfrac{1}{2}(a_0+b_0) & b_1=\sqrt{a_0b_0} & c_1=\tfrac{1}{2}(a_0-b_0) \\
a_2=\tfrac{1}{2}(a_1+b_1) & b_2=\sqrt{a_1b_1} & c_2=\tfrac{1}{2}(a_1-b_1) \\
a_3=\tfrac{1}{2}(a_2+b_2) & b_3=\sqrt{a_2b_2} & c_3=\tfrac{1}{2}(a_2-b_2) \\
\vdots & \vdots & \vdots \\
a_n & b_n & c_n
\end{array}
$$

The values of a_n rapidly approach those of b_n, so that c_n approaches zero. When $\alpha=0.75$, a_4 differs from b_4 by less than a part in 10 million; for smaller values of α, the difference is less. In most cases the table does not need to be computed beyond a_3. Then by substituting, in equation 1, the values of the elliptic integrals, K and E, as given by the arithmetico-geometric means,

$$F_m=\frac{\pi^2 y_m k}{4a_n(1-k^2)\sqrt{\alpha}}\{k^4-2(2-k^2)(c_1^2+2c_2^2+4c_3^2+\cdots\cdots)\} \qquad (6)$$

The following example in which the value of α is the largest of any used, illustrates the method of computation.

Example: Computation for coils S1 M2

$$\alpha=0.6258941$$

By formula 3, four significant figures of the value of y_m can be obtained, so that for the computation

$$y_m=0.3141000$$

By formula 2

$$k^2=0.9129842$$

Forming the table of arithmetico-geometrical means

$a_0=1$	$b_0=0.2949845$	$c_0=0.9555021$
$a_1=0.6474922$	$b_1=0.5431247$	$c_1=0.3525078$
$a_2=0.5953085$	$b_2=0.5930169$	$c_2=0.0521837$
$a_3=0.5941627$	$b_3=0.5941616$	$c_3=0.0011458$
$a_4=0.5941621$	$b_4=0.5941621$	$c_4=0.0000006$

Since $a_4=b_4$; $a_4=a_5=\cdots\cdots=a_n$.

Substituting in formula 6

$$
\begin{aligned}
c_1^2 &= 0.1242617 \\
2c_2^2 &= 0.0054463 \\
4c_3^2 &= 0.0000053 \\
c_1^2+2c_2^2+4c_3^2 &= 0.1297133 \\
2(2-k^2) &= 2.1740316 \\
k_4 &= 0.8335401 \\
2(2-k^2)(c_1^2+2c_2^2+4c_3^2) &= 0.2820007 \\
k^4-2(2-k^2)(c_1^2+2c_2^2+4c_3^2) &= 0.5515394 \\
\pi^2 y_m k &= 2.962097 \\
4a_n(1-k^2)\sqrt{\alpha} &= 0.1636115 \\
\frac{\pi^2 y_m k}{4a_n(1-k^2)\sqrt{\alpha}} &= 18.10446 \\
F_m &= 18.10446\times 0.5515393 \\
&= 9.985322 \text{ dynes for a cgs unit of current in each filament.}
\end{aligned}
$$

The constants entering into the formula do not all have to be computed with the same accuracy. For instance, the value of y_m need be known only with an accuracy of a part in a thousand to give an accuracy of a part in a million in the value of F_m, as shown by formula 4. However, it is important that the value of y_m should be used in the computation as though it were known to a part in a million, since it enters as a factor in the final product. Other factors also depend on y_m so that the effect of small changes in its value will be compensated in the final result. The value of F_m also depends on the factor $(1-k^2)$. But since, in the example, k^2 is more than 0.9, its value must be given to a part in 10,000,000 if $1-k^2$ is to be known to a part in a million. In this case, the last figure in the value of k^2 is unimportant, but the value chosen must be used throughout the computation.

The value of F_m agrees with that obtained by using Nagaoka's formula [22] and also by interpolation in Grover's tables.[23] The value is doubtless correct to at least a part in a million. The value of α used in this computation required an unfavorable interpolation in the tables of Nagaoka and Sakurai [24] so that a slightly different result was obtained by their use.

2. MAXIMUM FORCE BETWEEN COAXIAL COILS

The maximum force, $\mathfrak{F}_m$, between two circular coils having rectangular cross sections and carrying unit current in each turn, is, to a first approximation, equal to the maximum force, F_m, between two filaments carrying unit current and located at the geometric centers of the cross sections of the coils multiplied by the product of the number of turns in one coil and the number in the second. A second approximation can be obtained by adding a function which includes, in addition to the ratio of the radii of the circular filaments, the ratio of each cross-sectional dimension to the radius of its filament. A further refinement takes account of the fact that, for coils of finite cross section, the axial distance for maximum force may not be the same as the corresponding axial distance for the filaments located at the centers of their cross sections. A formula which takes all these factors into account was derived [25] on the assumption that the coils were composed of a number of circular hoops of insulated wire arranged within a rectangle of dimensions 2b and 2c (see fig. 7), all the n hoops being cut by a radial plane in which the current is transferred from hoop to hoop and in which the current enters and leaves the coil. The same current flows in each hoop, and the magnetic field is symmetrical around the axis of the coil. The equation is [26]

$$\mathfrak{F}_m = n_1 n_2 F_m \left\{ 1 + \Delta_2 + \Delta_4 + \frac{(x\Delta'_2)^2}{2\lambda_2} \right\} \tag{7}$$

In this formula the symbols have the following significance:

n_1 and n_2 = number of turns of wire in the larger coil and the smaller coil, respectively.

[22] Phil, Mag., vol. 6, p. 19, 1903. Also given in Grover's paper.

[23] See table 4, p. 372, of Grover's paper. Reference footnote 15.

[24] Sci. Papers of Inst. of Phys. and Chem. Research, Tokyo, table no. 2, 1927. Values of $\frac{\sqrt{a_1a_2}}{z}F$ are tabulated as a function of k^2.

[25] C. Snow, The Attraction Between Coils in the Rayleigh Current Balance, B.S. Jour. Research, vol. 11, p. 681, 1933. This formula was developed as a part of this research.

[26] The subscripts 2 and 4 for the Δ's were chosen because Δ_2 represents the second-order terms in the Taylor expansion, while Δ_4 represents the fourth-order terms. The term in Δ'_2 takes account of the fact that the z_m used with the coils is not the same as the z_m calculated for the filaments.

F_m = the maximum force for unit current in two filaments located at the centers of the cross sections of the coils. (See equation 1 or its equivalent equation 6.)

a_1 and a_2 = the mean radii of the larger coil and the smaller coil, respectively,

b_1 and b_2 = one half the axial width of the larger coil and the smaller coil, respectively,

c_1 and c_2 = one half the radial depth of the larger coil and the smaller coil, respectively,

$$\alpha = a_2/a_1$$

$$A^2 = a_1^2 + a_2^2 \tag{8}$$

$$\beta = \frac{1-\alpha^2}{1+\alpha^2} \tag{9}$$

$$y_m = 0.5 - \frac{9}{20}\alpha^2 - \frac{1}{16}\alpha^4 \quad \text{if } 0 < \alpha < 0.75 \tag{10}$$

$$x = \frac{y_m}{\sqrt{1+\alpha^2}} \tag{11}$$

$$\lambda_1 = \text{zero}^{27}$$

$$\lambda_2 = \frac{3\beta^2 - 2x^2}{\beta^2 + 2x^2 + x^4} \tag{12}$$

$$\lambda_3 = \frac{4x^2 + \lambda_2(11x^4 + 10x^2 - \beta^2)}{\beta^2 + 2x^2 + x^4} \tag{13}$$

$$\lambda_4 = \frac{16x^2\lambda_3(x^2+1) - 3x^2\lambda_2(23x^2+8)}{\beta^2 + 2x^2 + x^4} \tag{14}$$

$$B_1 = \frac{c_1^2}{a_1^2}[5(b_1^2 + b_2^2 - c_2^2) - 3c_1^2] - \frac{c_2^2}{a_2^2}[5(b_1^2 + b_2^2 - c_1^2) - 3c_2^2] \tag{15}$$

$$B_2 = \frac{c_1^2}{a_1^2}[5(b_1^2 + b_2^2 - c_2^2) - 3c_1^2] + \frac{c_2^2}{a_2^2}[5(b_1^2 + b_2^2 - c_1^2) - 3c_2^2] \tag{16}$$

$$B_3 = \frac{c_1^2}{a_1^2}[9c_1^2 + 15c_2^2 - 10(b_1^2 + b_2^2)] + \frac{c_2^2}{a_2^2}[9c_2^2 + 15c_1^2 - 10(b_1^2 + b_2^2)] \tag{17}$$

$$B_4 = \frac{1}{A^2}[10(b_1^2c_1^2 + b_2^2c_2^2) - 10(b_1^2 - c_1^2)(b_2^2 - c_2^2) - 3(b_1^4 + c_1^4 + b_2^4 + c_2^4)] \tag{18}$$

$$\Delta_2 = \frac{1}{12x^2}\left\{\left[\frac{c_2^2}{a_2^2} - \frac{c_1^2}{a_1^2}\right]\beta + \frac{2[c_1^2 - b_1^2 + c_2^2 - b_2^2]\lambda_2}{A^2}\right\} \tag{19}$$

$$\Delta_4 = \frac{1}{360A^2x^4}\left\{B_4\lambda_4 + (\beta B_1 - x^2B_2)\lambda_3 + \left[3\beta B_1 - x^2B_3 - 5x^4\left(\frac{c_1c_2A}{a_1a_2}\right)^2\right]\lambda_2 - 6\beta B_1\right\} \tag{20}$$

$$x\Delta_2' = \frac{1}{12x^2}\left\{2\beta\left(\frac{c_1^2}{a_1^2} - \frac{c_2^2}{a_2^2}\right) + \left[(x^2-\beta)\frac{c_1^2}{a_1^2} + (x^2+\beta)\frac{c_2^2}{a_2^2}\right]\lambda_2 + \frac{2(b_1^2 - c_1^2 + b_2^2 - c_2^2)}{A^2}\lambda_3\right\} \tag{21}$$

---

[27] In the general equation for the force at any distance as developed by Snow, the coefficient λ_1 appears. For the position of maximum force which is considered in this paper, λ_1 equals zero, and the expressions for λ_2, λ_3, and λ_4 are simplified.

The values of F_m, Δ_2, Δ_4, and $\mathfrak{F}_m$ for each pair of fixed and moving coils used in this investigation are given in table 13, p. 727.

3. MAXIMUM FORCE FOR THREE MALADJUSTED COILS

In a current balance of 3 coils, the 2 fixed coils held rigidly together, it is impractical to adjust the coils so that each fixed coil is placed in the exact position relative to the moving coil which was assumed in deriving the formula for the maximum force ($\mathfrak{F}_m$ of equation 7). Some of the adjustments can be made with sufficient accuracy, but for others, corrections must be made to the maximum force to allow for the maladjustment. Hence a discussion of the effect on the maximum force of all the possible maladjustments of each of the coils will be given, together with methods for determining the corrections for these maladjustments.

Each of the three coils used in the current balance can be considered as a solid having 6 degrees of freedom. Any change in position of a coil can be completely described by giving the translations of the center in the direction of three coordinate axes and the rotations relative to the same three axes. However, the rotation of a circular coil around its axis does not influence its magnetic effect because of its symmetry with respect to this axis. Hence, if 1 of the coordinate axes coincides with the axis of the coil, only 5 constants (3 translations and 2 rotations) are required to completely describe any magnetic effect of the coil. For this reason, a coil as used in a current balance is said to have 5 degrees of freedom. It follows that with 3 coils there are 15 independent adjustments but of these the 3 translations of 1 of the coils can be given arbitrary values. Hence, there are only 12 independent adjustments which must be accurately made or for which corrections must be applied to the computed force because of their maladjustment. The adjustments for the two rotations of each coil relative to the horizontal axes were made by levels. The permissible error in each of these two adjustments is $\pm 5'$, a condition which was readily met.

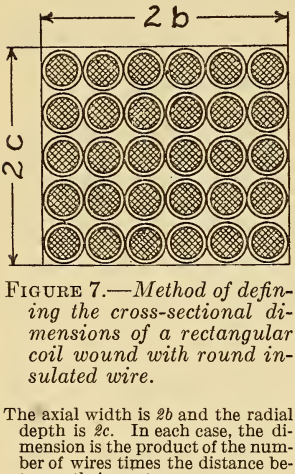

FIGURE 7.—*Method of defining the cross-sectional dimensions of a rectangular coil wound with round insulated wire.*

The axial width is $2b$ and the radial depth is $2c$. In each case, the dimension is the product of the number of wires times the distance between their centers.

With the 6 rotations correctly made, there remain the adjustments for 6 translations, 3 for the moving coil and 3 for one of the fixed coils. If the adjustments were perfect, the centers of the coils would be on the same vertical axis and at such a distance apart that the force between each fixed coil and the moving coil is a maximum. Instead of attempting to make perfect adjustments, it was more convenient, in some cases, to make approximate adjustments and to apply small corrections for any maladjustments that remained. In order to obtain the corrections for these maladjustments, a series of determinations of the force was made with different positions of the moving coil, but with a constant current in the coils. The data were interpreted by a graphical method.

The principle underlying the graphical method can be illustrated by considering the force acting on the moving coil when the three coils are coaxial and the distance between the fixed coils is so large that when the moving coil is in the midplane neither coil exerts its maximum force. The coil arrangement is shown in figure 8. The two fixed coils, 1 and 2, are so nearly identical, that the value of z_m may be taken as the same for both coils. The force, $\mathfrak{F}$, on the moving coil at a distance z from one of the fixed coils is less than the maximum

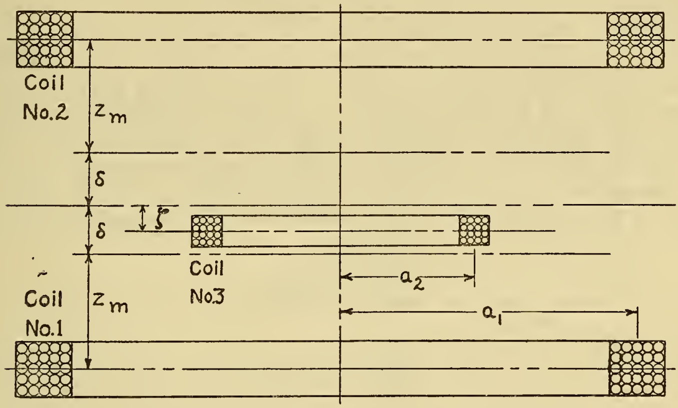

FIGURE 8.—*Nomenclature when the large coils are not correctly spaced.*

force, $\mathfrak{F}_m$, between these coils by an amount which, as a first approximation,[28], depends on the square of the displacement from the position of maximum force. Expressed as equations

$$\mathfrak{F}_{13} = \mathfrak{F}_m[1-\gamma(z_1-z_m)^2] = \mathfrak{F}_m[1-\gamma(\delta-\zeta)^2] \tag{22}$$

$$\mathfrak{F}_{23} = (\mathfrak{F}_m+\epsilon)[1-\gamma(z_2-z_m)^2] = (\mathfrak{F}_m+\epsilon)[1-\gamma(\delta+\zeta)^2] \tag{23}$$

where

$\mathfrak{F}_{13}$ = the downward force between coils 1 and 3 if the distance between them is z_1 and the current is in the same direction in both coils

$\mathfrak{F}_{23}$ = the downward force between coils 2 and 3 if the distance between them is z_2 and the current is in opposite direction in the two coils

$\mathfrak{F}_m$ = the computed maximum force between coils 1 and 3

ϵ = the small amount by which the computed force between coils 2 and 3 is larger than that between 1 and 3

z_m = the distance between a fixed and a moving coil for maximum force

γ = a positive constant, the value of which does not need to be known

δ = the distance from the midplane to the position of the maximum force. The sign of δ is positive if $z_1+z_2>2z_m$

ζ = the displacement of the moving coil below the midplane.

The above equations can readily be derived by Taylor's theorem since, at the maximum, all derivatives except the second are negligible.

The force was not measured for each coil separately, but first for the two fixed coils so connected that the force on the moving coil was the sum of the two forces and then with the current in one fixed

[28] On p. 380 et seq, of the 1911 paper, it is shown that this approximation is sufficiently accurate. A complete discussion is given there of these corrections. Reference footnote 8.

coil reversed (in coil 2 in equation 25), giving the difference of the two forces when the current is in the same direction in all three coils. Then, neglecting the term which is the product of ϵ and another small quantity, the equations of the sum and difference of the forces are:

$$\mathfrak{F}_s = \mathfrak{F}_{13} + \mathfrak{F}_{23} = 2\mathfrak{F}_m[1 - \gamma(\delta^2 + \zeta^2)] + \epsilon \tag{24}$$

$$\mathfrak{F}_d = \mathfrak{F}_{13} - \mathfrak{F}_{23} = \mathfrak{F}_m[4\gamma\delta\zeta] - \epsilon \tag{25}$$

For both the above arrangements, observations of the force were made at several values of ζ so that curves could be plotted with ζ as abscissa and $\mathfrak{F}_s$ or $\mathfrak{F}_d$ as ordinates. For the mathematical treatment, each of these quantities ($\mathfrak{F}_s$, $\mathfrak{F}_d$, and ζ) will be considered as a variable. Hence, equation 24 is a parabola with the maximum value of $\mathfrak{F}_s$ where $\zeta = 0$. Equation 25 is a straight line the slope of which ($4\,\mathfrak{F}_m\,\gamma\delta$) is positive if δ is positive. Taking derivatives of equations 24 and 25

$$\frac{d\mathfrak{F}_s}{d\zeta} = -4\mathfrak{F}_m\gamma\zeta \tag{26}$$

$$\frac{d\mathfrak{F}_d}{d\zeta} = 4\mathfrak{F}_m\gamma\delta \tag{27}$$

When these derivatives are equal, $\zeta = -\delta$. To find the value of ζ at which the two derivatives are numerically equal, the experimental data obtained for both $\mathfrak{F}_s$ and $\mathfrak{F}_d$ by varying ζ are plotted on the same sheet using the same scale in each plot, but not necessarily with the same origin of coordinates. Typical curves are shown in figure 9. A straight line is drawn parallel to the line representing $\mathfrak{F}_d$ as a function of ζ at such a position that it is tangent to the parabola representing $\mathfrak{F}_s$ as a function of ζ. The value of ζ at the point of tangency gives the value of δ, since at this point the derivative of $\mathfrak{F}_s$ is numerically equal to the derivative of $\mathfrak{F}_d$. In figure 9, the value of δ is given by the distance *bc*.

The same curves can also be used to determine the difference between the computed maximum force $2\mathfrak{F}_m + \epsilon$ and the measured maximum force $\mathfrak{F}_{sm}$. When $\zeta = \pm\delta$, the moving coil is at the position for maximum force with respect to one fixed coil, and at a distance 2δ from the position for the other coil. Substituting the value of ζ in equation 24, the following equations result:

$$\text{When } \pm\zeta = \delta,\ \mathfrak{F}_s = 2\mathfrak{F}_m\,(1 - 2\gamma\delta^2) + \epsilon \tag{28}$$

$$\text{When } \zeta = 0,\ \mathfrak{F}_{sm} = 2\mathfrak{F}_m\,(1 - \gamma\delta^2) + \epsilon \tag{29}$$

Eliminating γ between these equations, and rearranging

$$2\mathfrak{F}_m + \epsilon = \mathfrak{F}_{sm} + (\mathfrak{F}_{sm} - \mathfrak{F}_s) = \mathfrak{F}_{sm} + \Delta\mathfrak{F} \tag{30}$$

The value of $\Delta\mathfrak{F}$ is given by the distance *ab* on the curve of figure 9. The correction $\Delta\mathfrak{F}$ is the amount that must be added to the maximum value of the measured force to obtain the value that would have been measured had the fixed coils been correctly spaced.

The reasoning used above can be applied when the two fixed coils are not exactly coaxial. In this case, the sum of the forces is a minimum when the axis of the moving coil lies midway between the axes of the fixed coils, so that the sign of the correction terms in the force equations is positive instead of negative as given in equations 22 and 23 and hence the sign of $\Delta\mathfrak{F}$ is negative instead of positive. The displacement of the fixed coils from the coaxial position was resolved along two horizontal axes, and corrections applied for each of these axes. The method of finding the amount of displacement and the correction for this displacment is the same along each horizontal axis as the method described for the distance between the coils, i.e., the vertical axis.

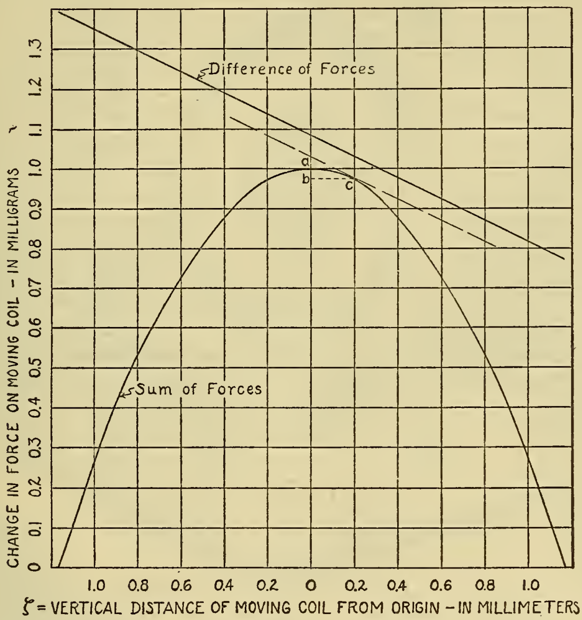

FIGURE 9.—*Variation of the sum and difference of the forces with vertical displacement of the moving coil. Vertex of the parabola is a, point of tangency of line is c.*

The experimental curve for the sum of the forces was taken with coils S1, S2, M2 with a current of 0.678 amperes, giving a total doubled force of 5.29 grams. The line for the difference of the forces was drawn arbitrarily, as the actual line corresponding to the difference of the forces was too nearly horizontal to make a suitable illustration.

The plotted data would be interpreted as follows: The length *bc* gives the distance, δ (0.2 mm), which each fixed coil would have to be moved to attain perfect spacing and the length *ab* gives the correction, $\Delta\mathfrak{F}$ (0.03 mg), to the doubled force for the assumed incorrect spacing.

V. DETERMINATION OF THE RATIO OF THE RADII OF THE COILS

The theory of the current balance shows that the ratio of the radius of the moving coil to that of each of the fixed coils must be known in order to compute the current from the measured force. These ratios, while very difficult to determine by mechanical measurements, may be very precisely determined by a relatively simple electromagnetic method. This method is based on the fact that the magnetic field at the center of a circular turn of wire is directly proportional to the current in the wire, is inversely proportional to the radius of the circle, and is in a direction perpendicular to the plane of the circle. If two circular turns are coplanar and concentric, and carry currents flowing in opposite directions, the resultant magnetic field at their common center is zero for a particular value of the ratio of the two currents. When the common plane of the two circular turns is vertical, and in the magnetic meridian, a small magnet hung at the common center of the turns will be undeflected when the field resulting from the currents is zero. With zero field at the center, the ratio of the currents is equal to the ratio of the radii of the turns.

If each circular turn is replaced by a coil having a number of turns and a relatively small cross section, the above procedure may be used to obtain the ratio of the effective mean radii of the two coils. This ratio of the radii is not identical with the ratio of the geometric mean radii but is more directly applicable to the current balance. Since only the magnetic effects of the coil are under consideration, the features of the coil must be defined in terms of the magnetic field rather than from geometric considerations. Hence the plane of the coil is a plane so located that the magnetic intensity is symmetrical with respect to it. The axis of the coil is that straight line perpendicular to the plane of the coil in which the magnetic field at every point has the same direction as the line. The center of a coil is the point where its axis cuts its plane. It is apparent that the plane, axis, and center of a coil, as defined above, will coincide with the same features as geometrically defined, provided the coil is a perfect one.

As a first approximation, when the resulting magnetic field at the common center of two coplanar and concentric coils is zero

$$\alpha \equiv \frac{a_2}{a_1} = \frac{n_2 I_2}{n_1 I_1} \tag{31}$$

where α = the ratio of the effective mean radii

a_1, n_1, I_1 = the radius, number of turns, and current respectively for the larger coil

a_2, n_2, I_2 = the same quantities for the smaller coil.

The ratio of two unvarying currents can be measured with an accuracy of a part in a million, and the number of turns determined with about the same accuracy. However, corrections must be made for the finite cross sections of the coils.

1. OUTLINE OF METHOD [29]

A large coil and a small coil were mounted as nearly concentric and coplanar as feasible, with their common plane vertical and approximately in the earth's magnetic meridian. A small magnet was suspended at the common center. A suitable current was sent through one coil, and the current in the other was varied until the magnetic needle showed no deflection. The ratio of the currents was then measured. The position of one coil relative to the magnet was then changed by a measured amount, the current varied, and the ratio of currents again measured. After three or more such readings, corresponding to a particular type of adjustment, had been obtained, a curve was plotted with positions of the coil as abscissas and ratios of currents as ordinates. The curve was a portion of a parabola, and

[29] The general method was described by Bosscha, Pogg. Ann. vol. 93, p. 392, 1854.

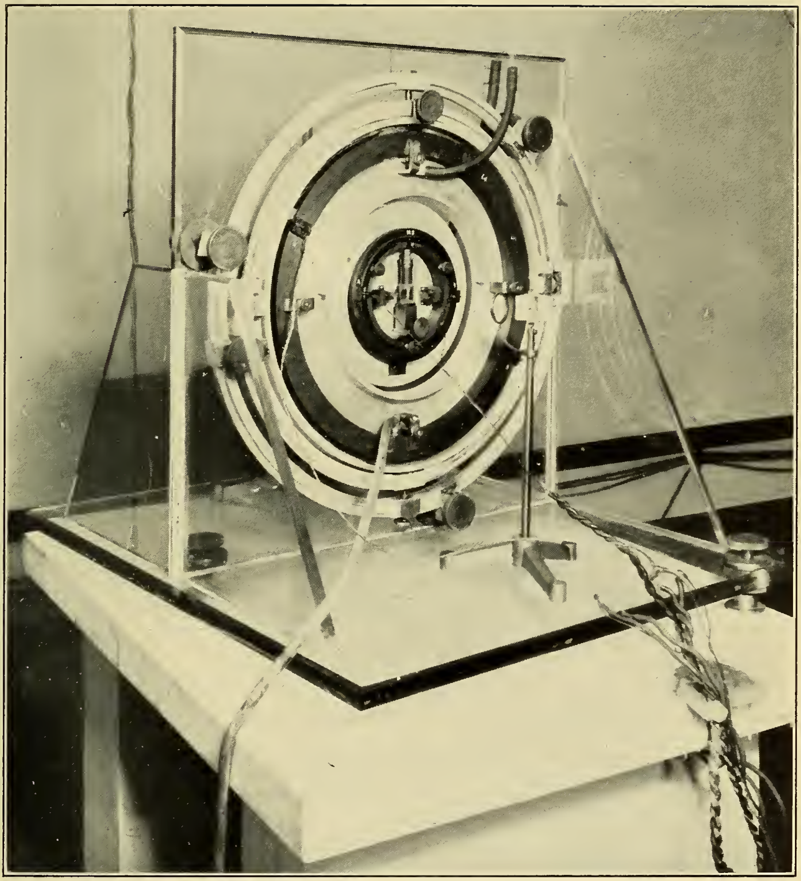

FIGURE 10.—*Photograph showing the way in which two coils were mounted when measuring the ratio of their radii.*

The large coil was held in a double gimbal system by which it could be independently given two rotations and three translations. The small coil was held in a single gimbal by which it could be independently given one rotation and one translation. The magnet could be given one rotation and two translations. The coils could be placed in magnetic meridian by turning the entire stand.

The mountings of the coils and all the connections to them, as well as the tripod stand and all of the optical parts carried by it, were tested by the astatic magnetometer and each piece was accepted only if its magnetic susceptibility differed but little from unity.

the abscissa for maximum or minimum ordinate indicated the correct position for this adjustment. The coil was then placed at the position indicated by the maximum or minimum and a second adjustment made. This procedure was continued for all necessary translations and rotations of each coil. The maximum or minimum point of the final adjustment indicated the point at which the coils were exactly coplanar and concentric with the magnet at the center. Then the product of the ratio of the currents and the ratio of the number of turns gives a first approximation to the ratio of the effective mean radii of the coils. Corrections could then be made for the finite length of the needle and for the cross sections of the coils.

2. MOUNTING OF THE COILS

The stand for holding the coils was constructed of plate glass which was chosen because it was nonmagnetic, could be procured in large pieces, and could be readily cemented together. The large coil was mounted on the glass upright in such a manner that the coil could be given 3 translations and 2 rotations by means of a double gimbal made of an aluminum alloy. The small coil was mounted on an aluminum disk rigidly attached to the glass upright by means of a single brass gimbal in such a way that the coil could be given 1 translation and 1 rotation. All the materials used in connection with the mounting of the coils were tested for their magnetic properties by an astatic magnetometer. (See p. 710.) A photograph of the assembled stand is shown in figure 10.

3. MAGNETOMETER

The magnetometer consisted of an outer case, a suspended system, a fiber for suspending this system, and adjustment screws for raising and lowering the suspended system and for turning it around a vertical axis. The magnetometer was mounted on a slide so attached to the plate glass that the entire magnetometer could be moved in the direction perpendicular to the upright glass plate.

The magnetometer case consisted of a tube of nonmagnetic brass. The tube was closed at the lower end and had a window made of glass having optically flat faces. The moving system consisted of a small glass rod to which was attached a small magnet, a small mirror, and the wing of an insect to act as a damping vane. All parts of the moving system were made with as small a moment of inertia as possible, in order that the period would be short. The size of the damping vane was adjusted until the moving system was slightly under-damped. A quartz fiber was used for suspending the moving system.

4. OPTICAL SYSTEM

The optical system consisted of the conventional lamp and scale in which the light was reflected from the small mirror of the moving system of the magnetometer to a scale 3.7 meters from the mirror. The mirror was mounted in the same plane as the magnet, and the light source was ordinarily placed along the axis of the coils. In some cases readings were required with the magnet deflected about 30° from the plane of the coils. For such readings a second light source was employed, the beam of which made an angle of about 60° with the axis.

5. ELECTRICAL CONNECTIONS

The electrical connections used in determining the ratio of the radii are shown in figure 11. A storage battery controlled by a single switch supplied currents to two parallel circuits, each having a rheostat for adjusting the current and a switch for reversing it. The 2 reversing switches and the battery switch were mechanically connected so that 1 switch handle operated them all. The battery switch opened first and closed last so that the currents in the two coils were broken and made simultaneously. An adjustable iron-cored

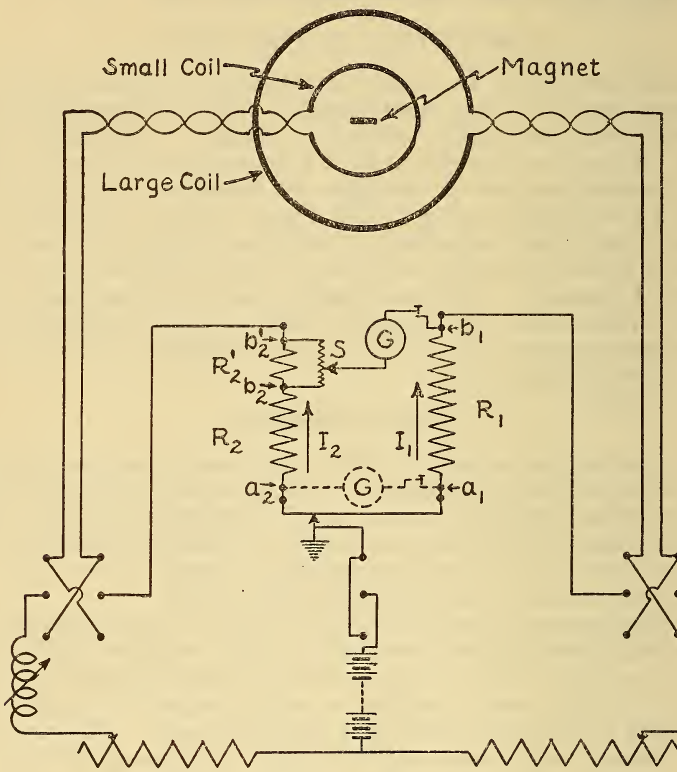

FIGURE 11.—*Electrical circuit used in measuring the ratio of the radii of the coils.*

The currents in the two coils were adjusted until their resultant magnetic field at the common center was zero, as indicated by the zero deflection of the small magnet at the center. The ratio of the two currents could be measured by the circuit shown and the ratio of the radii computed from the ratio of the currents. The two reversing switches and the battery switch were connected together mechanically so that the currents could be quickly reversed.

inductor in the small-coil circuit made the time constants of the two circuits nearly the same. This inductor was placed so far from the coils that its magnetic field did not deflect the magnet by a readable amount. Without this inductor, the magnet received a large deflection when the circuit was opened or closed; with it, the swing was reduced to a few degrees.

Observations were made with the currents in both directions in order to eliminate the effect of slow changes in the zero position of the magnet. In order to measure the ratio of radii with an accuracy of one part in a million, the deflection of the magnet, with 1 ampere in the small coil, must be observed to 70 microradians or 0.5 mm on the scale.

The zero position of the magnet must not have changed more than this amount during the 10-second interval necessary to make two readings, between which the currents in both coils were reversed. The zero position of the magnet depended almost entirely on the direction of the horizontal component of the external magnetic field, changes in which were caused either by variations in the earth's field or by fluctuations in the current in outside circuits. The most troublesome of the latter was a street-car line which was 300 meters distant. When this line was in operation, the zero of the magnet was not sufficiently steady. Consequently, all ratio-of-radii adjustments and measurements had to be made between the hours of 2:30 and 4:30 a.m., when the street cars were not running. It was not always possible to work even at these hours on account of magnetic storms. While this work was in progress, the daily reports on terrestrial magnetic conditions were received from the Cheltenham Magnetic Observatory and from Science Service; the reports reaching this laboratory several days after the observations were made. It was usually possible to correlate a poor set of results on the ratio of the radii with poor magnetic conditions, thus justifying their rejection.

The ratio of the currents was determined by observing that the potential drop over a resistance carrying a current I_1, was the same as that over another resistance carrying a current I_2. Then the ratio of the currents was equal to the inverse ratio of the two resistances. The arrangement of circuits for making the measurement is shown in figure 11. The battery current divided into the two currents I_1 and I_2 at a terminal which could be adjusted until the potential terminals a_1 and a_2 of the standard resistors were at the same potential. This adjustment was not appreciably altered by other changes in the circuit, so that it had to be made only once or twice during the measurements on a pair of coils. The galvanometer connection could be moved along the slide wire, S, to make the potential of some point between b_2 and b_2' the same as that of the terminal b_1. If the slide wire has s divisions, and if the reading of the slide wire is x, the ratio of currents is given with sufficient accuracy, with the resistances used, by the equation

$$I_1/I_2 = (R_2 + \frac{x}{s} R_2')/R_1 \qquad (32)$$

where R_2 is the resistance between a_2 and b_2; R_2' that between b_2 and b_2', and R_1 that between a_1 and b_1.

The resistor (R_2+R_2') in series with the small coil was usually a 1-ohm standard and carried 1 ampere. A potential tap was taken off at 0.002 ohm and this part of the resistance was shunted by a Kohlrausch slide wire, S, having a resistance of 7.6 ohms. The slide wire had 1,000 divisions, each corresponding to 0.000002 ohm change in the resultant resistance of the resistor so that one half division on the slide wire corresponded to a part in a million in the ratio of the currents.

The standard resistors for use as R_1 and R_2 were constructed from manganin strip in the same manner as those already described for use with the current balance. They were frequently compared with the primary resistance standards of the Bureau, generally before and after use. They were mounted in an oil bath, which was maintained at or near 25° C; the temperature at which they were tested. However, the current used in determining the ratio of the radii was much larger than the current used in testing. The larger current generated more heat in the resistance material and hence caused a greater temperature difference between this material and the surrounding oil. By means of the load coefficient of the resistance, a correction was applied to the value of the resistance obtained when it was tested to obtain the value of the resistance as used. The load coefficients of the resistors often introduced larger corrections than the temperature coefficients.

A moving-coil galvanometer having a low external critical damping resistance (6 ohms) was used in obtaining the setting of the slidewire. It was mounted so that both the galvanometer and the magnetometer could be read on the same scale. The sensitivity of the galvanometer, when critically damped, was 10 milliradians per microvolt, which gave a deflection of 3 centimeters per microvolt at the scale distance of 3 meters. In order to measure the ratio of the currents to one part in a million, the spot of light from the galvanometer would have to be adjusted only to within 3 centimeters of the correct position.

6. PROCEDURE IN TAKING OBSERVATIONS

In order that thermal conditions would be steady, the current was passed through the coils for several hours before observations were begun. The room was heated electrically, and thermostatically controlled to within 0.2° C. This was necessary because there was no other temperature control for the small coil. Thermostated water, circulating through the large coil, controlled its temperature. Fluctuations in battery voltage did not affect the ratio of currents.

The procedure in making an observation was as follows: The battery switch was momentarily opened to permit the observation of the zero position of the magnetometer; with the switch closed, the current in one of the coils (usually the smaller) was varied until the reading of the magnetometer was the same as the zero position; and almost simultaneously with the variation of the current, the contact on the slidewire was moved until the closing of the galvanometer key did not produce a deflection of the galvanometer. After the reading of the slidewire had been recorded the currents were reversed and the process was repeated. This procedure constituted one observation and took about 10 seconds. Several observations were made for each step in the adjustment of a coil to its correct position, and after the coils had been completely adjusted, about 10 observations were made for the purpose of determining the value of the ratio of the radii. The mean deviation from the mean for 10 observations was seldom as great as one part in a million. The whole arrangement was very sensitive and easily operated by one observer.

7. ADJUSTMENT OF THE COILS AND MAGNET

The adjustment of the coils and magnet was for the purpose of making the coils vertical, coplanar, and concentric, and of placing the magnet with its center at the common center of the coils and with its axis in the common plane of the coils. The complete adjustment was accomplished by translating and rotating the coils and magnet with respect to a chosen coordinate system and by determining for each translation or rotation (called an individual adjustment) the position at which the ratio of currents was a maximum or minimum, in the manner already described. To determine the number of individual adjustments that must be made, each coil and the magnet may be considered as a rigid body. Since every rigid body has 6 degrees of freedom (3 rotations and 3 translations), the system of 2 coils and a magnet has 18 possible degrees of freedom with respect to a fixed system of coordinates. But the magnetic field of a coil is symmetrical about its axis so that from a magnetic point of view each of the coils has only 5 degrees of freedom. Hence, if the coordinate system is fixed in the magnet, it is evident that only 10 individual adjustments are required for a complete adjustment. This arrangement of coordinates is useful to show the minimum number of individual adjustments that must be made, but is not as suitable as some others for actual application. The most convenient arrangement of coordinates is that in which the origin is at the center of the magnet, one axis vertical, one axis horizontal and parallel to the plane of the small coil, and the third axis perpendicular to the other two. The 10 adjustments relative to these axes are given below:

1. Translation of the *small coil* in a direction perpendicular to its plane until its center lies in a vertical plane through the center of the magnet.

2. Translation of the *small coil* in a vertical direction until its center and the center of the magnet are in a horizontal line.

3. Translation of the *small coil* in a horizontal direction in its plane until its center coincides with the center of the magnet.

4. Translation of the *large coil* in a direction perpendicular to its plane until its center lies in a vertical plane through the center of the magnet.

5. Translation of the *large coil* in a vertical direction until its center and the center of the magnet are in a horizontal line.

6. Translation of the *large coil* in a horizontal direction in its plane until its center coincides with the center of the magnet.

7. Rotation of the *small coil* about a horizontal diameter until its plane is vertical.

8. Rotation of the *large coil* about a horizontal diameter until its plane is vertical.

9. Rotation of the *large coil* about a vertical diameter until its plane coincides with that of the small coil.

10. Rotation of the *magnet* about a vertical line through its center until its axis coincides with the common plane of the coils.

For maximum sensitivity, the axis of the magnet should approximately coincide with the magnetic meridian. In this position there is no torque in the suspending fiber. Sufficient sensitivity was obtained by placing the plane of the small coil approximately in the magnetic meridian by comparison with a compass needle and turning the magnet to coincide approximately with this plane. The latter was accomplished by sending a current of a few milliamperes through the small coil and adjusting the torsion head of the magnetometer until a reversal of the current produced an equal and opposite deflection of the magnet.

The plate of glass supporting the coils was placed essentially vertical and in the magnetic meridian, and adjustments 1 to 6 were made by moving the coils in directions perpendicular and parallel to that plate, those directions coinciding quite closely with those previously specified in the catalog of the adjustments. In each case the position of the coil at which the ratio of the currents was a maximum or a minimum was determined, and the coil was placed in that position. The apparatus for making the translations was so designed that the small coil could be given only one translation, namely, in a horizontal line in its plane. Hence, for the purpose of making adjustments 1 and 2 which required other translations of the small coil relative to the coordinate system, the large coil and the magnet were moved as a unit, thus moving the coordinate system and those parts that should remain stationary with respect to it, rather than moving the small coil.

The rotational adjustment of the coils to make their planes vertical (adjustments 7 and 8) was accomplished as follows: With the large coil in a definite position, the ratio of the currents giving zero deflection of the magnet was determined for each of several positions of the small coil as it was rotated about a horizontal diameter. The plane of the small coil was made vertical by placing the coil in the position in which the ratio of the current in the small coil to that in the large coil was a minimum. With the small coil vertical, similar observations were made when the large coil was rotated around a horizontal diameter. In this way, the planes of the two coils were made vertical.

To secure data for adjusting the large coil so that its plane coincided with that of the small coil (adjustment 9), the ratio of the currents which produced zero *deflection* of the magnet was measured for two or more displaced zero *positions* of the magnet. The displacement of the zero position of the magnet was usually accomplished by employing the magnetic field of an auxiliary coil, the axis of which approximately coincided with those of the other coils and the center of which was at a distance of 50 cm or more from the center of the coils. By reversing the current in this auxiliary coil, two displaced zero positions of the magnet were obtained which were about equally spaced (usually 2 centiradians, or about 1°) on each side of the initial zero position. By plotting, as ordinates, the slide-wire readings, which were proportional to the ratio of the currents, and, as abscissas, the displaced zero positions of the magnet as read on the scale, a line was obtained, the slope of which was proportional to the angle between the two coils.

Data for determining the proportionality factor between the slope of the plotted line and the angle between the coils was obtained by first turning the large coil around a vertical diameter by a known angle of 1 or 2 centiradians and then making a second set of measurements of the ratio of the currents at two or more displaced zero positions of the magnet. The results were plotted on the same sheet as the first set. The dashed lines in figure 12 show a typical plot of the first two sets of measurements made in the adjustment of a pair of coils. Since, for each line, the slope was proportional to the angle between the coils, the difference in the slopes of the two lines was proportional to the angle through which the large coil was turned, so that the angle required to turn the large coil to make the two coils parallel was determined by proportion.

The rotation of the magnet to place its axis in the plane of the coils (adjustment 10) would have been unnecessary if the preceding adjustment of the large coil into the plane of the small coil could have been made perfectly. Since this ideal condition could not be

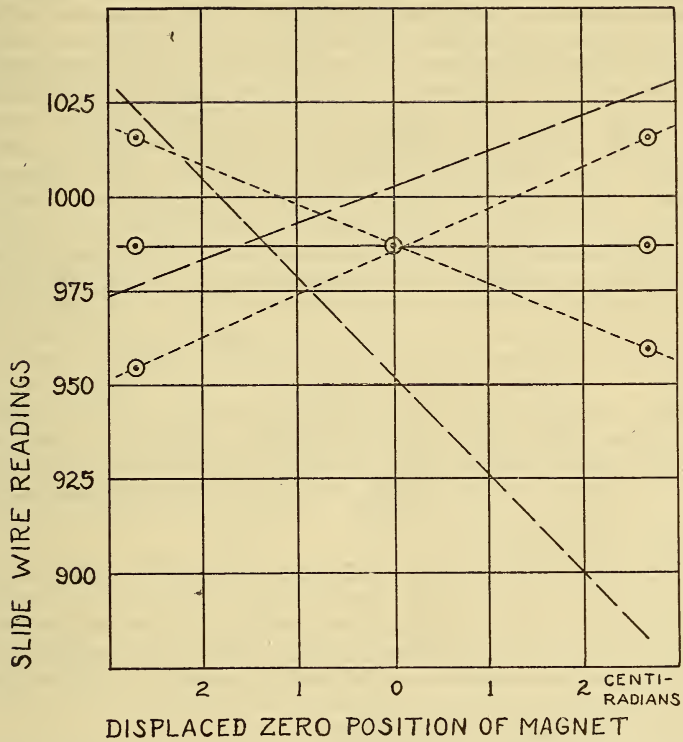

Figure 12.—*Data used in turning the plane of the large coil and the axis of the magnet into the plane of the small coil*

The ordinates are slide-wire divisions, each of which is approximately equal to a part in a million in the ratio of the currents. The abscissas are angular positions of the magnet which are measured from the initial zero position. The angle between any line and the horizontal is proportional to the angle between the coils at the time the observations used in plotting the line were made. The horizontal distance between zero and the intersection of two lines, plotted from data taken with the same initial zero of the magnet but with different angles between the coils, is proportional to the angle between the axis of the undeflected magnet and the plane that bisects the angle between the two positions of the plane of the fixed coil.

The dashed lines represent the first set of measurements for determining the angle between the coils and the angular position of the magnet. The dotted lines are from actual data taken to check the accuracy attained in adjusting the magnet. The coils used in this case were S2 and M3 and the angle between them was 1 milliradian (3.6′). The solid line is from actual data taken on the same coils to check the parallelism after adjustment.

attained, an adjustment of the magnet was always made, for the effect of an error in the adjustment of the planes of the coils to coincidence was minimized if the axis of the magnet was ajdusted nearly to coincide with the plane of the large coil. Such coincidence could not be satisfactorily secured by the approximate method used in setting the magnet before starting any of the adjustments. It was secured by the following procedure: The large coil was turned through a small angle (1 or 2 centiradians) from the position where it was left in adjustment 9, and the ratio of currents was determined for two or more displaced zero positions of the magnet, as in the previous adjustment. The large coil was then turned by the same amount in the opposite direction. The ratio of currents was again determined for two or more displaced zero positions of the magnet. The data were then plotted as before with slide-wire readings as ordinates and zero positions of the magnet as abscissas, giving two straight lines. The zero position corresponding to the point where the two lines crossed was the zero position at which the axis of the magnet coincided with the initial plane of the large coil. The angle through which the torsion head had to be turned to bring the magnet to the zero position indicated by the plot was determined by proportion after the change in the zero position of the magnet which was produced by turning the torsion head through a known angle (generally 10 divisions) had been observed. While this adjustment has been described as though data were taken especially for it, actually the data of adjustment 9 were generally used. The dashed lines of figure 12 can be applied both to adjustment 9 and adjustment 10.

As a check to insure that adjustment 10 had been correctly made, two additional sets of observations were made. The angle between the coils had the same value for each set, but opposite signs for the two sets. In each set three zero positions of the magnet were used, the initial one with the axis parallel to the plane of the small coil, the other two with displaced zero positions which were produced by a current in the auxiliary coil. If the adjustment had been correctly made, the plot of the data gave two lines which intersected at the initial zero position of the magnet. Such sets of data are plotted in the dotted lines of figure 12.

As a final check on adjustment 9, the planes of the coils were again made coincident, and a set of observations was made on the ratio of the currents with the new initial zero position of the magnet and two displaced zero positions. If adjustment 9 had been correctly made, the ratios of the currents were the same (within experimental error) for all three zero positions. The data obtained in such a check are plotted in the solid line of figure 12.

The above methods do not exactly conform to the conditions stated in describing the necessary adjustments. Hence, after the first series of 10 adjustments had been completed, an entirely new series was made to obtain a closer approximation to the theoretically correct position. In the second series, the positions of the coils and of the magnet were not changed from those in which they were placed by the first adjustment, except as required in making each individual adjustment. If, in the second series, any adjustment was changed by an appreciable amount, a third series was made. After the coils were in perfect adjustment, several careful determinations of the ratio of the currents which produced no deflection of the magnet were made. Since this ratio of currents was used to compute the ratio of the radii of the coils, all necessary precautions were taken to insure an accurate value of the ratio of the currents. This required a procedure somewhat different from that used in the adjustment of the coils, since in those measurements only changes in the ratio of the currents were required. The various measurements of the ratio of the currents for each pair of coils, when in perfect adjustment, were made at different times, but always with the external conditions approximately those under which the coils would be used in the current balance. Some measurements were usually made just after the complete adjustment had been finished, then additional ones both after the measurement of the temperature coefficient and of the load coefficient and still others after the length of the magnet had been determined.

Under favorable conditions about an hour was required to make an individual adjustment. Since it was possible to work only 2 hours a day, each series of 10 adjustments required about a week. A determination of the ratio of the radii of two coils, consisting of 2 or 3 series of individual adjustments and several final determinations of the ratio of the currents, took at least 2 weeks.

8. THEORY OF THE ADJUSTMENT OF THE COILS

The theory of each of the individual adjustments of the coils depends on the change which that adjustment produces in the torque on the magnet, and hence on the horizontal component of the magnetic field perpendicular to the axis of the magnet. The magnetic field of a coil along its axis is expressed with sufficient approximation by the equation

$$H_z = \frac{2\pi n I}{a}\left[1 - \frac{3}{2}\frac{z^2}{a^2} + \quad . \quad . \quad . \quad . \right] \tag{33}$$

Where H_z is the magnetic field at a distance z along the axis from the center of the coil, a is the radius of the coil, n the number of turns, and I the current in each turn. The magnetic field in the plane of the coil, H_x (or H_y) is given by the equation

$$H_x = \frac{2\pi n I}{a}\left[1 + \frac{3}{4}\frac{x^2}{a^2} + \quad . \quad . \quad . \quad . \right] \tag{34}$$

These equations show that the translational adjustments when made with respect to the coordinate axes as described, place the centers of the coils at the center of the magnet. The equations show that an error of a part in a thousand in x/a and z/a will produce an error of only a part in a million in the magnetic field. The smallest coil used had a radius of 10 cm, so that with it a maladjustment of 0.1 mm was permissible. This accuracy was easily attained.

The torque that a coil carrying a current exerts on a magnet suspended at the center of the coil is proportional to the cosine of the angle which the plane of the coil makes with the vertical, and hence is a maximum when the coil is vertical. The adjustment of each coil to be vertical could be made with ample precision, since the cosine does not change appreciably for small angles.

The theory of the adjustment of the coils to have coincident planes by the rotation of one of them around a vertical axis requires that the torques, around a vertical axis, which the magnetic fields of the two coils exert on the magnet shall be equal and opposite. To simplify the equations for the torques, the magnet is assumed to be so short that, at its poles, the magnetic field of each coil is the same as that at its center. The current I_1 in the large coil (radius a_1, number of turns of wire n_1) produces, on the magnet, a torque T_1, which is given by the equation

$$T_1 = \frac{2 m l n_1 I_1 \cos\ (\phi_m - \phi_c)}{a_1} \tag{35}$$

where l is the effective length of the magnet, m its pole strength, ϕ_m the angle which the magnet makes with the plane of the small coil and ϕ_c the angle between the two coils. (See fig. 13.)

A current I_2 in the small coil (radius a_2, number of turns of wire n_2) produces, on the magnet, a torque T_2 which is represented by the equation

$$T_2 = \frac{2mln_2I_2 \cos \phi_m}{a_2} \tag{36}$$

Equating the torques, and rearranging

$$\frac{a_2}{a_1} = \frac{I_2n_2 \cos \phi_m}{I_1n_1 \cos (\phi_m - \phi_c)} \tag{37}$$

It follows from this equation that the ratio of the radii is equal to the ratio of the current-turns if $\cos \phi_m = \cos (\phi_m - \phi_c)$, which is the

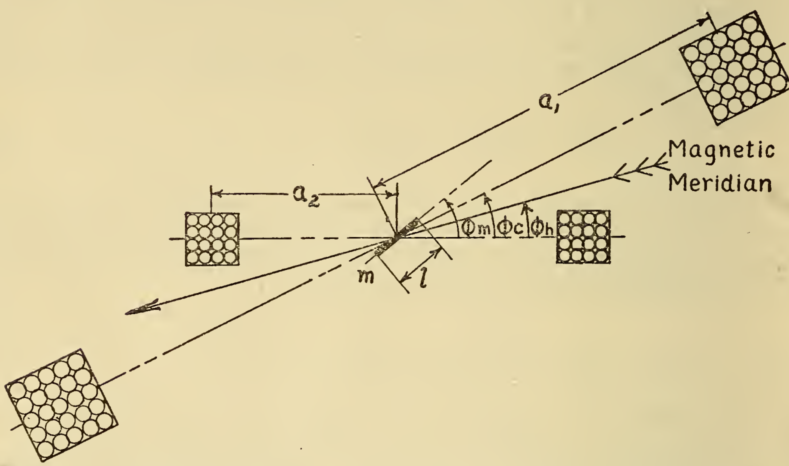

FIGURE 13.—*Horizontal crosssection of the coils and magnet as mounted for measuring the ratio of the radii*

The diagram shows the symbols used in describing the method for adjusting the coils to be coplanar with the axis of the magnet in the common plane.

case when $\phi_c = 0$ (the planes of the coils coincident) or when $\phi_m = \phi_c/2$ (the magnet bisects the angle between the planes).

In general, a small error in setting ϕ_c to zero is likely to occur. However, equation 37 shows that the effect of this error may be minimized by making ϕ_m small. For example, if $\phi_m = 0$, a value of ϕ_c of 1.4 milliradians (5′) introduces an error in the ratio of the radii of one part in a million but if $\phi_m = 2$ centiradians (1°) then this same value of ϕ_c introduces an error of 29 parts in a million.

Equation 37 can be written as

$$\frac{I_2}{I_1} = \frac{n_1a_2}{n_2a_1} (\cos \phi_c + \sin \phi_c \tan \phi_m) \tag{38}$$

Considering ϕ_c as a constant, this is a linear equation between I_2/I_1 and $\tan \phi_m$, in which $\sin \phi_c$, and hence for small angles ϕ_c, is proportional to the slope of the line. This shows the validity of the method employed to adjust the planes of the coils to coincidence.

The equations connected with the above-described adjustment for placing the magnet in the plane of the coils can be obtained from equation 38 by assuming that after the completion of adjustment 9, there is a very small angle, ϵ, between the planes of the coils so that instead of $\phi_c = 0$, $\phi_c = \epsilon$. Then with the large coil turned from this position by $+\theta$, equation 38 may be written in the form

$$\frac{I_2}{I_1} = \frac{n_1 a_2}{n_2 a_1} \left\{ \cos\,(\epsilon + \theta) + \sin\,(\epsilon + \theta)\,\tan\,\phi_m \right\} \tag{39}$$

Likewise, when the large coil is turned through an angle $-\theta$

$$\frac{I_2}{I_1} = \frac{n_1 a_2}{n_2 a_1} \left\{ \cos\,(\epsilon - \theta) + \sin\,(\epsilon - \theta)\,\tan\,\phi_m \right\} \tag{40}$$

These equations can be solved to determine the point of intersection of the lines which they represent. Equating the right-hand sides of the equations 39 and 40 and simplifying

$$\tan\,\epsilon = \tan\,\phi_m \tag{41}$$

Hence, when the magnet is adjusted by this method, it lies in the plane of the adjusted large coil and makes an angle ϵ with the plane of the small coil. It follows that equation 37 becomes

$$\frac{a_2}{a_1} = \frac{I_2 n_2 \cos\,\epsilon}{I_1 n_1} \tag{42}$$

However, ϵ is a very small angle, much smaller than the allowable value of the angle ϕ_m. It follows in making this adjustment that ϵ may be considered as zero. In fact, changes in the earth's field may at any time change ϕ_m by an angle larger than ϵ, but such changes were not sufficiently large to affect the value obtained for the ratio of the radii.

The value of ϕ_m was measured by readings on the magnetometer scale. By turning the torsion head through x divisions, the reading on the magnetometer scale was changed by y divisions. Hence, the number of divisions through which the torsion head had to be turned to make the axis of the magnet coincide with the plane of the large coil was $\phi_m x/y$.

9. TEMPERATURE COEFFICIENT OF EXPANSION

The termperature coefficient of expansion was determined experimentally for each coil, since so many elements entered that it could not be computed from the coefficients of expansion of the component parts. The method employed was to measure the ratio of the radii of two coils, one of which was kept at a constant temperature, while that of the other was varied. In the case of the large coils, the change in temperature was accomplished by changing the temperature of the water circulating through the coil forms. The temperature change was usually from about 7° C above to 5° C below the usual room temperature of 22 C. In the case of the small coils, the temperature was changed by changing the temperature of the air in the room. In each case the temperature of a coil was determined by measuring the resistance of the windings and assuming that the temperature coefficient of resistance of the copper winding was 0.0039 per degree centigrade. The results of these measurements, as well as those obtained in 1911, are given in table 3.

TABLE 3.—*Temperature coefficient of expansion of the coils*

[Values expressed as relative change in radius per degree centigrade]

Coil	1911 values	1933 values (average)	Maximum variation in 1933 values	Number of determinations in 1933
M2	20.3×10^{-6}	18.9×10^{-6}	0.2×10^{-6}	2
M3	19.7	16.4	.5	7
S1	17.0	17.0	.1	2
S2	17.6	17.4	.2	2
L3	18.9	17.6	.2	2
L4	18.5	18.9	.3	2

In a few cases the difference between the 1911 values and the corresponding 1933 values are somewhat larger than the experimental errors of the determinations.

The coils were somewhat constrained by their mountings, so that their expansion might have been influenced thereby. To test this effect, a special mounting was made for coil M3 using the hole-slot-plane system so that the coil could expand without constraint. The temperature coefficient with this special mounting was the same as with the regular mounting.

10. LOAD COEFFICIENT OF EXPANSION

The load coefficient of expansion of a coil is the relative change in radius for unit change in load (a power of 1 watt continuously transformed into heat) the temperature of the windings being the same in both cases. The load correction is required because, in most coils, the radius of the windings is, to some extent, affected by the radius of the form. If the wire of the windings is under tension, and if the temperature of the winding is higher than the temperature of the form, an increase in the temperature of the form only to make it equal to that of the winding will increase the tension in the winding and increase its radius. The load on a coil causes the temperature of the windings to be higher than the temperature of the form. Since the temperature of the winding only is determined by measuring its resistance, the radius of a coil which is carrying current and which has tension in the wire of the winding is less than would be the case if the form were at the temperature of the winding. Hence the correct radius of the winding is obtained by subtracting from the radius, as determined when the winding and form have the same temperature, a quantity which is proportional to the load that is converted into heat in the windings. The proportionality factor by which the load must be multiplied is called the load coefficient.

The method of measuring the load coefficient made use of the fact that each coil had 2 identical windings. When the 2 windings of a coil carried equal currents but in opposite directions, there was no magnetic field at the center of the coil, whereas if the 2 currents were in the same direction, the magnetic field was twice that for a single winding. Accordingly, a circuit was arranged so that the current ordinarily used in measuring the ratio of radii flowed as usual, i.e., produced a field at the center of the coil, while a special heating current flowed through the windings in such a way as to produce no magnetic field at the center. The circuit in figure 14, which is a Wheatstone bridge circuit, shows the way in which this was accomplished. In setting up this arrangement a galvanometer was first connected in place of the heating battery and the circuit balanced as an ordinary Wheatstone bridge by varying either of the resistances P or Q (about 12 ohms each). Later a better balance was obtained with the magnetometer itself, this being done by disconnecting the main battery and adjusting the resistance P (or Q) until the magnetometer zero was not changed when the heating battery was connected. The resistances R_a and R_b (usually 1 or 5 ohms) were inserted so that the current through each winding (and hence the total load) could be determined by measuring the fall in potential over them. Temperature measurements of each winding were made by comparing the fall in potential over the windings with the fall in potential over the resistance standards carrying the same current. Care was taken to have the heating battery well insulated from the main battery.

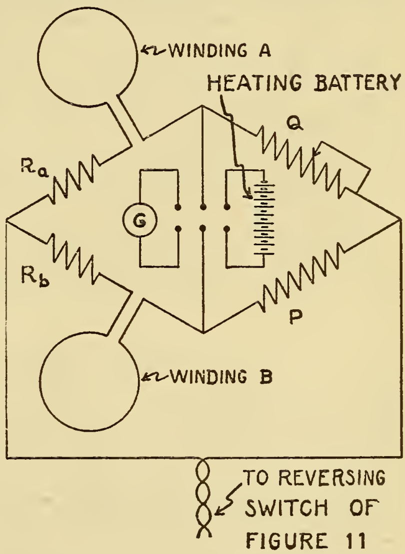

FIGURE 14.—*Auxiliary electrical circuit used for measuring the load coefficient of a coil.*

The auxiliary circuit was used in connection with the measurement of the ratio of the radii and was substituted for the regular circuit of the coil being studied at the reversing switch shown in figure 11. The 2 windings A and B are shown separately, whereas in figure 11 they are considered as a single winding. The total load on the windings was varied by changing the voltage of the heating battery.

The currents in the 2 windings were not the same during these measurements and hence the 2 windings carried different loads. In the special case where the voltage of the heating battery was adjusted so that the heating current and the measuring current were equal, 1 winding carried all the current and the other carried none. Careful temperature measurements on the 2 windings when there was a considerable difference in load gave no indication that 1 winding was at a different temperature from the other. The total load on the coil was always taken as the sum of the loads of the 2 windings.

In making a determination of the ratio of the radii, the temperature and load of 1 coil was kept constant, while the heating current on the other coil was varied from zero to the maximum permissible. The observed relative increase in diameter when the load was increased was subtracted from the relative increase which would have occurred if the form had also been raised to the temperature of the winding (this latter was computed from the temperature coefficient) and the remainder was divided by the load in watts to obtain the load coefficient.

For the large coils the load coefficients were small, and could be determined with sufficient accuracy by this method although the normal loads on them when used in the current balance were large (20 to 50 watts) so that the load corrections were appreciable. For the small coils the load coefficients were large so that the load corrections were appreciable although the normal loads when used in the current balance were small (1 or 2 watts). Moreover, the determination of the load coefficients was difficult, since the normal loads when measuring the ratio of the radii were nearly as large as the coils could safely carry. The above-described method was used with the small coils, but a more convenient method of obtaining the load coefficients of the small coils was to measure the ratio of the radii in the regular manner, but with various measuring currents. In this way the load on both coils was changed, but since the load coefficient of the large coil was known, the one for the small coil could be computed. The value of the load coefficient of each coil is given in table 4, together with the value obtained in 1911.

TABLE 4.—*Load coefficients of the coils*

Coil	1911 value	1933		
		Value	Average deviation from mean	Number of determinations
M2	2.9×10^{-6}	3.3×10^{-6}	0.1	2
M3	8.2	[a]1.0	1.0	6
S1	.47	[b].61	.11	3
S2	.93	[b].52	.16	5
L3	.81	[b].62	.14	3
L4	.84	[b].66	.26	6

[a] The values of the load coefficient of coil M3 had wide variations which were partly caused by the large uncertainty in the temperature coefficient of this coil. (See table 3.)
Both the temperature coefficient and the load coefficient of this coil have decreased since the 1911 measurement. This can be explained by assuming that the winding strains have been relieved.
[b] The load coefficients of the large coils were measured with about 1.2 liters of water per minute flowing through them. No measurements were made to determine whether the rate of flow affected the load coefficient, but this same rate of flow was used when the coils were mounted in the current balance.

In the case of the moving coils, where the load coefficient was the most uncertain, the effect of an error in it on the absolute value of the current was reduced by using the same value of the current in the measuring circuit of the ratio of the radii set-up as was later used in the current balance circuit.

The temperature coefficient and the load coefficient are related to the radius of a coil by the following equation:

$$a_0=a\{1-\tau\ (t-22\ \mathrm{C})+\psi p\} \tag{43}$$

where

τ = temperature coefficient
ψ = load coefficient
p = load in watts
t = temperature of winding measured electrically
a_0 = radius of coil at 22 C and no load
a = radius of coil at temperature t and load p

In this equation the standard conditions are taken as 22 C and zero load.

11. CORRECTION FOR THE LENGTH OF THE MAGNET

A correction to the observed ratio of the radii for the length of the magnet was required, because the components of the magnetic fields that were perpendicular to the magnet at its poles and hence which must be equal and opposite in order that the magnet should not be deflected, do not have the same ratio as the magnetic fields at the common center of the coils and magnet. The magnetic field produced by a current in a circular coil can be obtained at any point by differentiating the formula for magnetic potential [30] at that point. Near the center of a coil, the differentiation to give the component which is perpendicular to a line through the center of the coil at a given distance from the center and which lies in a plane that is perpendicular to the plane of the coil results in the equation

$$H_\theta = -\frac{d\Omega}{rd\theta} = \frac{2\pi n\, I \sin\theta}{a}\left[1 - \frac{3}{4}\frac{r^2}{a^2}(5\cos^2\theta - 1) + \cdot\cdot\cdot\right] \tag{44}$$

where H_θ = the component of the magnetic intensity at the point (r, θ) which is at right angles to the line connecting the point (r, θ) to the center of the coil and which lies in a plane that is perpendicular to the plane of the coil
r = the distance of the point from the center of the coil
θ = the angle between the axis of the coil and the line connecting (r, θ) to the center of the coil
a = the radius of the coil
n = the number of turns of wire in the coil
I = the current through the coil

If a magnet, the distance between the poles of which is $2r$, is suspended by a fiber with the center of the magnet at the center of the vertical coil, H_θ is the component of the magnetic field which tends to turn the magnet around the fiber. If the magnetic fields of two concentric and coaxial coils are equal and opposite at the poles of the magnet, it follows from equation 44 that

$$\left(\frac{I_2}{I_1}\right)_\theta = \frac{a_2 n_1}{a_1 n_2}\left[1 + \frac{3r^2}{4}\left(\frac{1}{a_2^2} - \frac{1}{a_1^2}\right)(5\cos^2\theta - 1) - \cdot\cdot\cdot\cdot\cdot\cdot\cdot\right] \tag{45}$$

where $\left(\frac{I_2}{I_1}\right)_\theta$ is the ratio of the currents which produce no deflection of the magnet when it makes an initial angle θ with the common plane of the coils. The second term in the brackets is the correction which must be applied on account of the length of the magnet. If $5\cos^2\theta = 1$, the correction term is zero, but this requires an angle ($\theta = 63°26'$) that is not convenient for regular use. Observations were regularly made with $\theta = 90°$, in which case the ratio of the radii was computed from the ratio of the currents by the equation

$$\left(\frac{I_2}{I_1}\right)_{90°} = \frac{a_2 n_1}{a_1 n_2}\left[1 - \frac{3r^2}{4}\left(\frac{1}{a_2^2} - \frac{1}{a_1^2}\right) - \cdot\cdot\cdot\cdot\cdot\cdot\right] \tag{46}$$

in which the length of the magnet is required. The length of the magnet was determined by measuring the ratio of currents for two

[30] Maxwell, Electricity and Magnetism, 3d ed., vol. 2, p. 334, eq. 8. Oxford University Press.

markedly different values of θ. Substituting the two sets of values in equation 45, subtracting the resulting equations, and solving for r, the following equation is obtained:

$$r^2 = \frac{4}{15}\frac{n_2 a_1}{n_1 a_2}\left\{\frac{a_1^2 a_2^2}{a_1^2 - a_2^2}\left[\left(\frac{I_2}{I_1}\right)_{\theta_1} - \left(\frac{I_2}{I_1}\right)_{\theta_2}\right]\frac{1}{\cos^2\theta_1 - \cos^2\theta_2}\right\} \tag{47}$$

The values of θ_1 and θ_2 were generally about 60° and 90°.

In the above theory, the assumption has been made that the coils were in perfect adjustment. Slight errors in certain of the adjustments had a very pronounced effect on the measured length of the magnet. The length of the magnet was always measured with at least two pairs of coils. Such measurements were a check not only on the length of the magnet but also on the perfection of the adjustment of the coils.

As the value of the correction term increases as the square of the length of the magnet, r, the magnet should be made as short as possible. The smallest magnet used was about 1.4 mm long. With it, a change in the ratio of the currents of a part in a million could be readily measured.

Four different magnets were employed in the course of this investigation. They were made of cobalt magnet steel and given a special heat treatment. All of them were 0.86 mm wide and 0.24 mm thick, and they varied in length from 1.41 mm to 2.95 mm. The effective length (distance between poles) of each of these magnets has been measured by the method outlined above. The ratios of the effective length to the actual length of these magnets are given in table 5. These ratios agree well with those obtained in 1911, when the method was developed.

Table 5 also gives the corrections to the ratios of the radii which had to be made on account of the lengths of the magnets. The advantage in using short magnets is clearly shown. Although the effective length of a small magnet can not be determined with the same relative accuracy as that of the longer magnets, yet the correction for the shorter magnets is so much smaller that it is advantageous to use them in the determinations of the ratio of the radii.

TABLE 5.—*Lengths of the magnets, and the resulting corrections to the measured ratio of the radii for the different pairs of coils*

Actual length	Ratio of effective to actual length	Correction to the measured ratio of radii			
		Pairs S1 M2 and S2 M2	Pairs S1 M3 and S2 M3	Pairs L3 M2 and L4 M2	Pairs L3 M3 and L4 M3
mm					
1.41	0.80±0.014	9×10^{-6}	18×10^{-6}	11×10^{-6}	20×10^{-6}
1.80	.80± .002	15	29	19	33
2.20	.84± .001	25	48	31	54
2.95	.86± .005	47	91	58	101

For three pairs of coils, determinations of the ratio of the radii were made with magnets of different lengths. The results, given in table 6, are affected by all the errors of a determination, since the coils had to be readjusted whenever the magnet was changed. The values used in the final computation were those measured with the shorter magnets, since with them the chance of an error was smaller.

TABLE 6.—*Ratio of radii of coils as measured with magnets of different lengths*

Date	Coils	Length of magnet	Correction	Number of observations in set	Ratio of radii	Average deviation from mean of set
1931		*mm*				
July	M3:S2	2.95	91×10^{-6}	4	[a] 0.5(1.005038)	3×10^{-6}
August	do	2.95	91	4	[a] .5(1.005059)	2
October	do	2.21	48	5	[a] .5(1.005040)	2
February–March	do	1.80	29	8	[b] .5(1.005022)	2
September	do	1.80	29	11	[a] .5(1.005037)	2
November	M3:L4	2.21	54	8	.4(1.003259)	5
1932						
January	do	2.21	54	4	.4(1.003266)	1
March	do	1.41	20	8	.4(1.003258)	1
January	M3:L3	2.21	54	10	.4(1.003444)	2
February	do	1.41	20	8	.4(1.003440)	1

[a] These results were obtained when the temperatures were high, a condition unfavorable for precise electrical measurements. Also, one of the resistance coils was changing erratically with time, making these results somewhat uncertain.
[b] This result was obtained in the winter, soon after the coils were used in the current balance and is the one used in the final computations.

12. CORRECTION FOR THE SECTIONAL DIMENSIONS OF THE COILS

The correction for the sectional dimensions arises from the fact that the magnetic field at the center of a coil carrying a current is not the same as if the current were concentrated in a filament located at the center of the cross section of the coil. The field at the center of a coil of rectangular cross section can be determined by the equation [31]

$$H=\frac{2\pi nI}{a}\left(1-\frac{b^2}{2a^2}+\frac{c^2}{3a^2}+\frac{3b^4}{8a^4}+\frac{c^4}{5a^4}-\frac{b^2c^2}{a^4}+\cdot\cdot\cdot\cdot\cdot\right) \tag{48}$$

$$=\frac{2\pi nI}{a}(1-\Delta)$$

where n = number of turns in the coil
I = current in each turn of the coil
$2b$ = axial dimension of the cross section of the coil
$2c$ = radial dimension of the cross section
a = mean radius of the coil
Δ = correction for reducing the magnetic field of a coil of rectangular cross section to that of a filament having a radius equal to the mean radius of the coil

The values of Δ are given for each coil in table 7, together with the values of a, b, and c, repeated from table 1, as measured when the coils were originally wound.

[31] See p. 314 of paper by Rosa, Dorsey, and Miller, reference footnote 8, or eq. 24 of Snow's paper, reference footnote 25.

TABLE 7.—*Corrections to the magnetic fields at the centers of the coils for their finite cross section*

Coil	a	$2b$	$2c$	Δ [a]
	cm	*cm*	*cm*	
M2	12.499	0.956	0.955	246×10^{-6}
M3	10.030	.997	1.029	359
S1	19.97	1.580	1.528	295
S2	19.96	1.579	1.522	298
L3	25.00	1.969	1.943	273
L4	25.00	1.965	1.925	279

[a] The values of Δ in this table were obtained from table VI of the paper by Rosa, Dorsey, and Miller (reference footnote 8) by subtracting the numbers in the last column from unity.

In deriving equation 48 the assumption has been made that the turns of wire in each layer are equally spaced and that corresponding turns in the different layers lie in planes which are perpendicular to the axis of the coil. The axial dimension of such a coil is equal to the number of turns per layer multiplied by the distance between the centers of the turns. If the wires are equally spaced and if the width of the channel in which they are wound is greater than the product of the number of turns per layer by the diameter of the wires, the axial dimension, $2b$, may be greater or less than the width of the channel in which the coil is wound. In an actual coil it is generally impossible to determine the value that should be taken. However, the corrections for cross section have all been computed on the assumption that the width of the winding channel is equal to the axial dimension of the coil. In the same way the radial dimension, $2c$, is equal to the number of layers multiplied by the distance between the layers. The distance between layers was measured at the time of the winding of the coils, and the average value was used in determining the radial dimension. Hence the radial depth, $2c$, of the coils has been determined as nearly as possible to correspond with the method required by theory.

If the magnetic fields of two coaxial coils are equal and opposite at their common center, the ratio of the radii, α, is given by the equation

$$\alpha \equiv \frac{a_2}{a_1} = \frac{n_2 I_2}{n_1 I_1}(1+\Delta_1-\Delta_2) \tag{49}$$

As the value of Δ for M2 is smaller than for any of the fixed coils, the correction term was always positive when M2 was being used. On the other hand, Δ for M3 is larger than for any of the fixed coils, so with this coil the correction was negative.

13. EFFECT OF THE LEADS

The correction for the leads to the coils was found by disconnecting the windings, short circuiting the leads with drops of solder (see fig. 3), and measuring the magnetometer deflection for normal current through the leads alone. Since the sensitivity of the magnetometer was known, the lead correction could be computed. It was always small, but measurable.

14. SOURCE OF ERROR

The most important sources of error in the determinations of the ratio of the radii were the following: (1) Error caused by maladjustment of the coils; (2) errors in temperature corrections; (3) errors in load corrections; (4) errors in the corrections for sectional dimensions; (5) errors caused by magnetization of the forms; and (6) errors in electrical measurements.

Each of these errors will now be separately discussed, and an estimate made of the effect of each error on the ratio of the radii.

(1) The error in the measured ratio of the radii of any pair of coils caused by their maladjustment was small. The series of individual adjustments was repeated until each individual adjustment agreed with that of the preceding series. After a complete adjustment had been made, a check was obtained in connection with the measurement of the effective length of the magnet. Lack of perfect adjustment probably did not in any case affect the measured value of the ratio of the radii by as much as two parts in a million.

(2) The error in the temperature correction, i.e., the error in reducing the ratio of radii from the measured temperature to the standard temperature, resulted from uncertainties in the measured temperature of the winding and from inaccuracies in the value of the temperature coefficient. The temperature of the winding was obtained by comparing the potential drop across it with the potential drop across a standard of resistance connected in series, and could be measured to 0.02° C. The temperature coefficient was known to about 1 percent. Hence the error in reducing the ratio of the radii to standard temperature for temperature differences less than 7° C (the maximum observed) was probably less than three parts in a million for each pair of coils.

(3) The error in the load correction was caused entirely by uncertainties in the load coefficient which have already been discussed. A consideration of all the factors involved has led the authors to the opinion that the error in the ratio of the radii caused by the load correction was not more than two parts in a million for any pair of coils.

(4) The error in the corrections for determining the radius of the equivalent filament from the cross-sectional dimensions of the coils depended on uncertainties in the values of the depth and the breadth of the coil. A formula for determining the error in the ratio of the radii from the errors in the dimensions of the coils can be obtained by differentiating equation 49. If terms of higher order than the second are neglected, the resulting equation is

$$\frac{\delta\alpha}{\alpha}=\frac{b_1^2}{a_1^2}\frac{\delta b_1}{b_1}-\frac{2}{3}\frac{c_1^2}{a_1^2}\frac{\delta c_1}{c_1}-\frac{b_2^2}{a_2^2}\frac{\delta b_2}{b_2}+\frac{2}{3}\frac{c_2^2}{a_2^2}\frac{\delta c_2}{c_2} \qquad (50)$$

If the error in any cross-sectional dimension is assumed to be 0.001 cm, the resulting error in the ratio of the radii is in no case more than three parts in a million for any pair of coils used in this investigation; for most pairs it is nearer one part in a million.

(5) The error caused by any magnetization of the forms was very difficult to estimate. In 1911 there were used two sets of fixed coils,

which were of the same size, but one of which (L1 L2) was about 100 times as magnetic as the other (L3 L4). The same result was obtained with both sets. The set L1 L2 has not been used in the present investigation.

A simple and convenient instrument for comparing the magnetic properties of the coil forms and of miscellaneous materials is a small astatic magnetometer. Such an instrument having one of the small magnets of the moving system about 2 inches below the other, was arranged with a series of mirrors in the optical system so that the operator could put the material close to one of magnets and at the same time watch the spot of light on the scale. All materials used in the construction of the current balance and the apparatus for determining the ratio of the radii were tested by this instrument. It was often difficult to get brass which would not cause a deflection of a few centimeters, but, by selection, stock brass could always be found that did not give a deflection greater than a few millimeters. This was, somewhat arbitrarily, considered satisfactory since coil L1, which gave a deflection of about 100 cm, had, in 1911, produced a negligible error in the result. A method for the production of non-magnetic brass castings has been developed by the Department of Terrestrial Magnetism of the Carnegie Institution [32] of Washington and has been found to be very useful for this work. The magnetic properties of the materials probably affected the final result by less than a part in a million.

(6) The error caused by the electrical measurements resulted from uncertainties in the values of the constants of the electrical circuit. The resistances (R_1 and R_2 of fig. 11) were frequently measured. Since only the ratio of resistances was needed, the temperature of the resistance standards was not important as they had approximately the same temperature coefficients of resistance and were kept in the same oil bath. The load coefficients of resistance were important since the two resistance standards carried different loads, but these standards were designed to make the load coefficients very small. The error introduced by the use of the calibrated slide wire was always less than a part in a million. No effect of changing the grounding connection of the circuit could ever be detected. Thermal electromotive forces were always small and errors from this source were avoided by using a false galvanometer zero. Insulation resistance was found to be high throughout the circuit. The error in the ratio of the radii resulting from the electrical measurements was estimated to be less than three parts in a million.

15. RESULTS

The results of the measurements of the ratio of the radii can be conveniently shown in the form of tables. Table 8 gives typical detailed results with one set of coils (M2 S1) after all adjustments had been made, and shows how the actual data were taken and reduced. The close agreement of the individual measurements and the small deviations from the mean value show that the observational

[32] Unpublished communication. Method was developed by C. Huff.

errors with this apparatus were small. In fact, individual results were often carried to 1 part in 10 million to get significant differences.

TABLE 8.—*Ratio of radii:* M2:S1

[Typical set of data for the determination of the ratio of the radii]

Date 1932	Average slide wire reading [a]	Ratio of currents [b] I_2/I_1	Coil temperatures ° C		Corrections for temperature		Total correction [c]	Ratio of radii (a_2/a_1) at 22 C and no load [d]	Deviation from mean
			M2	S1	M2	S1			
May 13	833.8±0.5	3.4(1.002225)	23.84	21.58	-35×10^{-6}	-7×10^{-6}	$+20\times10^{-6}$	0.625(1.001427)	-2×10^{-6}
13	833.8±0.7	2225	.83	.58	−35	−7	+20	1427	−2
13	834.4±0.6	2224	.81	.57	−34	−7	+21	1427	−2
14	835.7±0.4	2221	.61	.59	−30	−7	+25	1428	−1
14	836.3±0.7	2220	.63	.58	−31	−7	+24	1426	−3
14	835.7±0.5	2221	.66	.57	−31	−7	+24	1427	−2
15	833.6±0.7	2224	.80	.60	−34	−7	+21	1427	−2
15	834.1±0.5	2224	.81	.60	−34	−7	+21	1427	−2
21	834.3±0.4	2220	.26	.62	−24	−6	+32	1434	+5
21	835.0±0.4	2218	.25	.68	−24	−5	+33	1433	+4
24	839.3±0.6	2208	22.63	.55	−12	−8	+42	1432	+3
24	839.1±0.6	2209	.65	.55	−12	−8	+42	1433	+4
Average of the 12 measurements								0.625(1.001429) 0.6258931	$\pm3\times10^{-6}$

[a] The average of 10 observations. The slidewire, having a resistance of 7.6 ohms, was divided into 1,000 divisions and shunted across a resistance of 0.002097 ohm.

[b] The resistances used had the values 0.997090 ohm and 3.4 (1.001060) ohms with slight corrections for daily variations. The slidewire reading, divided by 1,000, gives the fraction of the 0.002097 ohm resistance to be added to the 0.997090 ohm resistance. This total is divided into the 3.4 (1.001060) ohms resistance to give the ratio of the currents.

[c] In addition to the temperature corrections, the total correction includes the following, in parts in a million: Load S1=−1; load M2=+4; leads =+1; length of magnet=+9; finite section of coils=+49.

[d] The ratio of the radii is obtained by multiplying the ratio of the currents by the ratio of the number of turns on the coils (72/392) and adding the total correction. The result is expressed as the product of the nominal ratio (0.625) times a number differing slightly from unity.

Table 9 gives all the results of the determinations of the ratio of the radii which were made on the eight combinations of coils. The 1911 values are given for comparison.

From the data given in table 9, four independent values of the ratio of the radii of the two moving coils have been computed. The average variation from the mean of these four values indicates the magnitude of the errors in the determinations. As shown in table 10, the average deviation from the mean is about three parts in a million. It can be shown that an error of three parts in a million in the ratio of the radii would have produced, in the most unfavorable case, an error of about

TABLE 9.—*Results of measurements of the ratio of the radii of the coils*

Coils	Ratio of radii of coils			Coils	Ratio of radii of coils		
	1911, corrected	1933, measured	1933, corrected [1]		1911, corrected	1933, measured	1933, corrected [1]
M2:L3	0.5001458	0.5001576	0.5001586	M2:S1		0.6258931	0.6258941
M2:L4	.5000498	.5000714	.5000701	M2:S2		.6261848	.6261844
M3:L3	.4013722	.4013764	.4013758	M3:S1	0.5022938	.5022786	.5022778
M3:L4	.4012952	.4013034	.4013045	M3:S2	.5025200	.5025107	.5025110

[1] The measured values were corrected to accord with the mean value of M2:M3 given in table 10.

four parts in a million in the absolute value of the current, which is much less than the observed inaccuracy. As the measurements of the ratio of the radii were sufficiently accurate for the purpose of this investigation, detailed data have been omitted.

In order to slightly improve the final result of this investigation, the mean ratio of the radii of the two moving coils (see table 10) was assumed to be correct, and the measured values of other ratios were corrected to make them consistent with this mean. These corrected values of the ratio of the radii are given in the last column of table 9, and are the ones used in the computations of the current in absolute measure.

TABLE 10.—*Computed ratio of the radii of the 2 moving coils using each fixed coil as intermediate*

Intermediate fixed coil	Ratio of radii M2:M3	Variation from mean
		Parts per million
S1	1.246107	−3
S2	1.246113	+2
L3	1.246107	−3
L4	1.246117	+5
Mean	1.246111	±3

VI. EXPERIMENTAL DETERMINATION OF THE MAXIMUM FORCE BETWEEN THE COILS

The experimental determination of the maximum force between the coils was accomplished by placing the two fixed coils in their correct positions and measuring the force at a number of positions of the moving coil. The fixed coils were placed by mechanical methods so that they were approximately coaxial and at the proper distance from each other. The axes of the coils were made vertical by leveling the coil forms. After making a set of measurements of the force, the fixed coils were sometimes readusted to reduce the corrections for maladjustment. Measurements of the force on the moving coil were made both with the currents in the two fixed coils in opposite directions and with them in the same direction. In the first case a large force was obtained (called the sum of the forces) and in the second case the resultant force was very small (called the difference of the forces). There would be no force in the second case if the two fixed coils were identical and the coils were in perfect adjustment.

1. MEASUREMENT OF THE FORCE FOR A GIVEN ARRANGEMENT OF THE COILS

For any given arrangement of the coils, the electromagnetic force produced by a current in the coils was compared with the gravitational force on a weight of known mass. A series of determinations of the rest point of the balance was made, but between each determination and the next the current in the fixed coils was reversed and at the same time the weight which counterbalanced the electromagnetic force was added or removed. The series of determinations of the rest point which is required for a measurement of the force is called a *run*. In each run there were made at least 6 rest-point determinations, every one of which was obtained by reading at least 5 turning points of the optical lever attached to the balance beam.

To obtain the force for a given arrangement of coils, the rest points of a run were plotted as ordinates with the times of determinations as abscissas. Normally the plotted points lay on two parallel straight lines, one containing the points obtained when the direction of the current in the fixed coils was such that the weight was placed on the pan of the balance, the other when the current was reversed in the fixed coils and the weight removed. The change in the rest point,

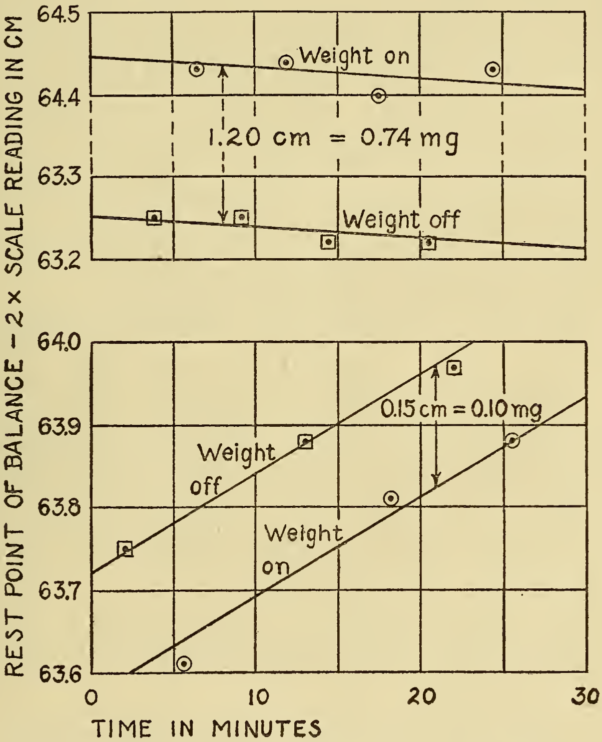

FIGURE 15.—*Data on typical runs with the current balance.*

The upper set, taken May 17, 1932, indicates that 0.74 mg should be added to the value of the weight (8.42986 g when corrected for the buoyancy of the air) to give the doubled electromagnetic force (8.43060 g) on the moving coil. The lower set, taken November 1, 1932, indicates that 0.10 mg should be subtracted from the value of the weight (11.91769 g when corrected for the buoyancy of the air) to give the doubled electromagnetic force (11.91759 g) on the moving coil. The above results are used in table 15.

obtained from the vertical distance between the two lines divided by the change in the rest point produced by one milligram, gave the difference between the electromagnetic force and the force produced by gravity on the mass of the weight which was added and removed from the pan of the balance during the determination. Figure 15 gives the plot of two typical runs.

In the normal case just described, the rate of change in the rest point during a run was generally constant (sometimes zero) as shown by the fact that the plotted points lay on straight lines. In some cases the plotted points did not lie on straight lines. In general, the data obtained in such cases were discarded, but occasionally the points lay on smooth curves, having a constant vertical distance between them. The data obtained in such cases were used.

2. ADJUSTMENT OF THE COILS

It has been shown in section IV that there are 12 important adjustments to be made in placing the coils in the current balance, 6 of which are rotations of the coils and 6 translations. The correctness of the rotational adjustments was determined by means of levels; that of the translational adjustments by electrical methods.

The lower fixed coil was supported on three leveling screws by means of which the coil could be made horizontal with the aid of an accurate level placed on the coil form. The upper fixed coil was held in place by three spacing rods which rested on the lower coil and the whole fixed coil assembly was clamped in position by three clamping screws above the upper coil. The three spacing rods were carefully made to have the same length and to have parallel ends so that, if the lower fixed coil was level, the upper one would be level also. Little trouble was experienced in setting both coils level within 30 seconds of arc, which was about 10 times the accuracy required.

The moving coil could not be leveled in the same way as the fixed coils because the mounting was not rigid enough to permit a level to be placed on the form. A machinist's surface gage was placed on the lower fixed coil, and the moving coil was adjusted until the pointer from the gage touched the moving coil form at corresponding points when placed at several positions around the circumference. This method was not as accurate as the one used for leveling the fixed coils. With care, the adjustment could be made to 0.2 mm or less, which, for the smaller coil, corresponded to an angle of less than 5 minutes. The maximum error of this adjustment produced an error of less than one part in a million in the final result.

In the above it has been assumed that the plane of the winding of each coil was parallel to the plane of the outside of the coil form. Since the coils were carefully and accurately made, this is a reasonable assumption. However, it was tested in 1911 when measurements were made with the fixed coils inverted as well as in two orientations, differing by 120°, and no measurable change in the result was produced by either change. These tests were not repeated in this investigation, since the same coils were used.

The translational adjustments are most easily described by referring them to a system of rectangular coordinates, having the origin midway between the planes of the fixed coils and midway between their vertical axes, and having one coordinate axis vertical, one north and south and the third east and west. The description is also simplified by assuming that the two fixed coils of each combination are exactly alike. The coils used are so nearly alike that this does not affect the conclusions. The moving coil was moved in the direction of each of the coordinate axes, and the sum of the forces for a given current was observed at several positions. For the adjustment along the vertical axis, the force was a maximum when the moving coil was midway between the two fixed coils. For the adjustment along either of the horizontal axes, the force was a minimum when the center of the moving coil coincided with the other horizontal axis. If the two fixed coils were so perfectly adjusted that their axes coincided and that they were at the correct distance apart for both coils to produce maximum force on the moving coil when placed midway between them, then the sum of the forces, when the center of the moving coil was at the origin of coordinates, would be the force required for the computation of the absolute value of the current.

The correction for the imperfect adjustment of the fixed coils was obtained by measuring the difference of the forces at several points along each of the three coordinate axes. If the adjustments were perfect, at all points near the origin the difference of the forces would be the same (zero if the fixed coils were identical). If the adjustments were not perfect, the difference of the forces would vary linearly with the displacement along the axes. The slope of the line for the difference of forces along the vertical axis could be changed by varying the distance between the two fixed coils. In every case the distance between the fixed coils was changed until the line had so little slope that the correction for lack of correct spacing was less than a part in a million. In the case of the two horizontal axes no attempt was made to adjust the fixed coils so that they would be more nearly coaxial, but the correction for lack of coaxiality was seldom more than one or two parts in a million. The theory by which the correction for imperfect adjustment of the coils was made has been given in section IV.

An actual series of adjustments was made as follows: The coils were leveled and placed as nearly as possible in their correct position by mechanical measurements. With the current through the coils maintained constant, a series of measurements of the sum of the forces was made for several different vertical positions of the moving coil. The vertical position was read by noting the rest point of the balance, and was changed by adding weights to a pan of the balance. The observed sum of the forces was then plotted as ordinate with the vertical position of the moving coil as abscissa, a series of observations giving a curve with a maximum ordinate. At the position of the maximum force, the moving coil was midway between the two fixed coils.

With the current maintained constant and the moving coil kept midway between the two fixed coils, the sum of the forces was measured for several positions of the moving coil along one horizontal axis (say the north-south direction). This adjustment was made by a displacement of the entire balance as described in section II and the position was measured in terms of the revolution of the screw by which the adjustment was made. The sum of the forces was plotted as ordinate and the position in the north and south direction as abscissa, a set of observations giving a curve with a minimum value of the sum of the forces at a definite north and south position. The moving coil was placed at the position indicated by the minimum point and an exactly similar adjustment made in the east-west direction. When the moving coil was placed at the minimum position in the east-west direction, the center was then midway between the axes of the two fixed coils and midway between their planes, i.e., at the origin of coordinates.

3. SUM OF THE FORCES

Following the adjustments outlined above, a set of measurements of the sum of the forces was made, to be used in determining the force from which the absolute value of the ampere could be computed. The set consisted in measuring the force with the center of the moving coil as near as possible to the origin of coordinates, then at two positions equally displaced from the origin, and located on the same coordinate axis, then again at the origin, followed by two positions equally displaced along another coordinate axis, etc. A complete set consisted of at least 4 measurements at the origin, and, for each of the coordinate axes, 2 measurements at points equally displaced from the origin. The horizontal position was relatively stable so that two measurements equally displaced from the origin were a sufficient check on each horizontal adjustment. The vertical position of the moving coil and also the optical system by which its position was measured depended on several variable factors so that the vertical adjustment was checked often, usually daily.

Each series of measurements made at a given position of the moving coil has been defined as a *run*. If the center of the moving coil is at the origin of coordinates, the run is called a *final run*. At other positions of the moving coil, the runs are called *adjustment runs*.

All of the final runs in each set of measurements were included in determining the value of the ampere. The adjustment runs served merely to insure that the center of the moving coil was at the origin of coordinates during the final runs. Since the shape of the curve connecting force and displacement was known from the preliminary set of measurements, three points were enough to determine the whole curve. The value of the electromagnetic force between the coils after all the adjustments had been made was thus determined from each of the final runs, and this value was used in computing the value of the current in absolute measure. The results are given in a following section.

4. DIFFERENCE OF THE FORCES

The principal reason for measuring the difference of the forces was to correct the sum of the forces for the lack of coaxiality of the fixed coils and for their incorrect spacing. In the region near the origin of coordinates, the difference of the forces is a linear function of the displacement of the moving coil along any coordinate axis. For obtaining the correction to the sum of the forces, only the slope of the line obtained by plotting the difference of forces against displacement of the moving coil is required. Hence for this purpose, measurements were necessary only at two positions along each coordinate axis, but, as a check, were made at three or more positions.

Additional reasons for measuring the differences of the forces were to obtain a check on the computations and to determine the effect of the magnetic properties of the surrounding medium on the measured sum of the forces. The latter reason will be considered in another section. In computing the force for unit current, the computations were made separately for each pair of coils. By subtracting the values, the computed difference of the forces for unit current was obtained. The difference of the forces was then measured with a known current in the coils and the result compared with the computed difference for this current. Since the computed difference is the result of the subtraction of two large and nearly equal quantities, a small error in measuring the ratio of the radii of the coils, or in computing the maximum force for either coil, or in adjusting the coils will produce a relatively large difference between the observed and computed difference of the forces.

The results obtained on the difference of the forces are given in the next section. They were subject to two possible errors, which were peculiar to these measurements only. The lead correction was made only for measurements of the sum of the forces. Since the connections were slightly different when the difference of the forces was measured, the lead effect might have been slightly different in the two cases (sum and difference of the forces). This was not realized at the time the measurements were made but as it could not affect the absolute value of the current, the measurements were not repeated.

In 1911 it was found that the value of the difference of the forces changed slightly when the two fixed coils were interchanged, the mean of the two values differing less from the calculated values than either value alone. The effect was small and did not seem important enough to justify the extra work which would have been required to investigate this point.

5. SOURCES OF ERROR AND CORRECTIONS

There are several sources of error in the absolute measurement of the current by means of the current balance. Corrections could be made for most of them, and those for which no correction could be made were thought to be small. Many of them are the same as those already discussed in connection with the measurement of the ratio of the radii. Some of these were of such a nature that they tended to cancel errors made in the measurement of the ratio of the radii, while others were independent of the previous measurements. The major sources of error and the corrections to be made for them will be considered in the following paragraphs.

(a) ADJUSTMENT OF THE COILS

Since there were 12 adjustments which were independent, one might expect that the total error would be relatively large. However, at various times, the coils were removed, replaced, and readjusted, and whenever this was done, the results of the measured force in the two cases agreed within a few parts in a million. It is estimated that the error caused by maladjustment of the coils never exceeded six parts in a million in the determination of the force or three parts in a million in the measurement of the current.

(b) SENSITIVITY OF THE BALANCE

The sensitivity of the balance was used in determining the small difference between the force of gravity on the weight and the electromagnetic force between the coils. In practice the weights and resistances used with a particular set of coils were adjusted so that the difference between the gravitational and electromagnetic forces was very small, in which case the sensitivity of the balance did not need to be known accurately. Nevertheless, the sensitivity was measured frequently and found to change very little after the balance and coils were set up and adjusted. It is estimated that the error in the measured sensitivity did not produce an error in the measured force as large as one part in a million.

(c) WEIGHTS AND BUOYANCY CORRECTION

The masses of the weights were measured by the mass section of this Bureau and the results certified to be correct to 0.01 mg. They were measured before and after each complete series of force determinations and the change in mass of any weight was never larger than 0.01 mg.

The density of the gold-platinum weights was 19.5 and of the platinum-iridium weights was 21.5 grams per cubic centimeter. The correction to the force for the buoyancy of the air at a temperature of 22 C, a pressure of 760 mm, and a relative humidity of 50 percent, was 63 parts in a million for the gold-platinum weights and 57 parts in a million for the platinum-iridium weights. The high density of the weights not only decreased the correction for buoyancy of the air, but minimized the changes in the buoyancy correction caused by changes in the temperature, pressure, and humidity of the air. The temperature, pressure, and humidity of the air were always measured, and the standard buoyancy correction modified, if necessary. The error introduced into the measurement of force by errors in the determination of the mass of the weights and in the buoyancy correction was estimated to be less than two parts in a million, which would make an error of less than one part in a million in the value of the current.

(d) ACCELERATION OF GRAVITY

The value of the acceleration of gravity was as important a factor as the mass in determining the gravitational force. The value of gravity which has been used in this investigation depended directly on the determination of the absolute value made in Potsdam,[33] Germany, where a value of 981.274 $\pm$0.003 cm/sec^2 was obtained. The value at a base station in Washington has on three occasions been accurately compared with that at Potsdam by means of pendulums, once [34] in 1900, a second time [35] in 1928–29, and again [36] in 1933. The value of gravity at the present base station in the Commerce Building,[37] in Washington, resulting from each of these determinations is as follows:

1900	980. 113 cm/sec^2
1929	980. 118
1933	980. 118

The transfer in Washington from the Coast and Geodetic base stations to the Bureau of Standards was made in 1910 and in 1933. The determination made in 1910 at this Bureau gave a value of

[33] Veroeffentlichung des Konigl. Preus. Geodätische Institutes, Neue Folge no. 27, 1906.
[34] U.S. Coast and Geodetic Survey Report for 1901, appendix 5.
[35] A determination of the relative value of gravity at Potsdam, Greenwich, Ottawa, and Washington, by A. H. Miller. Pub. of the Dominion Observatory, Ottawa, vol. 11, no. 2, 1931.
[36] The report on this comparison has not yet been published, but the result has been communicated by letter to the authors.
[37] The values here given for 1900 and 1929 are 0.001 cm/sec^2 higher than values published when the determinations were made, since those applied to the former Coast and Geodetic base station which is now abandoned.

980.095 cm/sec² for the acceleration of gravity at a pier in the basement of the south building. This value was based on the transfer made in 1900 from Potsdam. The measurements made in 1933, following the 1933 transfer from Potsdam, gave a value of 980.100 cm/sec² at a pier in the constant temperature room of the east building, which pier is 6.4 meters below the level of the old pier. Correcting for this difference in level, the 1933 value at the level of the old pier is 980.099 cm/sec².

The value assigned to the acceleration of gravity at this Bureau depends not only on the transfer from Potsdam to the Washington base station but also on the transfer from the Washington base station to the Bureau. Since in neither case do the 1933 values agree with the older values, four possible values may be assigned to the acceleration of gravity at any place at this Bureau. The four values at the elevation of the current balance (103 meters above sea level which is also the level of the old pier) are:

Washington base station to Bureau	Potsdam to Washington	
	1933	1900
1910	980.100 cm/sec²	980.095 cm/sec
1933	980.099	980.094
Mean	980.100 cm/sec	980.095 cm/sec²

An absolute determination of the value of the acceleration of gravity is now in progress at this Bureau, but the result is not yet available. There are, however, indications that the value obtained from this determination may differ by several parts in a million from the value as transferred from Potsdam. Hence, pending a decision as to the most probable value at this Bureau, the value used in 1911 has been retained in this investigation, making a correction for the difference in elevation of the two laboratories. The value which has been used is 980.095 cm/sec². Had the result of the recent transfer from Potsdam been used, the final value of the current would have been changed by only two parts in a million, which is negligible in comparison with other errors.

(e) CORRECTION FOR THE TEMPERATURE OF THE COILS

To correct the measured force for the change in radius produced by the difference between the temperature of the coils and the standard temperature (22 C), the change in the radius of each coil produced by this temperature difference was computed from the temperature coefficient of expansion. If a is the radius of a coil, τ its coefficient of expansion, and Δt the amount by which the measured temperature exceeds the standard temperature, then from equation 43 the increase in its radius, Δa is given by the equation

$$\Delta a = a\tau\Delta t \quad \text{or} \quad \Delta a/a = \tau\Delta t. \tag{51}$$

It follows that the relative change in the ratio of the radii, $\Delta\alpha/\alpha$, of the two coils having, at the standard temperature, radii a_1 and a_2, respectively, is

$$\Delta\alpha/\alpha = \Delta a_2/a_2 - \Delta a_1/a_1 = \tau_2\Delta t_2 - \tau_1\Delta t_1 \tag{52}$$

where all symbols having a subscript two apply to the small coil, and those having subscript one to the large coil.

The relative change in the maximum force corresponding to a known relative change in the ratio of the radii is given by the formula (see equation 5)

$$\frac{\Delta \mathfrak{F}_m}{\mathfrak{F}_m} = \epsilon \frac{\Delta \alpha}{\alpha} \tag{53}$$

where $\mathfrak{F}_m$ is the maximum force, α the ratio of radii and ϵ is a constant for any particular value of the ratio of the radii. The value of ϵ can be computed from the ratio of the radii by the formula [38]

$$\epsilon = \frac{\beta}{2x^2} = \frac{1-\alpha^2}{2y_m^2} = \frac{(1-\alpha^2)}{2\left(\frac{1}{2} - \frac{9}{20}\alpha^2 - \frac{1}{16}\alpha^4\right)^2} \tag{54}$$

where y^m is the ratio of the distance for maximum force to the radius a_1 of the large coil. A table of the values of ϵ for different values of α from 0 to 1 had been prepared by Grover [39] before the development of the above formula. The values which were used in the present investigation were computed from equation 54 and are given in the following table.

TABLE 11.—*Values of the constant ϵ used in the equation*

$$\frac{\Delta \mathfrak{F}_m}{\mathfrak{F}_m} = \epsilon \frac{\Delta \alpha}{\alpha}$$

α	ϵ
0.4013	2.312
.5001	2.549
.5023	2.555
.6260	3.083

From equations 52 and 53 the expression for determining the maximum force at the standard temperature, $(\mathfrak{F}_m)_{22}$, from the force at the observed temperature, $(\mathfrak{F}_m)_{obs}$ is

$$(\mathfrak{F}_m)_{22} = (\mathfrak{F}_m)_{obs}\,[1 - \epsilon(\tau_2 \Delta t_2 - \tau_1 \Delta t_1)]. \tag{55}$$

The errors made in applying the temperature correction of the coils were the same as those in the determination of the ratio of the radii which have already been discussed. It is to be noted, however, that if the temperatures of the coils in the measurement of the ratio of the radii were the same as when the coils were used in the current balance, the correction to standard temperature would be the same in both cases, so that no error would result. In practice it was not possible to have the two temperatures the same, but efforts were made to keep them as close as possible. No error was caused by assuming that the value of the temperature coefficient of resistance of the copper wire in the coils was 0.0039 per degree centigrade since this value was used in the measurement of the coefficient of expansion as well as in

[38] This formula is adapted from equation 23 of Snow's paper. See footnote 25 for reference.
[39] Table 5, p. 373 of Grover's paper. Reference footnote 15.

its application. It is estimated that the total error introduced into the final value of the current because of errors in the temperature correction was not more than two parts in a million.

(f) CORRECTION FOR THE ELECTRICAL LOAD ON THE COILS

In order to correct the observed force for the electrical load on the coils a formula for the load correction was derived from equations 43 and 53 in the same manner as equation 55 for the temperature correction. The formula is

$$(\mathfrak{F}_m)_0 = (\mathfrak{F}_m)_{obs}[1+\epsilon(\psi_2 p_2 - \psi_1 p_1)] \tag{56}$$

where

$(\mathfrak{F}_m)_0$ = the maximum force per unit current that would have been obtained with two coils if their loads had been zero

$(\mathfrak{F}_m)_{obs}$ = the force per unit current with the observed load

ψ_1 and ψ_2 = the load coefficients of the large coil and the small coil respectively

p_1 and p_2 = the loads in watts.

The value of ϵ is the same as given in equation 54.

One of the best checks on the accuracy of the load corrections was the good agreement obtained between the results taken with different currents and hence different loads. The error that may have been introduced into the final value of the current because of errors in the load correction is estimated to be less than three parts in a million.

(g) LEAD CORRECTION

The correction for the leads to each of the coils of the balance was determined experimentally. The terminals of each winding of each coil were constructed so that the windings could be disconnected by melting drops of solder and the leads short-circuited by adding a drop of solder. (See fig. 3, p. 672.) When this was done for the fixed coils alone, an experimental determination of the force with normal current through the moving coil and the fixed-coil leads gave the correction for the fixed-coil leads. The correction for the moving-coil leads was determined in a similar manner. The algebraic sum of the two gave the total correction, which was seldom more than a few parts in a million. This was determined for every coil combination. The error in this measurement was about the same as in any force determination, producing an error on the final result of not more than two parts in a million.

(h) CHANGES IN THE REST POINT DURING A RUN

During an actual run, there were often slight irregularities superimposed upon the uniform rate of change of the rest point. (See fig. 15.) In such a case, the number of rest-point determinations was increased. The number of observations required for a run was determined by the operator. As the data were taken, the observer judged whether the points would lie on two parallel straight lines. If the first six points appeared to satisfy the above condition, the operator decided that sufficient observations had been made for a run. If the first six points did not appear to be satisfactory, several more were taken until it appeared certain that there were enough points to establish a satisfactory average line. Under normal conditions, an experienced operator could repeat results by this method to within 0.02 mg. When the total electromagnetic force was 10 grams, the maximum error from this source was two parts in a million in the determination of the force, causing an error in the measurement of the current of one part in a million. When the force was smaller, that is, when smaller currents were used, the relative error was larger.

(i) CONVECTION CURRENTS IN THE AIR

Air currents which affected the balance were set up around both the moving coil and its flexible leads by the heat produced by the current in them. These air currents caused, on the parts with which they were associated, an upward force, equivalent to a few tenths of a milligram. However, the method of comparing the electromagnetic and gravitational forces, by reversing the current in the fixed coils and at the same time adding or removing a weight, allowed the heating of all parts to remain constant and therefore the air currents should have remained constant. However, these air currents were very susceptible to changes in the temperature of surrounding objects. The effect of any change in the air currents was to alter the rest point of the balance. Methods of minimizing the effect of slight irregularities in the rest point have already been discussed and the error produced by such irregularities has been estimated in the preceding paragraph.

(j) MAGNETIC BODIES

Magnetic bodies may, in considering their effect on the measured force, be divided into the three following classes:

(1) *Magnetic masses attached to the moving system.*—The magnetic masses which were attached to the moving system produced a force by virtue of the attraction between them and the fixed coils. If this force had been appreciable, the rest point of the balance would have changed when the current in the fixed coils only was changed from zero to its normal value. No definite effect was ever found. The only magnetic masses in the moving system were the steel knife edges of the balance beam and they were located a little more than a meter from the nearest coil. The magnetic field of the moving coil would tend to magnetize the moving masses, so that all material in the moving system that was near the coil was carefully tested and was found to be nonmagnetic.

(2) *Magnetic masses outside the balance.*—The magnetic masses which were outside the balance produced a force because of the attraction between them and the moving coil, and also a possible attraction between them and the moving magnetic masses on the balance. However, in these measurements, the field of the fixed coils was reversed so that only that portion of the attraction which reversed when the field of the fixed coils reversed caused an extraneous force that was included in the measured force. Permanent magnets produced no error because the sign of the force caused by them did not change when the field of the fixed coils was reversed. However, soft-iron masses which were magnetized by the field of the fixed coils reversed the sign of their force on the moving coil when the magnetizing field of the fixed coils was reversed, and if unsymmetrical with respect to the coils, would affect the measured force. Unsymmetrical

soft-iron masses generally produced a greater error in measuring the difference of the forces than in measuring their sum, since the external magnetic field above and below the coils was greater in the former case. The agreement between the computed and observed difference of the forces indicated that the presence of existing unsymmetrical soft-iron masses (if any were present) had little effect on the difference of the forces. If the fixed masses were symmetrical, the measurement of the difference of forces would not have detected them.

The only iron masses which were approximately symmetrical were the pipes in a shaft at a distance of about 3 meters from the balance. These were so distant that no effect appeared probable.

In addition to the iron pipes, there was some iron machinery about 3 meters above the balance. In 1911 it was found that 75 kg of iron placed 116 cm below the moving coil had an effect of not more than one part in a million on the sum of the forces and about 10 times as large an effect on the difference of the forces. The effect of a given mass of iron decreased rapidly with increase in distance. Since there was no appreciable amount of iron within 3 meters of the balance, the error caused by fixed magnetic masses must have been much less than a part in a million. The measurements on the difference of the forces support this conclusion.

The water jacket of brass or copper which surrounded the moving coil was so close to the coil that a slight magnetic susceptibility might affect the force on the moving coil. The jackets were tested for their magnetic properties with the astatic magnetometer (see p. 710) and their susceptibilities were found to be essentially unity. In addition, weighings with and without the jackets gave no difference in the results.

(3) *Magnetic bodies as a part of the coils themselves.*—Magnetic bodies which were a part of the coils themselves might have caused the external fields to be different from those computed, and therefore have caused the measured force to be different from the computed force. This was tested in 1911 by sticking iron filings to a piece of cloth and wrapping it around the fixed coils. No effect was noted. Also, the two sets of large fixed coils L1 L2 and L3 L4 used in 1911 differed principally in their magnetic properties. When brought in turn near an astatic magnetometer, coils L1 and L2 produced a deflection of the needle about 100 times that produced by L3 and L4. The absolute value of the current obtained with the two sets of coils agreed closely. These two lines of evidence indicate that the magnetic properties of the particular coils used in this work had a negligible effect on the force.

(k) EXTERNAL MAGNETIC FIELDS

The effect of the external magnetic field was measured by observing the change of force when the current through the moving coil was reversed; there being no current in the fixed coils. On several occasions the change of force was about 0.7 mg when a current of 0.8 ampere in the moving coil was reversed. This force, when constant, produced no error in the comparison of the electromagnetic and gravitational forces, because the moving-coil current was not reversed during a measurement. The only chance for error was in case the external nonuniform magnetic field changed during a run. Such a change would cause an irregularity in the rest point which could not be distinguished from irregularitites caused by variations in the convection currents in the air. The graphical method of treating the rest-point data gave an opportunity for eliminating such an irregularity. In plotting the curve connecting the rest point with the time of observation, a single point, which did not lie on the curve through the remaining points, could be neglected on the assumption that the rest point had been displaced by a change in the nonuniform portion of the external magnetic field. The ease with which such unusual points were detected and eliminated was an added reason for using the graphical method for the treatment of the observed data. (See fig. 15.)

(l) CORRECTION FOR THE FINITE CROSS SECTION OF THE COILS

The errors which arose in applying the correction for the finite cross section of the coils were caused by the uncertainty in the actual cross sections. A formula [40] for the effect on the force of estimated errors in the cross-sectional dimensions has been obtained by adding the derivatives of the expressions for the effect of cross section on the ratio of the radii (equation 49) and on the force between the coils (equation 7). The formula is

$$\frac{\delta\mathfrak{F}_m}{\mathfrak{F}_m}=\frac{1}{2x^2}\left\{\left[\frac{(1+\beta)\lambda_2}{3}-\beta\right]\left[\frac{c_1^2}{a_1^2}\frac{\delta c_1}{c_1}-\frac{b_1^2}{a_1^2}\frac{\delta b_1}{b_1}\right]+\left[\frac{(1-\beta)\lambda_2}{3}+\beta\right]\left[\frac{c_2^2}{a_2^2}\frac{\delta c_2}{c_2}-\frac{b_2^2}{a_2^2}\frac{\delta b_2}{b_2}\right]\right\} \quad (57)$$

The symbols have the same significance as in equation 7 p. 684. For a coil with square cross section ($b=c$) the error in the force for a given error in b is the same, but of opposite sign, as for a like error in c.

The errors, computed by the above formula for $\delta_b=\delta_c=+0.0005$ cm (each dimension in error by +0.01 mm), are given in table 12 for each pair of coils that was used. They are in substantial agreement with those obtained in 1911 by a different procedure.[41]

TABLE 12.—*Effect of errors in the sectional dimensions of the coils on the computed value of the force per unit current.*

Radii		Dimensions of square section		$\delta\mathfrak{F}_m/\mathfrak{F}_m$—parts per million for $\delta b=\delta c=+0.0005$ cm	
Fixed a_1	Moving a_2	Fixed $2b_1=2c_1$	Moving $2b_2=2c_2$	Fixed coil [a]	Moving coil [a]
cm	*cm*	*cm*	*cm*		
20	10	1.58	1.0	±0.59	±8.33
25	12.5	2.00	1.0	±.47	±5.33
25	10	2.00	1.0	±.29	±6.64
20	12.5	1.58	1.0	±1.20	±7.61

[a] The error is positive for the radial dimensions (c) and negative for the axial dimensions (b).

The errors are independent, that is, they may have either sign, so that the actual error caused by all errors in the sectional dimensions is not equal to the resultant of the separate errors.

[40] Equation 27 of Snow's paper. Reference footnote 25. The term $\delta\tau/\tau$ of Snow's equation has been assumed to be zero.

[41] See table XIII on p. 334 of reference in footnote 8.

The importance of accurately knowing the sectional dimensions is evident from the table. Also, the dimensions of the moving coils are more important than those of the fixed coils. From an analysis of the data which were obtained when the coils were wound, the authors have concluded that each dimension of each coil may have been in error by as much as 0.02 mm. Hence the maximum possible error, in $\mathfrak{F}_m$, even with the pair of coils in which a knowledge of the sectional dimension was most important, was only 36 parts in a million, corresponding to an error of 18 parts in a million in a determination of the current.

(m) ERRORS RESULTING FROM DIFFERENCES BETWEEN THE ACTUAL AND THE THEORETICAL COILS

The actual coils differed somewhat from those which were assumed in developing the theory of the current balance. The conditions that the actual coils failed to meet were (1) the wires were not uniformly distributed throughout the cross sections of the coils; (2) the currents in the two windings of a coil when connected in parallel were not exactly the same; (3) the distribution of the current in the cross section of any wire may not have been uniform, and (4) the coils were not exactly circular. No estimate has been made of the error introduced by the first of these conditions. The second has been tested experimentally and the third and fourth have been treated theoretically.

The theory of the current balance has been developed on the assumption that the wires of each layer are uniformly spaced and that the wires of different layers all lie in planes which are perpendicular to the axis of the coil. Such a condition cannot be met in any actual coil, and in coils with a double winding, such as used in this investigation, the deviation from the theoretical condition was necessarily appreciable. The wires of each layer were wound in spirals, and because of slight irregularities in the wires were never perfectly spaced. But perhaps the greatest difficulty arose in transferring the wires from one layer to the next. According to theory the wires should be transferred from one layer to the next on a radius of the coil. In the actual coils a small fraction of a turn was required for this transfer. No satisfactory method for estimating the effect of these imperfections on the final result has been developed.

Each coil had two windings which were placed side by side as the coil was wound. In using the coil these two windings were connected in parallel. As the resistance of the two differed slightly, one winding carried more current than the other. The current could be made the same in the two windings by adding a small resistance in series with the winding having the lower resistance. The effect on the force of equalizing the currents in the two windings was too small to be detected experimentally.

The effect of a nonuniform current distribution in each wire has been investigated theoretically.[42] It was shown that the distribution produced no effect on the absolute value of the current provided the ratio of the radii was measured by the electromagnetic method.

The effect of the lack of circularity of the coils on the force can be estimated by computing the force that would have been exerted if each coil had been composed of two semicircles, one having a radius

[42] See conclusion of Snow's paper. Reference footnote 25.

equal to the largest radius of the actual coil and the other having a radius equal to the smallest radius of the coil. Such a computation has been made for the coils L3 and M2, which were the two that showed the greatest departure from circularity when measured during construction. The correction to the force as computed for a circular coil was less than a part in a million.

(n) PREDICTED ERROR IN THE ABSOLUTE VALUE OF THE CURRENT

The maximum error in a single determination of the absolute value of the current could be predicted by summing the estimated errors for the different effects that have just been described. For convenience, the estimated errors will be divided into two classes, (1) those which are apparent from measurements on a single set of coils and (2) those which are disclosed only when the results on two or more sets of coils are available. Summing the estimated errors of the first class, no result of the absolute value of the current should differ from the mean obtained with that set of coils by more than 10 parts in a million. The results given later show that so large a deviation was seldom encountered. Hence it is probable that all the important sources of error which are disclosed by measurements with a single set of coils have been considered.

In the preceding discussion, an estimate has been made of the maximum possible effect on the absolute value of the current of three errors, the influence of which, on the absolute value of the current, can only be detected by measurements with different combinations of coils. These errors are those affecting the ratio of the radii, the adjustment of the coils in the current balance, and the cross-sectional dimensions of the coils. The maximum possible difference from the mean of the average result on any set of coils was estimated from these 3 errors to be 24 parts in a million in the absolute value of the current. This maximum possible difference is much less for 1 combination of coils than the observed difference from the weighted mean result, is very nearly equal for 2 combinations of coils, and is much greater for only 1 combination of coils. (See table 16.) Of the 3 errors which might cause this difference, the first 2 were checked experimentally, and the errors found to be approximately those estimated. Those concerning the cross-sectional dimensions of a coil can be checked by assuming an error in the dimensions of a coil that will make the result for a coil combination in which it was used the same as the mean result of all the coils. No reasonable assumption concerning cross-sectional errors will provide corrections that will give the same absolute value of the current for each set of coils. Hence there appears to be some source of error for which no estimate has been made, perhaps some of those discussed in (m).

VII. THE VALUE OF A CURRENT IN B.S. INTERNATIONAL AMPERES AND IN ABSOLUTE AMPERES

The value, in absolute amperes, of one B.S. international ampere, has been obtained from observations on four combinations of coils. For each coil combination, a number of sets of observations were made and the results averaged to obtain a value for that combination of coils. The final value is the weighted average of the results for the four combinations of coils.

The first step in determining the absolute value of the current from the measured force was to compute the constant $\mathfrak{F}_m$ for each pair of coils. This was done by the methods given in section IV using the corrected values of the ratio of radii as given in table 9. The results of the computation are given in table 13.

Table 13.—*Constants for computing the current from the force*

[For nomenclature see equation 7,[a] p. 684]

Coils	Ratio of radii	y_m	F_m	Δ_2	Δ_4	$\mathfrak{F}_m$	Increase from 1911 value[b]
							Parts per million
M2:L3	0.5001586	0.3835	5.356342	-88×10^{-6}	$+0.5\times10^{-6}$	5.355877	−6
M2:L4	.5000701	.3836	5.353925	−110	+.3	5.353340	−2
M3:L3	.4013758	.4259	3.146497	+413	+.6	3.147797	−2
M3:L4	.4013045	.4259	3.145207	+395	+.5	3.146452	0
M2:S1	.6258941	.3141	9.98532	−225	+8.1	9.98315	----------
M2:S2	.6261844	.3139	9.99959	−241	+8.0	9.99726	----------
M3:S1	.5022778	.3825	5.414473	+404	+1.0	5.416654	−1
M3:S2	.5025110	.3823	5.420902	+393	+1.9	5.423043	+2

[a] The term $\frac{(x\Delta_2')^2}{2\lambda_2}$ in equation 7 was computed and found to be less than 0.3×10^{-6} for each combination of these coils.

[b] The 1911 values were corrected, by equation 53, for the observed change in the ratio of the radii.

The results of all the final runs taken with the four sets of coils are given in table 14 for the difference of the forces and in table 15 for the sum of the forces. There are given in those tables the corrections that were computed from the data, and, for each set of observations, the result that was obtained. The sum of the forces is used to compute the absolute value of the current while the difference of the forces is used merely as a check on the other work as a basis for making small corrections and as a guide in weighing results.

In table 14 is given a summary of the results on the differences of the forces. It is self-explanatory except for the last column in which is listed the indicated error that would be introduced into the absolute value of the current if the computed difference in forces minus the observed difference was all caused by an error in the computed value of one of the forces. As the value for each coil combination has been computed at least twice by different computers, the indicated error is not the result of a numerical mistake. As it is appreciably larger for the coil combination S1 S2 M2 than for any of the others, the absolute value of the current obtained with that coil combination is probably less reliable than the values obtained with the other combinations. For this reason the indicated error has been used as a guide in weighting the results on the absolute value of the current as obtained with different coil combinations.

TABLE 14.—*The difference of the forces*

Date	Coils	Nominal ratio of radii	Current	Computed difference	Number of observations	Average observed difference	Average deviation between observed values	Computed minus observed values	Indicated error in final value of current
			Amperes	*Milligrams*		*Milligrams*	*Milligrams*	*Milligrams*	*Parts per million*
1932	L3 L4 M3	0.4	1.018	1.80	5	1.67	0.02	+0.13	7
1932	do	.4	0.678	0.80	2	0.82	.00	−.02	2
1930	S1 S2 M3	.5	.679	2.30	6	2.26	.03	+.04	4
1931	L3 L4 M2	.5	1.018	2.50	3	2.43	.04	+.07	4
1927	do	.5	0.768	1.42	3	1.36	.01	+.06	6
1931	do	.5	.678	1.11	4	1.10	.01	+.01	1
1931	do	.5	.509	0.63	1	0.64		−.01	2
1932	S1 S2 M2	.626	1.018	8.42	3	8.92	.01	−.50	25
1932	do	.626	0.678	3.74	3	3.92	.02	−.18	18

In table 15 are given the essential data for all final runs for determining the value of the current in absolute amperes from the sum of the forces, and the relation between that value and the value of the current in international amperes. In the first three numbered columns are given the coil combination and the temperatures of the moving coil and of the fixed coils. In the case of the fixed coils the average temperature of the two coils is given, there being seldom any significant difference between them. The temperatures of the coils were always determined immediately after the measurement of the sum of the forces by measuring the resistance of the windings with the same current in them as was used in measuring the force. The temperature of the moving coil depended somewhat upon the type and position of the water jacket, but the results were independent of the water jacket so that no detailed discussion has been given.

In the fourth column is given the observed doubled force in grams, which resulted from the reversal of the current. The difference in the rest points when the current was reversed (at the same time that the weight was added or removed) was multiplied by the sensitivity of the balance to obtain the amount that should be added to, or subtracted from, the value of the actual weight in order that the electromagnetic and gravitational forces should be equal. (See fig. 15, p. 713.) The gravitational force on the weight was corrected for the buoyancy of the air.

In the fifth column is given the correction to the observed doubled force for the temperature of the coils. A standard temperature of 22 C was chosen to conform with the work in 1911.

In the sixth column of the table is given the sum of the corrections for the loads on the coils and for the leads to the coils. For a given set of coils carrying a given current, the load correction was a constant and the lead correction had a definite value so long as the position of the leads was not changed. The latter was determined experimentally whenever a set of coils was installed in the balance.

In the seventh column is given the corrected doubled force corresponding to 22 C and no load which is obtained by adding, for each row, the values in columns 5 and 6 to the observed doubled force.

The eighth column contains the absolute value of the current as computed from the doubled force (column 7) and the dimensions of the coils. To compute the current, the force of gravity on the weights is equated to the electromagnetic force between the coils, giving the formula

$$2\,M\,g = 2\,n_1 n_2\,(I_a/10)^2\,(\mathfrak{F}_1 + \mathfrak{F}_2) \tag{58}$$

where

$2\,M$ = observed doubled force in grams corrected to 22 C and no load

g = acceleration of gravity in cm/sec^2

$2\,M\,g$ = observed doubled force in dynes corrected to 22 C and no load

n_1 = number of turns of wire on each of the fixed coils

n_2 = number of turns of wire on the moving coil

I_a = current in absolute amperes (1 absolute ampere = 1/10 cgs electromagnetic unit of current)

$\mathfrak{F}_1$ = computed constant for the moving coil and one of the fixed coils at 22 C including corrections for the cross sections of the coils

$\mathfrak{F}_2$ = computed constant for the moving coil and the other fixed coil

Solving for the current

$$I_a = \frac{10\sqrt{2\,M\,g}}{\sqrt{2\,n_1 n_2\,(\mathfrak{F}_1 + \mathfrak{F}_2)}} \tag{59}$$

In the ninth column is given the value of the same current in B.S. international amperes. This was obtained by dividing the numerical value of the potential difference which the current produced at the terminals of a standard resistor by the numerical value of the resistance, both electrical quantities being measured in terms of the units as maintained [43] by the Bureau of Standards.

In the tenth column are listed the differences between the value of the current in B.S. international amperes and in absolute amperes, divided by the value of the current in absolute amperes. These relative differences are related to the ratio of the two units as experimentally determined for each current and hence should all have the same value regardless of the particular value of the current used and of the external conditions of measurement. These numbers have been averaged by groups to obtain the final result of this investigation.

In the eleventh and last column is given the difference between each number in the tenth column and the mean value of the particular group to which it belongs. The numbers of this column indicate the uniformity of the experimental values with a given set of coils.

[43] The values given in this paper are based on the units of resistance and electromotive force as maintained at the time the individual measurements were made. On September 19, 1933, after the observational work was completed and after most of the data had been reduced, the unit of electromotive force of the Bureau of Standards was increased by five microvolts, thus decreasing the values assigned to individual cells by this amount. This change is smaller than the experimental error of this investigation so that the values have not been revised to take account of this change.

TABLE 15.—*Data and results on the absolute determination of the ampere*

[All final runs have been included]

Date	Coil combination[1]	Temperature of coils: Fixed; average of two	Temperature of coils: Moving	Observed doubled force	Correction for: Temperature	Correction for: Load and leads	Doubled force corrected to 22 C and no load	Values of the current: I_A	Values of the current: I_{BS}	$\frac{I_{BS}-I_A}{I_A}$	Deviation from mean of set
	1	2	3	4	5	6	7	8	9	10	11
		°C	*°C*	*Grams*	*Milligram*	*Milligram*	*Grams*	*Absolute amperes*	*B.S. int. amperes*	*Parts per million*	*Parts per million*
1927											
Feb. 12	L4 L3 M2	23.35	21.90	5.99868	+0.41	−0.23	5.99886	0.767638	0.767701	82	−8
15	do	23.38	22.04	810	.37	−.23	824	598	662	83	−7
Mar. 8	do	23.33	22.00	931	.37	−.23	945	675	743	89	−1
22	do	23.39	22.03	903	.38	−.23	918	658	725	87	−3
24	do	23.34	22.01	896	.37	−.23	910	653	727	96	+6
25	do	23.34	22.03	906	.37	−.23	920	660	727	87	−3
26	do	23.29	22.00	977	.36	−.23	990	704	776	94	−4
28	do	23.33	21.98	955	.38	−.23	970	691	751	78	−12
28	do	23.33	21.98	885	.38	−.23	6.00000	711	786	98	+8
29	do	25.28	23.80	978	.40	−.23	5.99995	707	786	103	+13
30	do	25.21	23.76	946	.39	−.23	962	686	761	98	+8
31	do	27.98	26.42	946	.40	−.23	963	687	762	98	+8
Apr. 1	do	28.20	26.42	946	.46	−.23	969	691	761	91	+1
2	do	23.39	22.00	961	.39	−.23	977	696	761	85	−5
2	do	23.39	22.00	895	.39	−.23	911	654	724	91	+1
5	do	23.28	21.93	899	.38	−.23	914	656	724	88	−2
5	do	23.35	21.95	856	.39	−.23	872	629	693	83	−7
	Average of 17 runs									90	±6
1930											
Aug. 25	S1 S2 M3	25.41	25.86	3.91376	−.05	0	3.91371	0.678650	0.678687	55	+2
26	do	25.43	25.86	373	−.04	0	369	648	687	58	+5
28	do	25.45	26.04	310	−.07	0	303	591	629	56	+3
Sept. 3	do	25.67	25.58	364	+.04	+.01	369	648	684	53	0
4	do	25.36	25.58	367	−.01	.01	367	646	686	59	+6
5	do	25.39	25.60	369	−.01	.01	369	648	685	55	+2
8	do	24.63	24.77	369	0	.01	370	649	686	55	+2
8	do	24.61	24.76	372	0	.01	373	652	686	50	−3
10	do	24.56	24.73	369	−.01	.01	369	648	684	53	0
10	do	24.54	24.70	371	−.01	.01	371	650	684	50	−3
11	do	24.55	24.71	370	−.01	.01	370	649	684	52	−1
11	do	24.56	24.70	371	0	.01	371	650	684	50	−3
15	do	24.75	24.82	366	+.01	+.04	371	650	684	50	−3
15	do	24.75	24.81	368	+.01	.04	373	652	684	47	−6
	Average of 14 runs									53	±3
1931											
Feb. 17	L3 L4 M2	24.85	23.55	4.68254	+.27	−.08	4.68273	0.678221	0.678294	107	+7
18	do	24.77	23.68	261	.23	.08	276	224	293	102	+2
21	do	24.75	23.48	262	.26	.08	280	227	294	99	−1
25	do	24.85	23.60	262	.26	.08	280	227	294	99	−1
Mar. 17	do	26.97	25.51	256	.29	.08	277	224	294	103	+3
20	do	27.02	25.44	256	.32	.08	280	227	293	97	−3
Apr. 3	do	23.65	22.45	266	.26	.08	284	230	293	93	−7
8	do	25.69	24.23	260	.30	.08	282	228	294	97	−3
	Average of 8 runs									100	±3
Mar. 23	L3 L4 M2	22.81	22.01	2.63906	+.10	−.02	2.63914	0.509159	0.509205	90	
30	do	22.58	21.85	909	+.09	−.02	916	161	205	86	
	Average of 2 runs									88	
Apr. 2	L3 L4 M2	26.79	23.92	10.55022	+1.38	−.53	10.55107	1.018052	1.018155	101	
17	do	28.92	25.92	25	1.42	.53	114	57	152	94	
24	do	28.91	25.93	20	1.41	.53	108	53	154	99	
	Average of 3 runs									98	

[1] In each case the first coil listed is the upper fixed coil.

TABLE 15.—*Data and results on the absolute determination of the ampere*—Contd.

[All final runs have been included]

	1	2	3	4	5	6	7	8	9	10	11
Date	Coil combination[1]	Temperature of coils: Fixed; average of two	Temperature of coils: Moving	Observed doubled force	Correction for: Temperature	Correction for: Load and leads	Doubled force corrected to 22 C and no load	Values of the current: I_A	Values of the current: I_{BS}	$\frac{I_{BS}-I_A}{I_A}$	Deviation from mean of set
1932		°*C*	°*C*	*Grams*	*Milligram*	*Milligram*	*Grams*	*Absolute amperes*	*B.S. Int. amperes*	*Parts per million*	*Parts per million*
Jan. 7	S2 S1 M2	25. 03	23. 90	11. 91728	+. 61	−. 27	11. 91761	1. 017659	1. 017664	5	−4
11	do	24. 86	23. 78	724	. 58	−. 27	755	656	665	9	0
12	do	24. 93	23. 84	719	. 59	−. 27	751	655	664	9	0
18	do	25. 75	24. 64	723	. 56	−. 27	752	655	665	10	+1
19	do	25. 36	24. 21	718	. 61	−. 27	752	655	665	10	+1
23	do	25. 42	24. 25	726	. 62	−. 27	761	659	666	7	−2
25	do	25. 51	24. 34	726	. 61	−. 27	760	658	669	11	+2
25	do	25. 51	24. 35	730	. 60	−. 27	763	660	669	9	0
26	do	25. 43	23. 65	681	1. 04	−. 27	758	657	669	12	+3
28	do	24. 65	24. 32	782	. 08	−. 27	763	660	670	10	+1
29	do	25. 60	24. 74	745	. 39	−. 27	757	657	668	11	+2
	Average of 11 runs									9	±2
30	S2 S1 M2	23. 30	23. 24	5. 29360	−. 02	−. 05	5. 29353	0. 678235	0. 678240	7	+1
Feb. 9	do	22. 14	22. 16	363	−. 01	−. 05	357	238	241	4	−2
9	do	22. 14	22. 14	362	0	−. 05	357	238	241	4	−2
15	do	22. 35	22. 27	354	+. 01	−. 05	350	234	241	11	+5
20	do	22. 44	22. 42	361	−. 02	−. 05	354	236	241	7	+1
22	do	22. 46	22. 61	365	−. 06	−. 05	354	236	241	7	+1
27	do	22. 60	22. 82	365	−. 08	−. 05	352	235	241	9	+3
29	do	23. 36	23. 01	350	+. 07	−. 05	352	235	240	7	+1
Mar. 7	do	22. 55	22. 11	349	+. 11	−. 05	360	240	239	−1	−7
8	do	23. 50	22. 90	345	+. 14	−. 05	359	230	239	0	−6
	Average of 10 runs									6	±3
12	S2 S1 M2	22. 82	22. 50	2. 95985	+. 04	−. 01	2. 95988	0. 507160	0. 507165	10	
14	do	21. 52	21. 28	985	+. 05	−. 01	989	161	165	9	
15	do	21. 55	21. 61	990	0	−. 01	989	161	165	9	
	Average of 3 runs									9	
Apr. 5	L3 L4 M3	24. 85	24. 25	3. 74713	+. 13	−. 10	3. 74716	0. 678318	0. 678369	75	−7
9	do	24. 55	24. 48	718	+. 05	−. 10	713	315	370	81	−1
11	do	24. 32	24. 14	715	+. 06	−. 10	711	313	369	83	+1
12	do	24. 29	24. 17	654	+. 05	−. 10	649	258	314	83	+1
18	do	23. 21	23. 13	719	+. 03	−. 10	712	314	370	83	+1
19	do	23. 20	23. 19	717	+. 02	−. 10	709	311	370	87	+5
19	do	23. 20	23. 21	720	+. 02	−. 10	712	314	370	83	+1
23	do	23. 26	23. 30	724	+. 01	−. 10	715	317	370	78	−4
23	do	23. 26	23. 31	690	+. 01	−. 10	681	286	342	83	+1
25	do	23. 22	23. 67	697	−. 04	−. 10	683	288	343	81	−1
26	do	23. 22	23. 77	698	−. 05	−. 10	683	288	342	80	−2
	Average of 11 runs									82	±2
May 2	L3 L4 M3	26. 76	26. 70	8. 43070	+. 20	−. 48	8. 43042	1. 017435	1. 017524	87	+3
10	do	26. 94	26. 51	3064	. 32	−. 48	3048	438	526	87	+3
12	do	26. 99	26. 24	3056	. 42	−. 48	3050	440	526	85	+1
16	do	26. 93	26. 17	3061	. 42	−. 48	3055	444	528	83	−1
17	do	26. 91	26. 19	3060	. 41	−. 48	3053	442	528	85	+1
26	do	27. 11	26. 72	2994	. 32	−. 48	2978	398	484	85	+1
27	do	27. 02	26. 62	2998	. 32	−. 48	2982	400	484	83	−1
June 4	do	27. 37	26. 58	2992	. 45	−. 48	2989	403	487	83	−1
6	do	27. 35	26. 58	2995	. 44	−. 48	2991	405	489	83	−1
7	do	27. 36	26. 65	2996	. 43	−. 48	2991	405	487	81	−3
8	do	27. 07	26. 57	3003	. 35	−. 48	2990	404	488	83	−1
9	do	27. 02	26. 40	2995	. 39	−. 48	2986	402	488	85	+1
10	do	27. 03	26. 40	2999	. 37	−. 48	2988	403	488	84	0
10	do	27. 03	26. 39	3074	. 37	−. 48	3063	448	535	86	+2
	Average of 14 runs									84	±2

[1] In each case the first coil listed is the upper fixed coil.

TABLE 15.—*Data and results on the absolute determination of the ampere*—Contd.

[All final runs have been included]

	1	2	3	4	5	6	7	8	9	10	11
Date	Coil combination[1]	Temperature of coils		Observed doubled force	Correction for		Doubled force corrected to 22 C and no load	Values of the current		$\frac{I_{BS}-I_A}{I_A}$	Deviation from mean of set
		Fixed; average of two	Moving		Temperature	Load and leads		I_A	I_{BS}		
1932		°C	°C	*Grams*	*Milligram*	*Milligram*	*Grams*	*Absolute amperes*	*B.S. Int. amperes*	*Parts per million*	*Parts per million*
June 23	L3 L4 M3	24. 20	24. 26	2. 08778	+. 01	−. 02	2. 08777	0. 506318	0. 506357	77	0
24	do	24. 17	24. 24	778	+. 01	−. 02	777	318	357	77	0
24	do	24. 17	24. 24	781	+. 01	−. 02	780	322	361	77	0
24	do	24. 17	24. 25	789	+. 01	−. 02	788	330	372	83	+5
29	do	25. 39	25. 45	776	+. 02	−. 02	776	317	356	77	0
29	do	25. 39	25. 45	778	+. 02	−. 02	778	319	356	73	−4
	Average of 6 runs									77	±2
Oct. 25	S2 S1 M2	27. 20	26. 04	11. 92045	+. 45	−. 32	11. 92058	1. 017786	1. 017805	19	+4
26	do	27. 20	25. 95	1762	+. 55	−. 32	1785	770	683	13	−2
Nov. 1	do	25. 59	24. 51	1759	+. 53	−. 32	1780	665	683	18	+3
1	do	25. 60	24. 52	1761	+. 53	−. 32	1782	667	683	16	+1
8	do	25. 52	24. 35	1751	+. 60	−. 32	1779	664	683	19	+4
14	do	25. 50	24. 42	1786	+. 54	−. 32	1808	679	687	8	−7
	Average of 6 runs									15	±4

[1] In each case the first coil listed is the upper fixed coil.

A summary of the results of the measurements with the four sets of coils is given in table 16. The first three columns are self-explanatory. In column 4 are given the averages, for each combination of coils, of the mean results, as recorded in column 10 of table 15, for that combination. Each number in column 4 is subtracted from unity to obtain the numbers which are given in column 5, each number expressing, for any current, the ratio of its value in absolute amperes to its value in B.S. international amperes. The mean of these values, each weighted as indicated in column 6, is the final result of this investigation. The numbers in column 6 for weighting the values in column 5 are approximately proportional to the reciprocals of the average indicated error for each coil combination as given in table 14. Since only for one coil combination was the indicated error appreciably different from the others, only this combination was given a low weight. The difference between the final result and the value for each set of coils is given in column 7.

TABLE 16.—*The average results with each combination of coils*

1	2	3	4	5	6	7
Fixed coils	Moving coil	Number of independent observations	$\frac{I_{BS}-I_A}{I_A}$	Ratio of values of a current in Absolute Amperes to the value in B.S.Int. Amperes	Weight	Deviation from weighted mean
L3 L4	M3	31	81×10^{-6}	0. 999919	5	-9×10^{-6}
L3 L4	M2	30	94	. 999906	5	−22
S1 S2	M3	14	53	. 999947	5	+19
S1 S2	M2	30	10	. 999990	1	+62
Weighted means				. 999928		$\pm20\times10^{-6}$
Probable error of weighted mean						$\pm8\times10^{-6}$

The result of this investigation may be expressed by stating that a current which has a value of 1 B.S. international ampere has a value of 0.999928 absolute ampere. This may be expressed by the equality

1 B.S. International Ampere = 0.999928 Absolute Ampere

The authors estimate that this result differs from the true value by less than 20 parts in a million.

The result obtained by Rosa, Dorsey, and Miller in 1911 may be expressed[44] as

1 B.S. international ampere = 0.999926 absolute ampere

The close agreement between the results (two parts in a million) is fortuitous, since the difference between the results is less than the probable error of either. The recent measurements[45] with the silver voltameter also indicate that there has been no important change in the unit of current at the Bureau of Standards since 1911.

A recent estimate[46] of the value of the B.S. international ohm in absolute measure is

1 B.S. international ohm = 1.000460 absolute ohms

From this estimate and the results on the determination of the ampere here reported, the value of any electrical quantity can be estimated in absolute units from its value as measured in international units. For example:

1 B.S. international volt = 1.000388 absolute volts
1 B.S. international watt = 1.000316 absolute watts

VIII. DISCUSSION OF RESULTS

The results on the ratio of the radii were self-checking, as shown in table 10, where the possible errors in the measurements on the coils are indicated. The largest indicated error in the ratio of the radii affects the absolute value of the current as measured with any coil combination by only a few parts in a million. The observed difference in the results obtained with different coil combinations is much larger than can be accounted for in that way.

The uniformity of the results in the absolute value of the current obtained on different days with a given coil combination, as shown in column 10 of table 15, indicates that the weighings and reduction to standard conditions caused an error of only six parts in a million in the most unfavorable case. In the cases where the coils were twice assembled in the balance, the results from the two assemblies differed by 5 and 7 parts in a million, respectively. This shows that errors in the adjustments of the coils did not cause all of the observed differences.

The observed difference between the mean of all the results and the result with any one coil combination might be explained by assuming that there is an error in the cross-sectional dimensions of one of the coils. However, no reasonable assumption concerning the errors in the cross sections of the coils will bring the results with the different

[44] In that work the result was stated in terms of the electromotive force of the Weston normal cell at 20 C. The value obtained was 1.018225 semiabsolute volts, a semiabsolute volt being the potential drop produced by 1 absolute ampere in a resistance of 1 B.S. international ohm. The Weston normal cell at 20 C. has, by international agreement, a value of 1.018300 international volts. Since, for two systems of units which have the same unit of resistance, the units of current and the units of electromotive force are directly proportional, the value obtained in 1911 for the ratio of the international to the absolute ampere is 1.018225/1.018300 = 0.999926.

[45] G. W. Vinal, International comparison of electrical units, B.S. Jour. Research, vol. 8, p. 729, 1932.

[46] See p. 92 of paper referred to in footnote 10.

coil combinations into substantially better agreement. Another possible explanation of the variation in the results when different coil combinations were used is that the coil windings did not strictly conform to the conditions assumed in deriving the theoretical formula. This can be tested by using a different type of coil. At present there is no satisfactory explanation of the rather large discrepancy in results with different coil combinations. It may be noted, however, that observations with the coil combination S1 S2 M3 were made under less favorable conditions than the others. The work was done in the late summer when the humidity was high and the electrical measurements were difficult. The result with the coil combination S1 S2 M2 differs from the mean by a much larger amount than any other coil combination. That combination was not used in 1911, probably because the computation of the force by the formulas then available would have been very laborious. As the computation can be readily made by the formula now in use, there seemed to be no reason why that combination should not be used in this investigation. The reason why the result with that combination is so different from the others has not been found, and nothing has been discovered that would justify the discarding of the value. The low weighting of the result for that combination is justified because the weighting factors were deduced from an entirely independent set of measurements.

The authors are of the opinion that, to obtain a more accurate result for the absolute value of the ampere, the present current balance should be improved and some entirely different method [47] should be perfected in which the systematic errors would probably be quite different from those in the current-balance method. The most obvious improvement of the current balance is the construction of new coils, the cross sections of which can be more accurately measured and the windings of which conform more nearly to the conditions assumed in deriving the equation for the force.

The authors wish to express their indebtedness to many members of the staff of this Bureau for the helpful cooperation without which this work would have been much more difficult. The resistance section and the electrochemical section made many tests of resistance coils and standard cells. N. P. Case gave assistance in the design and setting-up of much of the apparatus; G. B. Schubauer made the earlier measurements with the current balance, and V. H. Goerke made most of the measurements of the ratio of the radii of the coils.

WASHINGTON, March 24, 1934.

[47] See footnote 11 p. 668.

NOTE.—After the manuscript of the above paper was sent to press decision was reached to use the name National Bureau of Standards (abbreviated NBS) instead of Bureau of Standards, (abbreviated B.S.) as given in this paper.